On the Finsler modules over $H^*$-algebras

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Abstract. In this paper, applying the concept of generalized $A$-valued norm on a right $H^*$-module and also the notion of $\phi$-homomorphism of Finsler modules over $C^*$-algebras we first improve the definition of the Finsler module over $H^*$-algebra and then define $\phi$-morphism of Finsler modules over $H^*$-algebras. Finally we present some results concerning these new ones.

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1. Introduction and Preliminaries

Generalized $A$-valued norm on a right $H^*$-module has been introduced by [12]Zalar (1995), also Finsler module over a $C^*$-algebra has been investigated by [7] Phillips and Weaver (1998), then many mathematicians developed these subjects in several directions. The authors of [3] Amyari and Niknam (2003) and [11] Taghavi and Jafarzadeh (2007), studied $\phi$-homomorphisms of Finsler modules over $C^*$-algebras. Taking idea from these notions we are motivated to improve the concept of Finsler module over $H^*$-algebra (see [1]Ambrose (1945), [4] Balachandran and Swaminathen (1986)) and define $\phi$-morphism of Finsler modules over $H^*$-algebras and investigate some properties for these new ones. A $H^*$-algebra, introduced by [1]Ambrose (1945) in the associative case, is a Banach algebra $A$ satisfying the following conditions:

(i) $A$ is itself a Hilbert space under an inner product $\langle.,.\rangle$;

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(ii) For each $a$ in $A$ there is an element $a^*$ in $A$, the so-called adjoint of $a$, such that we have both $\langle ab, c \rangle = \langle b, a^*c \rangle$ and $\langle ab, c \rangle = \langle a, cb^* \rangle$ for all $b, c \in A$.

**Example 1.1** The Hilbert space $l^2 = \{\{a_n\}_n : a_n \in \mathbb{C}, \sum_n |a_n|^2 < \infty \}$ is a $H^*$-algebra, where for each $\{a_n\}_n$ and $\{b_n\}_n$ in $l^2$, $\{a_n\}_n\{b_n\}_n = \{a_nb_n\}_n$ and $\{a_n\}_n^* = \{\overline{a_n}\}_n$.

**Example 1.2** Any Hilbert space is a $H^*$-algebra, where the product on each pair of elements to be zero. Of course in this case the adjoint $a^*$ of $a$ need not be unique, in fact every element, is an adjoint of every element.

Recall that $A_0 = \{a \in A : aA = \{0\}\} = \{a \in A : Aa = \{0\}\}$ is called the annihilator ideal of $A$. A proper $H^*$-algebra is a $H^*$-algebra with zero annihilator ideal. [1]Ambrose (1945), proved that a $H^*$-algebra is proper if and only if every element has a unique adjoint.

The trace class $\tau(A)$ of $A$ is defined by the set $\tau(A) = \{ab : a, b \in A\}$. It is known that $\tau(A)$ is an ideal of $A$ which is a Banach *-algebra under a suitable norm $\tau_A(\cdot)$. The norm $\tau_A$ is related to the given norm $\|\cdot\|$ on $A$ by $\tau_A(a^*a) = \|a\|^2$ ($a \in A$) and $\|a\| \leq \tau_A(a)$ for each $a \in \tau(A)$ (see [9]Saworotnow (1970)). If $A$ is proper, then $\tau(A)$ is dense in $A$ ([1, Lemma 2.7]). The trace functional $tr$ on $\tau(A)$ is defined by $tr(ab) = \langle a, b^* \rangle = \langle b, a^* \rangle = tr(ba)$ for each $a, b \in A$, in particular $tr(aa^*) = tr(a^*a) = \|a\|^2$ for all $a \in A$. A positive member of $A$ is an element $a \in A$ such that $\langle ax, x \rangle \geq 0$ for each $x \in A$. It is known in [9]Saworotnow (1970), that for each $a \in A$ there exists a unique positive member $[a]$ of $A$ such that $[a]^2 = a^*a$. A nonzero element $e \in A$ is called a projection, if it is self adjoint and idempotent. Two idempotents $e$ and $e'$ are doubly orthogonal if $\langle e, e' \rangle = 0$ and $ee' = e' = 0$. An idempotent is primitive if it can not be expressed as the sum of two doubly orthogonal idempotents. Every proper $H^*$-algebra contains a maximal family of doubly orthogonal primitive self adjoint idempotents ([1, Theorem 3.3]). If $\{e_i\}_{i \in I}$ is a maximal family of doubly orthogonal primitive self adjoint idempotents in a proper $H^*$-algebra $A$, then $A = \sum_{i \in I} e_iA = \sum_{i \in I} Ae_i$ ([1, Theorem 4.1]) and $a = \sum_{i \in I} e_ia = \sum_{i \in I} a_{i}$, for each $a \in A$. For, if $a \in A$, then $a = \sum_{i \in I} e_ib_i$ for some $b_i \in A$ and so for each $j \in I$, $\sum_{i \in I} e_j b_i = e_jb_\lambda$. The next part is proved similarly. We recall from [9]Saworotnow (1970), that if $a$ is a nonzero element in $A$, then there exists a sequence $\{e_n\}_n$ of doubly orthogonal projections and a sequence $\{\lambda_n\}_n$ of positive numbers such that $a^*a = \sum_n \lambda_n e_n$. In this case, $[a] = \sum_n \frac{1}{\lambda_n} e_n$ and if $a$ is in $\tau(A)$, then $\tau_A(a) = tr([a])$. Throughout this note we mean by a morphism a *-homomorphism of proper $H^*$-algebras.

The notion of Hilbert $H^*$-module is introduced by [8]Saworotnow (1968) under the name of generalized Hilbert space. It has been studied by Smith, Molnar, Cabrera, Martinez, Rodriguez and others.

**Definition 1.3** Let $A$ be a proper $H^*$-algebra. A Hilbert $H^*$-module is a left module $E$ over $A$ with a mapping $[\cdot, \cdot] : E \times E \rightarrow \tau(A)$, which satisfies the following conditions:

(i) $[ax, y] = a[x, y]$,  
(ii) $[x + y, z] = [x, z] + [y, z]$,  
(iii) $[ax, y] = a[x, y]$,  
(iv) $[x, y]^* = [y, x]$.  
(v) For each nonzero element $x$ in $E$ there is a nonzero element $a$ in $A$ such that $[x, x] = a^*a$.  
(vi) $E$ is a Hilbert space with the inner product $(x, y) = tr([x, y])$, for each $a \in \mathbb{C}$, $x, y, z \in E$, $a \in A$. For example every $H^*$-algebra $A$ is a Hilbert $A$-module whenever we define $[x, y] = xy^*$. We say Hilbert $A$-module $E$ is full, if the linear

Finsler modules over \(H^\ast\)-algebras are generalization of Hilbert \(H^\ast\)-modules. It first was introduced by [12] Zalar (1995) by defining a generalized \(A\)-valued norm on a right \(H^\ast\)-module. It is proved in [12] Zalar (1995), that a generalized \(A\)-valued norm \(\rho\) on a \(H^\ast\)-module \(E\) over a proper \(H^\ast\)-algebra \(A\) arises from a \(\tau(A)\)-valued inner product \([\cdot, \cdot]\) on \(E\), if and only if \(\rho\) satisfies the parallelogram low. In this paper, we improve and investigate some facts concerned with this concept. In the sequel, we extend the definition of \(\phi\)-homomorphism of Finsler modules over \(H^\ast\)-algebras by the name of \(\phi\)-morphisms and describe some basic properties of such class of module maps ([11] Taghavi and Jafarzadeh (2007)). This work is a reconstruction of some results appeared in [2] Amyari and Niknam (2003), [3] Amyari and Niknam (2003), [11] Taghavi and Jafarzadeh (2007), to Finsler modules over \(H^\ast\)-algebras and is also interesting in its own.

2. Main Results

Definition 2.1 ([12] Zalar (1995)) Let \(A\) be a proper \(H^\ast\)-algebra and \(E\) be a complex linear space which is a left \(A\)-module and \(\lambda(ax) = (\lambda a)x = a(\lambda x)\) where \(\lambda \in \mathbb{C}\), \(a \in A\) and \(x \in E\) equipped with a map \(\rho_A : E \to \{ a^*a : a \in A \}\) such that

(i) the map \(\| \cdot \|_E : x \mapsto tr(\rho_A(x))^{1/2}\) is a norm on \(E\);

(ii) \(\rho_A(ax) = a\rho_A(x)a^*\) for each \(a \in A\) and \(x \in E\). Then \(E\) is called a pre-Finsler module over \(H^\ast\)-algebra \(A\). If \((E, \| \cdot \|_E)\) is complete, then \(E\) is called a Finsler module. For instance, every Hilbert \(H^\ast\)-module \(E\) with the map \(\rho_A(x) = \|x\|_E(x)\) \((x \in E)\) is a Finsler module.

Example 2.2 The set \(A = \ell^2\), is a proper \(H^\ast\)-algebra and \(\tau(A) = A\) (since \(A\) is unital). It is easy to verify that \(\{e_i\}_{i \in \mathbb{N}}\) \((e_i, i \in \mathbb{N})\) is a maximal family of doubly orthogonal projections for \(A\). If \(\{a_n\}_n \in A\), then \(\{(a_n)\}_n = \{a_n\}_n^2 = \sum_n |a_n|^2 e_n\), \(\{(a_n)\}_n = \sum_n |a_n| e_n\) and \(\tau_A(\{a_n\}_n) = tr(\{(a_n)\}_n) = tr(\sum_n |a_n| e_n) = \sum_n |a_n| tr(e_n) = \sum_n |a_n|\). Since \(tr(e_n) = tr(e_n^2) = 1\).

Let \(E = A\) and \(\rho_A : E \to \{(a_n)\}_n^2 : \{a_n\}_n \in A\) be defined by \(\rho_A(\{a_n\}_n) = \{a_n\}_n^2\). Then \(E\) is a full (Hilbert module) Finsler module over \(A\). For fullness of \(E\), let \(\epsilon > 0\) be given and \(\{a_n\}_n \in \tau(A)\). Then by definition of \(\tau_A\), it is easy to find complex numbers \(\lambda_i\) and \(a_{i,n}\) \((n \in \mathbb{N}, i = 1, ..., k)\), in which \(\tau_A(\sum_{i=1}^k \lambda_i |a_{i,n}|^2 - a_{i,n}) < \epsilon\). Now surjectivity of \(\rho_A\) gives the desired result, i.e. \(\|\rho_A(E)\|_{\tau_A} = \tau(A)\).

The following lemmas which are interesting, will be used frequently later.

Lemma 2.3 Let \(E\) be a Finsler module over \(H^\ast\)-algebra \(A\). Then it is a Banach \(A\)-module.

Proof. By the definition of Finsler module, \(E\) is a Banach space. It remains to show that \(\|ax\|_E \leq \|a\|_E \|x\|_E\) for all \(a \in A\) and \(x \in E\). For, let \(x \in E\). Then \(\rho_A(x) = b^*b\) for some \(b \in A\) and \(\|x\|_E = tr(\rho_A(x)^2) = tr(b^*b^2) = \|b\|\). So \(\|ax\|_E^2 = tr(\rho_A(ax)) = tr(a\rho_A(x)a^*) = \|ba^*\|^2 \leq \|b\|^2 \|a\|^2 = \|x\|^2_\mathbb{C} \|a\|^2\).
As a consequence of the above lemma we have $\|ax\|_E \leq \tau_A(a)\|x\|_E$ for each $a \in \tau(A)$ and $x \in E$.

**Lemma 2.4** Let $E$ be a full Finsler module over $H^*$-algebra $A$ and $a \in A$. Then $ax = 0$ for all $x \in E$ if and only if $a = 0$.

**Proof.** Firstly, suppose that $a \in \tau(A)$ and also $b \in \tau(A)$ is arbitrary. Since $E$ is full, there exists a sequence $\{u_n\}_n$ in $\langle \rho_A(E) \rangle$ such that $b = \lim_{n \to \infty} \tau^* u_n$. Each $u_n$ is of the form

$$u_n = \sum_{i=1}^{k_n} \lambda_{i,n} \rho_A(x_{i,n})$$

in which $\lambda_{i,n} \in \mathbb{C}$, $x_{i,n} \in E$. Hence,

$$aba^* = \lim_{n \to \infty} \tau^* a u_n a^* = \lim_{n \to \infty} \tau^* (\sum_{i=1}^{k_n} \lambda_{i,n} \rho_A(x_{i,n}) a^*) = \lim_{n \to \infty} \tau^* \sum_{i=1}^{k_n} \lambda_{i,n} \rho_A(x_{i,n}) a^* = 0. \quad (1)$$

Relation (1) holds since if $x$ is an arbitrary element in $E$, then $\rho_A(ax) = c^* c$ for some $c \in A$ and by assumption $\|c\|^2 = \text{tr}(c^* c) = \text{tr}(\rho_A(ax)) = \|ax\|_E^2 = 0$. It implies that $c = 0$ and so $\rho_A(ax) = 0$. Replacing $b$ by $a^* a$ in (1) we get $\text{tr}(aba^*) = \text{tr}(aa^* a^*) = \|aa^*\|^2 = 0$. Consequently $aa^* = 0$ and by [1, Lemma 2.2], $a = 0$. Secondly, suppose that $a \in A$ and $ax = 0$ for all $x \in E$. Let $b \in A$ be arbitrary, then by Lemma 2.3, $bax = 0$ for all $x \in E$. By the above discussion and since $ba \in \tau(A)$, so $ba = 0$ for each $b \in A$. It implies that $Aa = 0$. Hence $a = 0$, because $A$ is proper. \[\square\]

**Remark 1** If $\phi : A \to B$ is an isometric morphism of $H^*$-algebras, then for each $a \in A$,

$$\|\phi(a)\|^2 = \|a\|^2$$

and so $\langle \phi(a), \phi(a) \rangle = \langle a, a \rangle$. Whence $\text{tr}(\phi(aa^*)) = \text{tr}(aa^*)$. If in addition $\phi$ is an isomorphism, then for each $b \in B$, $\text{tr}(\phi^{-1}(bb^*)) = \text{tr}(bb^*)$.

Taking idea from [2] Amyari and Niknam (2003), we have two following theorems.

**Theorem 2.5** Let $E$ be a full Finsler module over $H^*$-algebra $B$, $\phi : A \to B$ be a morphism of $H^*$-algebras such that $\phi|_{\tau(A)} : \tau(A) \to \tau(B)$ be a $\tau$-continuous isomorphism and isometric with respect to $\| \cdot \|$ Then by the module action, $ax = \phi(a)x$ and the map $x \mapsto \rho_A(x)$ defined by $\rho_A(x) = \phi^{-1}(\rho_B(x))$, $E$ is a full Finsler $A$-module.

**Proof.** It is clear that $E$ is a complex linear space, and by morphism of $\phi$, $E$ is a left $A$-module. Because of isometric isomorphism of $\phi|_{\tau(A)}$, for each $x \in E$ we have $\|x\|_E^A = \text{tr}(\rho_A(x))^\frac{1}{2} = tr(\rho_B(x)) = \|x\|_B^B$ (2). Furthermore, $\| \cdot \|_E^A$ is a norm on $E$ and so $\| \cdot \|_E^A$ is. Let $a \in A$, $x \in E$, then $\rho_A(ax) = \rho_A(\phi(a)x) = \phi^{-1}(\rho_B(x)) a^* = a \phi^{-1}(\rho_B(x)) a^* = \rho_A(ax) a^*$. Hence $E$ is a pre-Finsler module over $A$. On the other hand (2) and completeness of $(E, \| \cdot \|_B^B)$ imply that $(E, \| \cdot \|_E^A)$ is complete. Thus $E$ is a Finsler module over $A$. We will show that $E$ is a full Finsler module over $A$, i.e. $[\rho_A(E)]^\tau_A = \tau(A)$. Note that by the inverse mapping theorem $(\phi|_{\tau(A)})^{-1} : \tau(B) \to \tau(A)$ is a $(\tau_B, \tau_A)$-continuous isomorphism
Now let \( a \in A \) shows that 
\[
\langle \rho_A(E) \rangle^{\tau_A} = \left\{ \sum_{i=1}^{k_n} \lambda_{i,n} \rho_A(x_{i,n}) : \lambda_{i,n} \in \mathbb{C}, \ x_{i,n} \in E \right\}
\]
\[
= \sum_{i=1}^{k_n} \lambda_{i,n} \rho_B(x_{i,n}) : \lambda_{i,n} \in \mathbb{C}, \ x_{i,n} \in E \}
\]
\[
= \phi^{-1} \left( \sum_{i=1}^{k_n} \lambda_{i,n} \rho_B(x_{i,n}) : \lambda_{i,n} \in \mathbb{C}, \ x_{i,n} \in E \right)
\]
\[
= \phi^{-1}(\langle \rho_B(E) \rangle^{\tau_B}) = \phi^{-1}(\tau(B)) = \tau(A).
\]

In the following we shall establish a converse statement to the above theorem.

**Theorem 2.6** Let \( E \) be a both full Finsler module over \( A \) and a full Finsler module over \( B \) and let \( \phi : A \to B \) be a map such that \( ax = \phi(a)x \) and \( \phi(\rho_A(x)) = \rho_B(x) \), where \( x \in E, a \in A \). Then \( \phi \) is a continuous monomorphism, \( \phi \mid_{\tau(A)} : (\tau(A)) \to \tau(B) \) is a \((\tau_A, \tau_B)\)-continuous and it has dense range, i.e. \( \phi \mid_{\tau(A)}(\tau(A)) = \tau(B) \). If for each \( x \in E \), \( tr(\rho_A(x)) = tr(\rho_B(x)) \), then \( \phi \) is isometric on the set \( \{ a \in A : \text{there exists } x \in E \text{ in which } a^*a = \rho_A(x) \} \).

**Proof.** For simplicity in writing we put \( \phi_1 = \phi \mid_{\tau(A)} \). Assume that \( \{a_n\}_n \) is a sequence in \( \tau(A) \) such that \( \lim_{n \to \infty} a_n = 0 \) and \( \lim_{n \to \infty} \phi_1(a_n) = b \). Applying Lemma 2.4, \( b = 0 \). It follows from closed graph theorem that \( \phi_1 \) is \((\tau_A, \tau_B)\)-continuous. A similar argument shows that \( \phi \) is continuous. Since \( (\phi(a + b) - \phi(a) - \phi(b))x = (a + b)x - ax - bx = 0 \) for each \( x \in E \) and for each \( a, b \in A \), so by Lemma 2.4, \( \phi(a + b) = \phi(a) + \phi(b) \).

Similarly for each \( \lambda \in \mathbb{C} \) and for each \( a, b \in A \), \( \phi(\lambda a) = \lambda \phi(a) \) and \( \phi(ab) = \phi(a)\phi(b) \).

Now let \( a \in \tau(A) \), then we may assume that \( a = \lim_{n \to \infty} \tau_A u_n \), each \( u_n \) is of the form

\[
u_n = \sum_{i=1}^{k_n} \lambda_{i,n} \rho_A(x_{i,n}) \text{ for some } \lambda_{i,n} \in \mathbb{C} \text{ and } x_{i,n} \in E \text{. Hence } \phi_1(a^*) = \lim_{n \to \infty} \tau_B \phi_1(u_n^*) = \]

\[
\lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n} \rho_A(x_{i,n}^*) = \lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n} \rho_B(x_{i,n}) = \lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n} \rho_B(x_{i,n}) = \]

\[
(\lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n} \rho_A(x_{i,n}))^* = (\lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n} \rho_A(x_{i,n}))^* = \phi_1(a)^*. \]

Therefore \( \phi_1 \) is a morphism. Let \( a \in A \), then there exists a sequence \( \{a_n\}_n \subseteq \tau(A) \) such that \( a = \lim_{n \to \infty} a_n \). By morphism of \( \phi_1 \) and
continuity of $\phi$ we can write $\phi(a^*) = \phi(\lim_{n \to \infty} a_n^*) = \lim_{n \to \infty} \phi(a_n)^* = (\lim_{n \to \infty} \phi(a_n))^* = (\phi(a))^*$. If $\phi(a) = 0$, then $ax = \phi(a)x = 0$, for all $x \in E$. Hence $a = 0$, by Lemma 2.4. Therefore $\phi$ is a monomorphism. Given $\epsilon > 0$ and let $b \in \tau(B)$ be arbitrary. Since $E$ is a full Finsler module over $B$, so $\tau_B(b - \sum_{i=1}^{k_n} \lambda_{i,n} \rho_B(x_{i,n})) < \epsilon$, for some $\lambda_{i,n} \in \mathbb{C}$ and $x_{i,n} \in E$. Hence $\tau_B(b - \phi_1(\sum_{i=1}^{k_n} \lambda_{i,n} \rho_A(x_{i,n}))) < \epsilon$. Therefore $\phi_1$ has dense range in $\tau(B)$. Now suppose that for each $x \in E$, $tr(\rho_A(x)) = tr(\rho_B(x))$. Also assume that $a \in A$ and $a^*a = \rho_A(x)$ for some $x \in E$, then $\|a\|^2 = tr(a^*a) = tr(\rho_A(x)) = tr(\rho_B(x)) = tr(\phi(\rho_A(x))) = tr(\phi(a^*a)) = \|\phi(a)\|^2$.

We could not drop the condition of fullness. For instance, let $B = I^2$ and $A = E = \{\{a_n\}_n \in B : a_1 = 0\}$. Then $E$ is a full Finsler module over $A$, when $\rho_A(\{a_n\}_n) = \{||a_n||^2\}_n$ and $E$ is a Finsler module over $B$ when $\rho_B(\{a_n\}_n) = \{a_n^2\}_n$. $E$ is not full over $B$, because let $\{b_n c_n\}_n \in \tau(B)$ (= $B$) with $b_1 c_1$ be nonzero. If on the contrary $\langle \rho_B(E) \rangle^{\tau_B} = \tau(B)$, then there exist $\lambda_i \in \mathbb{C}$ and $\{a_{i,n}\}_n \in E$ ($i = 1, ..., k$) in which $\tau_B(\sum_{i=1}^{k} \lambda_i \{a_{i,n}\}_n - \{b_n c_n\}_n) < \epsilon$ (3). Put $\{d_n\}_n = \sum_{i=1}^{k} \lambda_i \{a_{i,n}\}_n - \{b_n c_n\}_n$. As we see in Example 2.2, the left side of (3) is equal to $\sum_{n=1}^{\infty} |d_n|$. Hence $|b_1 c_1| = |d_1| \leq \sum_{n=1}^{\infty} |d_n| < \epsilon$ by (3) and since $\epsilon > 0$ is arbitrary, so $b_1 c_1 = 0$, which is a contradiction. Now let $\phi : A \to B$ be the inclusion map, obviously $\phi$ satisfies in the conditions of Theorem 2.7, i.e, for each $x \in E$ and for each $a \in A$, $ax = \phi(a)x$ and $\phi(\rho_A(x)) = \rho_B(x)$. On the other hand $\phi(\tau(A))^{\tau_B} \neq \tau(B)$. Indeed, by a similar argument as above if $\{b_n c_n\}_n \in \tau(B)$ (= $B$) and $b_1 c_1 \neq 0$, then it is not in $\phi(\tau(A))^{\tau_B} (= A)$. Thus $\phi|_{\tau(A)}$ does not have dense range in $\tau(B)$.

The following theorem is a version of [3, Lemma 2.2] in the framework of Finsler modules over $H^*$-algebras.

**Theorem 2.7** Let $E$ be a Finsler module over $H^*$-algebra $A$, $I$ be a closed two sided ideal of $A$ and $x$ be in $E$ such that $\rho_A(x) \in I$. Then $x = \sum_{\lambda \in \Lambda} e_{\lambda} x$, where $\{e_{\lambda}\}_{\lambda \in \Lambda}$ is a maximal family of doubly orthogonal primitive self adjoint idempotents for $I$.

**Proof.** Let $\Lambda_0$ be a finite subset of $\Lambda$. We claim that

$$
\rho_A(x - \sum_{\lambda \in \Lambda_0} e_{\lambda} x) = \rho_A(x) - \sum_{\lambda \in \Lambda_0} e_{\lambda} \rho_A(x) - \sum_{\lambda \in \Lambda_0} \rho_A(x) e_{\lambda} + \sum_{\lambda \in \Lambda_0} e_{\lambda} [d] \sum_{\gamma \in \Lambda_0} [d] e_{\gamma}
$$

(4)

where $\rho_A(x) = d^* d = [d]^2$ for some $d \in A$ ([9, Lemma 2]). If $b$ is the left side and $c$ is the
right side of (4), then obviously $b$ and $c$ are self adjoint and for each $a \in A$, we have

$$ac^* = a\rho_A(x)a^* - a \sum_{\lambda \in \Lambda_0} e_\lambda \rho_A(x)a^* - a \sum_{\lambda \in \Lambda_0} \rho_A(x)e_\lambda a^* + a \sum_{\lambda \in \Lambda_0} e_\lambda [d] \sum_{\gamma \in \Lambda_0} [d]e_\gamma a^*$$

$$= \left(a - \sum_{\lambda \in \Lambda_0} ae_\lambda \right) \rho_A(x) \left(a - \sum_{\gamma \in \Lambda_0} ae_\gamma \right)^* = \rho_A \left(\left(a - \sum_{\lambda \in \Lambda_0} ae_\lambda \right)x\right)$$

$$= \rho_A \left(a \left(x - \sum_{\lambda \in \Lambda_0} e_\lambda x\right)\right) = a\rho_A \left(x - \sum_{\lambda \in \Lambda_0} e_\lambda x\right)a^* = aba^*.$$

Thus for each $a \in A$, $a(c - b)a^* = 0$, specially for $a = c - b$. Hence $(c - b)^3 = 0$ and so $c = b$ by [1, Lemma 2.3]. Consequently $\rho_A(x - \sum_{\lambda \in \Lambda_0} e_\lambda x) = 0$ and so $\text{tr}(\rho_A(x - \sum_{\lambda \in \Lambda_0} e_\lambda x))^\frac{1}{2} = \|x - \sum_{\lambda \in \Lambda} e_\lambda x\|_{E} = 0$ which implies that, $x = \sum_{\lambda \in \Lambda} e_\lambda x$.

**Definition 2.8** Let $E$ and $F$ be Finsler modules over proper $H^*$-algebras $A$ and $B$ respectively and $\phi : A \to B$ be a morphism of $H^*$-algebras. A linear operator $\Phi : E \to F$ is said to be a $\phi$-morphism of Finsler modules if the following conditions are satisfied:

(i) $\Phi(ax) = \phi(a)\Phi(x)$,

(ii) $\rho_B(\Phi(x)) = \phi(\rho_A(x))$,

where $x \in E$ and $a \in A$.

$\Phi$ is called a module map if it satisfies in the condition (i). If $E, F$ and $G$ are Finsler modules over proper $H^*$-algebras $A, B$ and $C$ respectively, $\phi_1 : A \to B$ and $\phi_2 : B \to C$ are morphisms of $H^*$-algebras, and $\Phi_1 : E \to F$ and $\Phi_2 : F \to G$ are $\phi_1$-morphism and $\phi_2$-morphism of Finsler modules respectively, then it is straightforward to show that $\Phi_2\Phi_1 : E \to G$ is a $\phi_2\phi_1$-morphism of Finsler modules.


**Theorem 2.9** Let $E$ and $F$ be Finsler modules over $H^*$-algebras $A$ and $B$ respectively, $\phi : A \to B$ be a morphism in which $\phi|_{\tau(A)} : \tau(A) \to \tau(B)$ be a $(\tau_A, \tau_B)$-continuous injective morphism and $\phi(\tau(A))$ be $\tau_B$-closed in $\tau(B)$. Also let $\Phi : E \to F$ be a $\phi$-morphism. If $\text{Im}(\Phi)$ is a full Finsler module over $\text{Im}(\phi)$, then $E$ is a full Finsler module over $A$.

**Proof.** Applying inverse mapping theorem, $(\phi|_{\tau(A)})^{-1} : \phi(\tau(A)) \to \tau(A)$ is a $(\tau_B, \tau_A)$-continuous morphism. We will show that $E$ is full. Let $a \in \tau(A)$ be arbitrary, then $a = a_1a_2$ for some $a_1, a_2 \in A$. Therefore $\phi(a) = \phi(a_1)\phi(a_2) \in \tau(\text{Im}(\phi))$. Since $\text{Im}(\Phi)$ is a full Finsler $\text{Im}(\phi)$-module, thus we have

$$\phi(a) = \lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n}\rho_B(\Phi(x_{i,n}))$$

$$= \lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n}\phi(\rho_A(x_{i,n})) (5)$$

for some $\lambda_{i,n} \in \mathbb{C}$, $x_{i,n} \in E$. Effecting $(\tau_B, \tau_A)$-continuous morphism $(\phi|_{\tau(A)})^{-1}$ to both sides of (5), we obtain that $a = \lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n}\rho_A(x_{i,n})$ by injectivity of the
morphism $\phi_{|\tau(A)}$. Thus $a \in \langle \rho_A(E) \rangle^{\tau_A}$ and therefore $\tau(A) \subseteq \langle \rho_A(E) \rangle^{\tau_A} \subseteq \tau(A)$. So $\tau(A) = \langle \rho_A(E) \rangle^{\tau_A}$. Note that $\phi$ cannot come out in (5).

The following lemma is proved in the framework of Finsler modules over $C^*$-algebras ([11, Lemma 3.1]). It is easy to show this lemma in the Finsler modules over $H^*$-algebras.

**Lemma 2.10** Let $E$ and $F$ be Finsler and full Finsler module over $H^*$-algebras $A$ and $B$ respectively, $\phi_i$ $(i = 1, 2)$ be maps from $A$ to $B$ and $\Phi : E \to F$ be a surjective map satisfies $\Phi(ax) = \phi_i(a)\Phi(x)$ $(i = 1, 2)$ for all $x \in E$ and $a \in A$. Then $\phi_1 = \phi_2$.

**Theorem 2.11** Let $E$ and $F$ be full Finsler modules over $H^*$-algebras $A$ and $B$ respectively and $\Phi : E \to F$ be a continuous isomorphism satisfies $\Phi(ax) = \phi(a)\Phi(x)$ and $\rho_B(\Phi(x)) = \phi(\rho_A(x))$, for all $x \in E$ and $a \in A$, where $\phi : A \to B$ be a map. Then $\phi$ is a continuous monomorphism, $\phi_{|\tau(A)}$ is $(\tau_A, \tau_B)$-continuous and has dense range in $\tau(B)$. Moreover, $\phi$ with these conditions is unique.

**Proof.** Applying a similar argument in the proof of Theorem 2.6. one can see that, $\phi$ is a continuous monomorphism and $\phi_{|\tau(A)}$ is $(\tau_A, \tau_B)$-continuous. We will show that $\phi$ is one to one. Let $\phi(a) = 0$ for some $a \in A$, so $\phi(a)\Phi(x) = 0$ for each $x \in E$. Hence $\Phi(ax) = 0$ and by injectivity of $\Phi$, $ax = 0$ for each $x \in E$. Then $a = 0$ by fullness of $E$. So $\phi$ is a monomorphism. In addition, $\tau(B) = \langle \rho_B(F) \rangle = \langle \rho_B(\Phi(E)) \rangle = \langle \phi(\rho_A(E)) \rangle \subseteq \langle \phi(\tau(A)) \rangle = \phi(\tau(A)) \subseteq \tau(B)$. Therefore $\phi(\tau(A)) = \tau(B)$ and so $\phi_{|\tau(A)}$ has dense range. Uniqueness of $\phi$ obtains from Lemma 2.10.

**Remark 2** Fullness condition can not be dropped in the above theorem. For example let $B = l^2$, $A = E = \{a_n\}_n \in B : a_1 = 0\}$ and $F = \{a_n\}_n \in B : a_1 = a_2 = 0\}$. Then $E$ is a full Finsler module over $A$, when $\rho_A(\{a_n\}_n) = \{a_n^2\}_n$ and $F$ is a Finsler module over $B$, when $\rho_B(\{a_n\}_n) = \{a_n^2\}_n$. As we mentioned before $F$ is not full Finsler module over $B$. Let $\Phi : E \to F$ defined by $\Phi(\{a_n\}_n) = \{b_n\}_n$, where $b_1 = 0$ and for $n = 2, \ldots, b_n = a_{n-1}$ and $\phi : A \to B$ defined by $\phi(\{a_n\}_n) = \Phi(\{a_n\}_n)$. Clearly $\Phi$ is a continuous isomorphism, $\Phi(\{a_n\}_n)\Phi(\{b_n\}_n)$ and $\rho_B(\Phi(\{a_n\}_n)) = \phi(\rho_A(\{a_n\}_n))$ for all $\{a_n\}_n \in A$ and $\{b_n\}_n \in E$. On the other hand $\phi(\tau(A)) (= \phi(A))$ dose not have dense range in $\tau(B)$ $(= B)$.

In the following we state [3, Theorem 3.4], in the framework of Finsler modules over the $H^*$-algebras.

**Theorem 2.12** Let $E$ and $F$ be Finsler modules over $H^*$-algebras $A$ and $B$ respectively, $\phi : A \to B$ be an isometric morphism and $\Phi : E \to F$ be a $\phi$-morphism of Finsler modules. Then

(i) $\text{Im}(\Phi)$ is a closed subspace of $F$.

(ii) $\text{Im}(\Phi)$ is a Finsler module over $H^*$-algebra $\text{Im}(\phi)$, such that $\rho_{\text{Im}\phi}(\Phi(E)) = \phi(\rho_A(E))$.

(iii) If $E$ is a full Finsler module and $\phi_{|\tau(A)} : \tau(A) \to \phi(\tau(A))$ is $(\tau_A, \tau_B)$-continuous, then $\text{Im}(\Phi)$ is a full Finsler module over the $H^*$-algebra $\text{Im}(\phi)$.

(iv) If $\Phi$ is surjective, $F$ is full Finsler module over $B$ and $\phi(\tau(A))$ is $\tau_B$-closed, then $\phi_{|\tau(A)}$ is surjective.

**Proof.** (i) We will show that $\Phi$ is isometry and so it has closed range. Let $x$ be an arbitrary element in $E$. Then $\rho_A(x) = a^*a$ for some $a \in A$, and since $\phi$ is isometric so $\|\Phi(x)\|_F = \text{tr}(\rho_B(\Phi(x))) = \text{tr}(\phi(\rho_A(x))) = \text{tr}(\phi(a^*a)) = \text{tr}(\rho_A(x)) = \|x\|_E.$

(ii) Straightforward.
(iii) We will show that $\text{Im}(\Phi)$ is a full Finsler module over the $H^*$-algebra $\text{Im}(\phi)$ i.e. $\langle \rho_B(\text{Im}\Phi) \rangle^{\tau_A} = \tau(\text{Im}\phi)$. For this, let $b \in \tau(\text{Im}\phi)$, then $b = \phi(a_1a_2)$ for some $a_1, a_2 \in A$. By fullness of $E$ and $(\tau_A, \tau_B)$-continuity of $\phi|_{\tau(A)}$ we have $b = \phi(a_1a_2) = \phi(\lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n} \rho_A(x_{i,n})) = \lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n} \phi(\rho_A(x_{i,n})) = \lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n} \rho_B(\Phi(x_{i,n}))$ for some $\lambda_{i,n} \in \mathbb{C}$ and $x_{i,n} \in E$. It gives the desired result.

(iv) It follows by the argument applied in the proof of Theorem 2.11.

References