$n$-Jordan homomorphisms on $C^*$-algebras

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Abstract. Let $n \in \mathbb{N}$. An additive map $h : A \rightarrow B$ between algebras $A$ and $B$ is called $n$-Jordan homomorphism if $h(a^n) = (h(a))^n$ for all $a \in A$. We show that every $n$-Jordan homomorphism between commutative Banach algebras is a $n$-ring homomorphism when $n < 8$. For these cases, every involutive $n$-Jordan homomorphism between commutative $C^*$-algebras is norm continuous.

Keywords: $n$-homomorphism; $n$-ring.

1. Introduction

Let $A$ and $B$ be two algebras. An $n$-ring homomorphism from $A$ to $B$ is a map $h : A \rightarrow B$ that is additive (i.e., $h(a + b) = h(a) + h(b)$ for all $a, b \in A$) and $n$-multiplicative (i.e., $h(a_1 a_2 \ldots a_n) = h(a_1) h(a_2) \ldots h(a_n)$ for all $a_1, a_2, \ldots, a_n \in A$). The map $h : A \rightarrow B$ is called $n$-Jordan homomorphism if it is additive and $h(a^n) = (h(a))^n$ for all $a \in A$. It is clear that every $n$-ring homomorphism is $n$-Jordan homomorphism but the converse is not true. There are some examples of $n$-Jordan homomorphisms which are not $n$-ring homomorphisms (for example refer to [2]). It is shown in [2] that every $n$-Jordan homomorphism between commutative Banach algebras is also $n$-ring homomorphism when $n \leq 4$. For $n = 2$, the proof is simple and routine. For the non-commutative case, Zelazko in [9] showed that if $A$ is a Banach algebra which need not be commutative, and $B$ is a semisimple commutative Banach algebra, then each Jordan homomorphism $h : A \rightarrow B$ is a ring homomorphism.

An $n$-ring homomorphism $h : A \rightarrow B$ between $C^*$-algebras is said to be an $n$-ring homomorphism if $h(a^n) = (h(a))^n$ for all $a \in A$. Similarly one can define an $n$-Jordan homomorphism. If, in addition, $h$ is linear, we say that $h$ is involutive $n$-ring (Jordan) homomorphism.

One of the fundamental results in the study of $C^*$-algebras is that if $T : A \rightarrow B$ is a homomorphism between $C^*$-algebras, then it is norm contractive [6, theorem 2.1.7]. In [4], authors ask: Is every involutive $n$-ring homomorphism between $C^*$-algebras continuous? Park and Trout in [7] answered this question and proved that every involutive $n$-ring homomorphism between $C^*$-algebras is in fact norm

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contractive. Some questions of automatic continuity for \( n \)-homomorphisms between Banach algebras were also investigated in [1, 5]. After that, Tomforde in [8, theorem 3.6] proved that if \( \mathcal{A} \) and \( \mathcal{B} \) are unital \( C^* \)-algebras and \( : \mathcal{A} \rightarrow \mathcal{B} \) is a unital \( \mathcal{B} \)-preserving ring homomorphism, then \( \mathcal{B} \) is contractive. Consequently, \( \mathcal{B} \) is also continuous.

In this paper, we prove that every \( n \)-Jordan homomorphism between commutative Banach algebras is \( n \)-ring homomorphism when \( n = 5 \) (for the case \( n = 5 \) this had been proved earlier by Eshaghi et al in [3] with a long proof). Finally, using these results, we show that every involutive \( n \)-Jordan homomorphism between commutative \( C^* \)-algebras is continuous.

2. Main Results

**Theorem 2.1.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be two commutative algebras, and let \( h : \mathcal{A} \rightarrow \mathcal{B} \) be an \( n \)-Jordan homomorphism. Then \( h \) is an \( n \)-ring homomorphism for \( n = 3, 4, 5, 6, 7 \).

**Proof** For the cases \( n = 3, 4 \), refer to [3]. As for \( n = 5 \), the map \( h \) is additive such that \( h(x^5) = (h(x))^5 \) for all \( x \in \mathcal{A} \). Using this equality, we have

\[
h \begin{align*}
4 \sum_{k=1}^{5} k^{-1} x^k y^{5-k} &= 4 \sum_{k=1}^{5} k^{-1} h(x)^k h(y)^{5-k} \\
&= h(x)^4 h(y)^1 + h(x)^3 h(y)^2 + h(x)^2 h(y)^3 + h(x) h(y)^4 + h(y)^5
\end{align*}
\]  

(1)

for all \( x, y \in \mathcal{A} \). Replacing \( x \) by \( x+z \) in (1), we obtain

\[
h \begin{align*}
4 \sum_{k=0}^{4} k^{-1} x^k z^{4-k} y + 2 \sum_{k=0}^{3} k^{-1} x^k z^{3-k} y^2
+ 2 \sum_{k=0}^{2} k^{-1} x^k z^{2-k} y^3 + x y^4 + z y^4
&= h(x)^4 h(z)^1 + h(x)^3 h(z)^2 + h(x)^2 h(z)^3 + h(x) h(z)^4 + h(z)^5 \\
&+ 2 \sum_{k=0}^{2} k^{-1} h(x)^k h(z)^{2-k} h(y)^2 + h(x) h(y)^4 + h(z) h(y)^4
\end{align*}
\]  

(2)

for all \( x, y, z \in \mathcal{A} \). Combining (1) and (2) gives

\[
h(2x^3 z y + 3x^2 z^2 y + 2x z^3 y + 3x^2 z y^2 + 3x z^2 y^2 + 2x z y^3)
= 2h(x)^3 h(z) h(y) + 3h(x)^2 h(z)^2 h(y) + 2h(x) h(z)^3 h(y)
+ 3h(x)^2 h(z) h(y)^2 + 3h(x) h(z)^2 h(y)^2 + 2h(x) h(z) h(y)^3
\]  

(3)

for all \( x, y, z \in \mathcal{A} \). Substituting \( z \) by \( x \) in (3), we obtain

\[
h(x^4 y + 2x^2 y^3) = h(x)^4 h(y) + 2h(x)^2 h(y)^3
\]  

(4)

for all \( x, y \in \mathcal{A} \). Now, if we replace \( y \) by \( y+w \) in (4) and employ the same equality, we get

\[
h(x^2 y^2 w + x^2 y w^2) = h(x)^2 h(y)^2 h(w) + h(x)^2 h(y) h(w)^2
\]  

(5)
for all $x \ y \ w \ \mathcal{A}$. Replacing $x$ by $x + u$ in (5), we have
\[ h(xuy^2w + xuyw^2) = h(x)h(u)h(y)^2h(w) + h(x)h(u)h(y)h(w)^2 \] (6)
for all $x \ y \ u \ w \ \mathcal{A}$. Now, if we change $y$ to $y + v$ in (6), we conclude
\[ h(xuyvw) = h(x)h(u)h(y)h(v)h(w) \]
for all $x \ y \ u \ v \ w \ \mathcal{A}$. Therefore $h$ is 5-ring homomorphism.

For the case $n = 6$, we assume that the map $h$ is additive and $h(x^6) = (h(x))^6$
for all $x \ \mathcal{A}$. This fact implies the following equality if we replace $x$ by $x + y$
\[ h \left( \sum_{k=1}^{5} k \ x^k y^{6-k} \right) = \sum_{k=1}^{6} k \ h(x)^k h(y)^{6-k} \] (7)
for all $x \ y \ \mathcal{A}$. Commuting $x$ by $x + z$ in (7), we obtain
\[ h \left( \sum_{k=0}^{5} k \ x^k z^{5-k} \ y + 15 \sum_{k=0}^{4} k \ x^k z^{4-k} \ y^2 + 20 \sum_{k=0}^{3} k \ x^k z^{3-k} \ y^3 + 6xy^5 + 6z^5 \right) = 6 \sum_{k=0}^{5} k \ h(x)^k h(z)^{5-k} \ h(y) + 15 \sum_{k=0}^{4} k \ h(x)^k h(z)^{4-k} \ h(y)^2 + 6h(x)h(y)^5 + 6h(z)h(y)^5 \]
for all $x \ y \ z \ \mathcal{A}$. Combining the above equality and (7), we get
\[ h \left( \sum_{k=1}^{5} k \ x^k z^{5-k} \ y + 15 \sum_{k=1}^{4} k \ x^k z^{4-k} \ y^2 + 20 \sum_{k=1}^{3} k \ x^k z^{3-k} \ y^3 + 30z^4 \right) = 6 \sum_{k=1}^{5} k \ h(x)^k h(z)^{5-k} \ h(y) + 15 \sum_{k=1}^{4} k \ h(x)^k h(z)^{4-k} \ h(y)^2 + 30h(x)h(z)h(y)^4 \]
for all $x \ y \ z \ \mathcal{A}$. Changing $z$ to $x$ in the last equality, we obtain
\[ h(x^4y^2 + x^2y^4) = h(x)^4h(y)^2 + h(x)^2h(y)^4 \] (8)
for all $x \ y \ \mathcal{A}$. Now, if we replace $y$ by $y + t$ in (8), we conclude
\[ h(x^4yt + 2x^2yt^3 + 3x^2y^2t^2 + 2x^2y^3t) = h(x)^4h(y)h(t) + 2h(x)^2h(y)^2h(t)^3 + 3h(x)^2h(y)^2h(t)^2 + 2h(x)^2h(y)^3h(t) \] (9)
for all $x \ y \ t \ \mathcal{A}$. Substituting $t$ by $t + u$ in (9), we have
\[ h(x^2 y^2 u + x^2 y tu^2 + x^2 y^2 tu) = h(x)^2 h(u)h(y)h(t)^2 h(u) + h(x)^2 h(u)h(y)h(t)h(u)^2 \\
+ h(x)^2 h(y)^2 h(t)h(u) \] 

(10)

for all \( x \ y \ t \ u \ A \). We replace \( u \) by \( u + v \) in (10) to obtain

\[ h(x^2 y t u v) = h(x)^2 h(u)h(y)h(t)h(v) \] 

(11)

for all \( x \ y \ t \ u \ v \ A \). Finally if we change \( x \) to \( x + w \) in (11), we get

\[ h(x y t u v w) = h(x)h(y)h(t)h(u)h(v)h(w) \]

The above equality shows that the map \( h \) is 6-ring homomorphism. Now, for \( n = 7 \). Replacing \( x \) by \( x + y \) in equality \( h(x^7) = (h(x))^7 \), we have

\[ h \sum_{k=1}^{6} \binom{7}{k} x^k y^{7-k} = \sum_{k=1}^{6} \binom{7}{k} h(x)^k h(y)^{7-k} \] 

(12)

for all \( x \ y \ A \). Commuting \( x \) by \( x + z \) in (12), we obtain

\[ h \sum_{k=0}^{6} \binom{7}{k} x^k z^{6-k} y + 21 \sum_{k=0}^{5} \binom{5}{k} x^k z^{5-k} y^2 + 35 \sum_{k=0}^{4} \binom{4}{k} x^k z^{4-k} y^3 + 35 \sum_{k=0}^{3} \binom{3}{k} x^k z^{3-k} y^4 \\
+ 21 \sum_{k=0}^{2} \binom{2}{k} x^k z^{2-k} y^5 + 7xy^6 + 7zy^6 \]

\[ = \sum_{k=0}^{6} \binom{7}{k} h(x)k h(z)^{6-k} h(y) + 21 \sum_{k=0}^{5} \binom{5}{k} h(x)k h(z)^{5-k} h(y)^2 \\
+ 35 \sum_{k=0}^{4} \binom{4}{k} h(x)k h(z)^{4-k} h(y)^3 + 35 \sum_{k=0}^{3} \binom{3}{k} h(x)k h(z)^{3-k} h(y)^4 \\
+ 21 \sum_{k=0}^{2} \binom{2}{k} h(x)k h(z)^{2-k} h(y)^5 + 7h(x)h(y)^6 + 7h(z)h(y)^6 \]

for all \( x \ y \ z \ A \). Combining (12) and the above equality, we get

\[ h \sum_{k=0}^{5} \binom{6}{k} x^k z^{6-k} y + 21 \sum_{k=1}^{4} \binom{4}{k} x^k z^{5-k} y^2 + 35 \sum_{k=1}^{3} \binom{3}{k} x^k z^{3-k} y^4 + 42xzy^5 \]

\[ = \sum_{k=1}^{6} \binom{6}{k} h(x)k h(z)^{6-k} h(y) + 21 \sum_{k=1}^{4} \binom{4}{k} h(x)k h(z)^{5-k} h(y)^2 \\
+ 35 \sum_{k=1}^{3} \binom{3}{k} h(x)k h(z)^{3-k} h(y)^4 \\
+ 42h(x)h(z)h(y)^5 \]

for all \( x \ y \ z \ A \). Letting \( z \) to be \( x \) in the above, we obtain

\[ h(3x^2 y^5 + 5x^4 y^3 + x^6 y) = 3h(x)^2 h(y)^5 + 5h(x)^4 h(y)^3 + h(x)^6 h(y) \] 

(13)

for all \( x \ y \ A \). Now, if we replace \( y \) by \( y + t \) in (13) and use the same equality, we conclude
\[ h(x^2y^4t + 2x^2y^3t^2 + 2x^2y^2t^3 + x^2yt^4 + x^4y^2t + x^4yt^2) \\
= h(x)^2h(y)^4h(t) + 2h(x)^2h(y)^3h(t)^2 + 2h(x)^2h(y)^2h(t)^3 \\
+ h(x)^2h(y)h(t)^4 + h(x)^4h(y)^2h(t) + h(x)^4h(y)h(t)^2 \] (14)

for all \( x \ y \ t \ \mathcal{A} \). Substituting \( t \) by \( t + u \) in (14), we have

\[ h(2x^2y^3tu + 3x^2y^2t^2u + 3x^2y^2tu^2 + 2x^2yt^4u + 3x^2yt^2u^2 + 2x^2ytu^3 + x^4ytu) \\
= 2h(x)^2h(y)^3h(t)h(u) + 3h(x)^2h(y)^2h(t)^2h(u) \\
+ 3h(x)^2h(y)^2h(t)h(u)^2 + 2h(x)^2h(y)h(t)^3h(u) \\
+ 3h(x)^2h(y)h(t)^2h(u)^2 + 2h(x)^2h(y)h(t)h(u)^3 + h(x)^4h(y)h(t)h(u) \]

for all \( x \ y \ t \ u \ \mathcal{A} \). We replace \( u \) by \( u + v \) in the last equality to obtain

\[ h(x^2y^2tuvw + x^2ytu^2v + x^2yuv + x^2ytuv^2) \\
= h(x)^2h(y)^2h(t)h(u)h(v) + h(x)^2h(u)h(y)h(t)^2h(u)h(v) \\
+ h(x)^2h(y)h(t)h(u)^2h(v) + h(x)^2h(y)h(t)h(u)h(v)^2 \] (15)

for all \( x \ y t u v \ \mathcal{A} \). Replacing \( v \) by \( v + w \) in (15), we deduce

\[ h(x^2ytuvw) = h(x)^2h(y)h(t)h(u)h(v)h(w) \] (16)

for all \( x \ y t u v w \ \mathcal{A} \). Finally, if we change \( x \) to \( x + z \) in (16), we get

\[ h(xyztuvw) = h(x)h(y)h(z)h(t)h(u)h(v)h(w) \]

for all \( x \ y z t u v w \ \mathcal{A} \). Hence the map \( h \) is 7-ring homomorphism.

3. Applications

An element \( a \) of a \( C^* \)-algebra \( \mathcal{A} \) is \textit{positive} if \( a \) is hermitian, that is \( a = a^* \), and

\( \mathbb{R}^+ \), where \( (a) \) is the spectrum of \( a \). We write \( a \geq 0 \) to mean \( a \) is positive. Also a linear map \( T : \mathcal{A} \rightarrow \mathcal{B} \) between \( C^* \)-algebras is positive if \( a \geq 0 \) implies \( T(a) \geq 0 \) for all \( a \ \mathcal{A} \). We say that the map \( T \) is \textit{completely positive} if, for any natural number \( k \), the induced map \( T_k : M_k(\mathcal{A}) \rightarrow M_k(\mathcal{B}); T_k((a_{ij})) \rightarrow (T(a_{ij})) \), on \( k \times k \) matrices is positive.

**Proposition 3.1.** Let \( n \in \mathbb{N} \) such that \( 2 \leq n \leq 7 \). Suppose \( \mathcal{A} \) and \( \mathcal{B} \) are commutative \( C^* \)-algebras. Let \( f \) be a map from \( \mathcal{A} \) into \( \mathcal{B} \), and let \( r \ s \) be real numbers such that either \((r - 1)(s - 1) > 0 \) and \( s \geq 0 \) or \((r - 1)(s - 1) > 0 \), \( s < 0 \), and \( f(0) = 0 \). Assume that \( f \) satisfies the system of functional inequalities

\[ f(x + y + z^*) \leq f(x) + f(y) + f(z)^r \]

\[ f(x^n) \leq x^{ns} \]
for all $x \, y \in \mathcal{A}$. Then, there exists a unique $-n$-ring homomorphism $h : \mathcal{A} \to \mathcal{B}$ such that

$$f(x) - h(x) \leq \frac{2}{2^r} x^r$$

for all $x \in \mathcal{A}$.

**Proof.** We can deduce the result from [3, theorem 2.1, theorem 2.2] and theorem 2.

The following theorem has been proved by Park and Trout in [7, theorem 3.2].

**Theorem 3.1.** Let $\mathcal{A} \to \mathcal{B}$ be an involutive $n$-homomorphism between $C^*$-algebras. If $n \geq 3$ is odd, then $h \leq 1$, i.e., is norm-contractive.

**Corollary 3.1.** Let $n = 3, 5, 7$, and let $\mathcal{A}$ and $\mathcal{B}$ be commutative $C^*$-algebras. If $h : \mathcal{A} \to \mathcal{B}$ is an involutive $n$-Jordan homomorphism, then $h \leq 1$, i.e., $h$ is norm contractive.

**Proof.** For $n = 3$, the result follows from [2, theorem 2.1] and theorem 2 and for $n = 5, 7$, we can use theorem 2 and theorem 3.

For the even case, we need the following theorem which is proved in [7, theorem 2.3].

**Theorem 3.2.** Let $\mathcal{A} \to \mathcal{B}$ be an involutive $n$-homomorphism between $C^*$-algebras. If $n \geq 2$ is even, then $h$ is completely positive. Thus, $h$ is bounded.

**Corollary 3.2.** Let $n = 4, 6$. If $h : \mathcal{A} \to \mathcal{B}$ is an involutive $n$-Jordan homomorphism between commutative $C^*$-algebras, then $h$ is completely positive. Thus, $h$ is bounded.

**Proof.** By using [2, theorem 2.1] and theorem 2 for $n = 4$ and theorems 2 and 3 for $n = 6$, we obtain the desired result.

**Question.** Let $n$ be an arbitrary and fixed natural number. Is every $n$-Jordan homomorphism between commutative algebras also a $n$-ring homomorphism? If this is true, then every involutive $n$-Jordan homomorphism between commutative $C^*$-algebras is norm contractive. Is this true in the non-commutative case?

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**References**


