Some notes on \( L \)-projections on Fourier-Stieltjes algebras

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Abstract. In this paper, we investigate the relation between \( L \)-projections and conditional expectations on subalgebras of the Fourier-Stieltjes algebra \( B(G) \), and we will show that compactness of \( G \) plays an important role in this relation.

Keywords: \( L \)-projection, conditional expectation, Fourier-Stieltjes algebra, spine of Fourier-Stieltjes algebra, Locally compact group.

1. Introduction

The concept of conditional expectation is fundamental for a large part of probability theory. Let \((X, S, \mu)\) be a probability space and \( T \) a \( \sigma \)-subalgebra of \( S \). The conditional expectation operator \( E^T : L^1(X, S, \mu) \to L^1(X, T, \mu) \) is determined by the relation \( \int_T E^T(f) \, d\mu = \int_T f \, d\mu \) for \( T \in \mathcal{T} \) and all \( f \in L^1(X, S, \mu) \). Existence and uniqueness of \( E^T \) follows from the Radon-Nikodym theorem. In [2], Douglas gave a complete characterization of norm one projections on \( L^1(X, S, \mu) \) related closely to the notion of conditional expectation.

The notion of conditional expectation (or quasi-expectation in [9]) is defined for any algebra. Tomiyama in [11], proved that if \( A \) is a unital \( C^* \)-algebra and \( P : A \to A \) is a norm one projection with \( P(1) = 1 \) and \( P(A) \) is a \( C^* \)-subalgebra of \( A \), then \( P \) is a conditional expectation. In view of this fundamental theorem, A.T.-M. Lau and R.J. Loy in [7], explored the relation between norm one projections and conditional expectations on Banach algebras related to locally compact groups.

In this paper, we investigate the relation between \( L \)-projections and conditional expectations on \( B(G) \) and its certain subalgebras, for instance \( A^*(G) \), and we will show that the compactness of \( G \) plays an important role in this relation.

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2. Preliminaries

Let $X$ be a Banach space and $P : X \to X$ be a projection, i.e. $P$ is a bounded idempotent operator, then $P$ is called $L$-projection if $\|x\| = \|Px\| + \|(I-P)x\|$ for all $x \in X$. It is clear that if $P$ is an $L$-projection then $\|P\| = 1$.

Let $A$ be an algebra. An idempotent operator $P : A \to A$ is a conditional expectation, if $P(b_1ab_2) = b_1P(a)b_2$ for all $b_1, b_2 \in P(A)$ and $a \in A$. The following proposition is a part of [7, Proposition 2.1], and its proof is a straightforward calculation.

**Proposition 2.1** Let $A$ be a Banach algebra and $P : A \to A$ an idempotent operator such that $P(A)$ is a subalgebra of $A$, then the following statements are equivalent:

1. $P$ is a conditional expectation.
2. If $b_1, b_2 \in P(A)$ and $a \in \ker P$ then $P(b_1ab_2) = 0$.

In [3], P. Eymard introduced $B(G)$ and $A(G)$, then proved that $A(G)$ is a closed ideal in $B(G)$. In [6], M. Ilie and N. Spronk introduced $A^*(G)$, the spine of Fourier-Stieltjes algebra, as a subalgebra of $B(G)$. We give a brief introduction of $A^*(G)$.

Let $G$ be a locally compact group. We will denote the topology on $G$ and the almost periodic compactification of $G$ by $\tau_G$ and $G^{ap}$ respectively. Let the continuous homomorphism $\eta_{ap} : G \to G^{ap}$ be the compactification homomorphism. It is clear that $\tau_{ap} := \eta_{ap}^{-1}(\tau_{G^{ap}})$ is a group topology on $G$. Suppose that $\tau$ is a group topology on $G$ such that there are locally compact group $G_\tau$ and continuous homomorphism $\eta_\tau : G \to G_\tau$ with the following three properties:

1. $\eta_\tau(G) = G_\tau$
2. $\tau = \eta_\tau^{-1}(\tau_{G_\tau})$
3. $\tau_{ap} \subseteq \tau$.

So $G_\tau$ is unique up to topological isomorphism between locally compact groups.

The set of such $\tau$ is shown by $T_{aq}(G)$. It is trivial that $\tau_G, \tau_{ap} \in T_{aq}(G)$. If $\tau_1, \tau_2 \in T_{aq}(G)$, we let $\tau_1 \vee \tau_2$ denote the smallest group topology on $G$ which includes $\tau_1$ and $\tau_2$. By [6], we know that $\tau_1 \vee \tau_2 \in T(G)$. Under this operation $T_{aq}(G)$ is a semigroup in which all elements are idempotent. From [3], we know that $A_{\tau}(G) := A(G_\tau) \circ \eta_\tau$ is a closed subalgebra of $B(G)$ such that $A(G_\tau)$ is isomorphic to $A_{\tau}(G)$ as Banach algebras.

**Theorem 2.2** If $\tau_1, \tau_2 \in T_{aq}(G)$ and $\tau_1 \neq \tau_2$, then we have

$$A_{\tau_1}(G)A_{\tau_2}(G) \subseteq A_{\tau_1 \vee \tau_2}(G), \quad A_{\tau_1}(G) \cap A_{\tau_2}(G) = \{0\}$$

**Proof**. This follows from [6, Lemma 3.4 and Proposition 3.1].

**Definition 2.3** We let

$$A^*(G) = \bigoplus_{\tau \in T_{aq}(G)} A_{\tau}(G) \quad \text{(in the sense of Banach spaces)}$$

and call this space the spine of $B(G)$, it is clear that $A^*(G)$ is a closed subalgebra of $B(G)$. We refer the reader to [6], for more details about $A^*(G)$. 
3. \textit{L-projections on }B(G)\textit{.}

Let \(G\) be a locally compact group. By [7, Proposition 3.8], if every positive contractive projection \(P : B(G) \to B(G)\) whose range is a *-subalgebra, is a conditional expectation, then \(G\) is compact. Now, we prove a similar result for \(L\)-projections.

\textbf{Proposition 3.1} Let \(G\) be a locally compact group. If every \(L\)-projection \(P : B(G) \to B(G)\) whose range is a *-subalgebra, is a conditional expectation, then \(G\) is compact.

\textit{Proof.} By [8, Theorem 2.1] or [1, Theorem 3.18, Corollary 3.13], there is a unique continuous unitary representation \(\pi\) of \(G\) such that \(B(G) = A(G) \oplus A_{\pi}(G)\), where \(A_{\pi}(G) = \overline{\text{span}} \left\{ (\pi(g)\xi, \eta) : \xi, \eta \in \mathcal{H}_{\pi}, g \in G \right\}\)

Furthermore this is an \(\ell^1\)-direct sum, that is if \(f \in B(G)\) then there are unique elements \(f_\rho \in A(G)\) and \(f_\pi \in A_{\pi}(G)\) such that \(f = f_\rho + f_\pi\) and \(\|f\| = \|f_\rho\| + \|f_\pi\|\).

Define \(P : B(G) \to A(G) : f \mapsto f_\rho\), since

\[
\|f\| = \|f_\rho\| + \|f_\pi\| = \|P(f)\| + \|(I - P)(f)\|
\]

\(P\) is an \(L\)-projection. By [3, Proposition 3.8], \(A(G)\) is a *-subalgebra of \(B(G)\). So \(P\) is a conditional expectation by the hypothesis. If \(f \in A(G)\) and \(g \in A_{\pi}(G)\), then \(P(fg) = 0\) by Proposition 2.1, and since \(A(G)\) is an ideal in \(B(G)\), then \(P(fg) = fg\).

Consequently

\[
\forall f \in A(G), \forall g \in A_{\pi}(G) : f^2g = 0 \quad (1)
\]

Let \(g \in A_{\pi}(G)\). By (1), for each \(x \in G\) and each \(f \in A(G)\), we have \(f(x)g(x) = 0\). But from [3, Lemma 3.2], we know that \(A(G)\) separates the points of \(G\). Thus \(g = 0\) and \(A_{\pi}(G) = \{0\}\). Therefore \(B(G) = A(G)\), so \(G\) is compact. \(\Box\)

We prove the preceding proposition for \(A^{\ast}(G)\).

\textbf{Proposition 3.2} Let \(G\) be a locally compact group. If every \(L\)-projection \(P : A^{\ast}(G) \to A^{\ast}(G)\) whose range is a *-subalgebra, is a conditional expectation, then \(G\) is compact.

\textit{Proof.} Suppose \(G\) is not compact. Since \(G\) is not topologically isomorphic with the compact group \(G^{\text{op}}\), by [12, Theorem 3] we know that \(A(G) \neq A_{\tau_{op}}(G)\), and by Theorem 2.2, \(A(G) \cap A_{\tau_{op}}(G) = \{0\}\). Let \(\tau_1, \tau_2 \in T_{\text{ng}}(G)\) and \(\tau_1 \neq \tau_{op}\). Thus \(\tau_1 \vee \tau_2 \neq \tau_{op}\). So by Theorem 2.2, we have:

\[
A_{\tau_1 \vee \tau_2}(G) \cap A_{\tau_{op}}(G) = \{0\}, \quad A_{\tau_1}(G)A_{\tau_2}(G) \subseteq A_{\tau_1 \vee \tau_2}(G).
\]

Therefore the Banach algebra

\[
A := \bigoplus_{\tau \neq \tau_{op} \in T_{\text{ng}}(G)} A_{\tau}(G)
\]

is an ideal in \(A^{\ast}(G)\). By [3, Proposition 3.8], \(A_{\tau}(G) \cong A(G_{\tau})\). So the Banach algebra \(A\) is a *-subalgebra of \(A^{\ast}(G)\) and we have \(A^{\ast}(G) = A \oplus_1 A_{\tau_{op}}(G)\). Let \(P : A^{\ast}(G) \to A\) be the canonical projection. Clearly \(P\) is an \(L\)-projection and \(P(A^{\ast}(G)) = A\) is a *-subalgebra of \(A^{\ast}(G)\). So by the hypothesis, \(P\) is a conditional
expectation. Since \( A(G) \subseteq A \), then \( A \) separates the points of \( G \), and since \( A \) is an ideal in \( A^*(G) \), by the same argument in the preceding proposition, we have \( A_{\tau_{ap}}(G) = \{0\} \). Since \( A_{\tau_{ap}}(G) \cong A(G_{ap}) = B(G_{ap}) \), the constant function \( 1_G \), is in \( A_{\tau_{ap}}(G) \) which is a contradiction. So \( G \) is compact.

The following theorem strengthens the conclusions of two preceding propositions.

**Theorem 3.3** Let \( G \) be a locally compact group, and \( A \) is a subalgebra of \( B(G) \).

1. Suppose that \( A(G) \subseteq A \). If every \( L \)-projection \( P : A \to A \) whose range is a *-subalgebra, is a conditional expectation, then \( G \) is compact and \( A = B(G) \).
2. Let \( A \) is a *-subalgebra and \( A_{\tau_{ap}}(G) \subseteq A \). If every \( L \)-projection \( P : A \to A \) whose range is a *-subalgebra, is a conditional expectation, then \( G \) is compact and \( A = B(G) \).
3. Let \( A_{\tau_{ap}}(G) \subseteq A \), if every \( L \)-projection \( P : A \to A \) whose range is a *-subalgebra, is a conditional expectation, then \( G \) is compact and \( A = B(G) \).

**Proof.** 1) As we discussed in the proof of Proposition 3.1, \( B(G) = A(G) \oplus_1 A_{\tau_{ap}}(G) \). Suppose that \( G \) is not compact. So \( A(G) \neq B(G) \) and \( A_{\tau_{ap}}(G) \neq \{0\} \). Let \( B := A \cap A_{\tau_{ap}}(G) \). Since \( A(G) \subseteq A \), then \( B \neq \{0\} \) and \( A = A(G) \oplus_1 B \). The canonical projection \( P : A \to A \) is an \( L \)-projection with range \( A(G) \). So \( P \) is a conditional expectation. Similar to the proof of Proposition 3.1, \( B = \{0\} \) which is a contradiction. So \( G \) is compact and consequently \( A(G) = A = B(G) \).

2) By [10], \( B(G) = A_{\tau_{L F}}(G) \oplus_1 A_{\tau_{ap}}(G) \), where \( A_{\tau_{L F}}(G) \) is a closed ideal in \( B(G) \), (note that in [10], \( A_{\tau_{ap}}(G) \) was shown by \( A_F(G) \)). If \( G \) is not compact, as it was shown in the Proposition 3.2, \( A(G) \cap A_{\tau_{ap}}(G) = \{0\} \) and by [10, p. 681, Remark (2)], we know that \( A(G) \subseteq A_{\tau_{L F}}(G) \). Since \( B(G) \) and \( A_{\tau_{ap}}(G) \) are closed under the complex conjugation, so is \( A_{\tau_{L F}}(G) \), i.e. \( A_{\tau_{L F}}(G) \) is a *-subalgebra of \( B(G) \). Let \( B := A \cap A_{\tau_{L F}}(G) \), since \( A \) and \( A_{\tau_{L F}}(G) \) are *-subalgebras of \( B(G) \), then \( B \) is a *-subalgebra, and since \( A_{\tau_{ap}}(G) \subseteq A \), then \( B \neq \{0\} \). Now, let \( P : A \to B \) be the canonical projection. Since \( A = B \oplus_1 A_{\tau_{ap}}(G) \), then \( P \) is an \( L \)-projection whose range is a *-subalgebra. So \( P \) is a conditional expectation, by the hypothesis. Since \( A_{\tau_{L F}}(G) \) is an ideal and \( A \) is a subalgebra of \( B(G) \), then \( B \) is an ideal in \( A \). Hence we have:

\[
\forall f \in B \quad \forall g \in A \quad : \quad f^2 g = f g f = P(f g f) = 0 \quad (1)
\]

Since \( A_{\tau_{ap}}(G) \cong A(G_{ap}) = B(G_{ap}) \), the constant function \( 1_G \), is in \( A_{\tau_{ap}}(G) \). By taking \( g = 1_f \) in the relation \((1)\), we have \( f = 0 \) for every \( f \in B \), i.e. \( B = \{0\} \), and this is a contradiction. Hence \( G \) is compact and \( A_{\tau_{ap}}(G) = A = B(G) \).

3) Proof of this part is similar to the proof of part (2), but it should be noted that since \( A \) is not necessarily closed under the complex conjugation, then \( B \) is just a subalgebra.

**Corollary 3.4** According to the part (2) of the preceding theorem, if every \( L \)-projection \( P : B_p(G) \to B_p(G) \) whose range is a *-subalgebra of \( B_p(G) \), is a conditional expectation, then \( G \) is compact.

**Lemma 3.5** Let \( G \) be an abelian locally compact group. \( G \) is compact and 0-dimentional iff \( \hat{G} \) is a discrete torsion group.

**Proof.** Let \( G \) be a compact 0-dimentional group. Since \( G \) is compact, \( \hat{G} \) is discrete by [5, Theorem 23.17]. Let \( \Phi \in \hat{G} \), by [5, Corollary 24.18], there is a compact
subgroup $H$ of $\hat{G}$ that contains $\Phi$. Since $\hat{G}$ is discrete, then $H$ is finite and therefore $\Phi$ is of finite order. Consequently $\hat{G}$ is a torsion group. Conversely, let $\hat{G}$ be a discrete torsion group. By [5, Theorem 23.17, 24.8], $G$ is compact, and since $\hat{G}$ is a torsion group, $G$ is 0-dimensional by [5, Theorem 24.21, 24.8].

Let $G$ be an abelian locally compact group. By Bochner’s theorem, [4, Theorem 33.3], the $\ast$-Banach algebras $M(\hat{G})$ and $B(G)$, are isomorphic. Now, by the preceding lemma and [7, Theorem 3.6], we have the following corollary. See also [7, Corollary 3.12].

**Corollary 3.6** Let $G$ be an abelian locally compact group. The following four statements are equivalent:

1. $G$ is a compact 0-dimensional group.
2. $\hat{G}$ is a discrete torsion group.
3. Each $L$-projection $P: B(G) \rightarrow B(G)$ whose range is a subalgebra, is a conditional expectation.
4. Each $L$-projection $P: B(G) \rightarrow B(G)$ whose range is a subalgebra and $P(1_G) = 1_G$, is a conditional expectation.

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**References**
