Solving the linear quadratic differential equations with constant coefficients using Taylor series with step size $h$

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Abstract. In this study we produced a new method for solving regular differential equations with step size $h$ and Taylor series. This method analyzes a regular differential equation with initial values and step size $h$. This type of equations include quadratic and cubic homogeneous equations with constant coefficients and cubic and second-level equations.

Keywords: Differential equation; initial value; step length; numerical methods; Taylor series.

1. Introduction

In the first and second sections of this paper, the numerical solution of a initial linear differential equations of cubic and quadratic homogeneous with constant coefficients is calculated by using Taylor series with length $h$. In this part the series $\sum_{n=0}^{\infty} a_n x^n$ will be replaced in Taylor expansion of $y(x_i), y'(x_i)$ and $y''(x_i)$, and then the obtained series replace in the given original differential equation, and we obtain $a_0, a_1, a_2, \ldots, a_n$. In Section 3 we will solve a quadratic homogenous differential equation

$$y'' + (A_0 x + B_0)y' + (A_1 x + B_1)y = 0,$$

using the mentioned method. Details are thoroughly discusses in the books [3] and [4].

An introduction to differential and their application of Zill et. al. [2], and numerical methods for partial differential equation of Ames[2].

2. Method of Solution

2.1 Case 1.

We consider the following initial value problem

$$y'' + Ay' + By = 0, \quad y(x_0) = y_0, \quad y'(x_1) = y_1. \quad (1)$$

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We assume that the solution of Equation (1) has the following form:

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n. \]

According to Taylor series respect to \( x_0 \), we have

\[ y(x_i) = y(x_0 + ih) = y(x_0) + ih y'(x_0) + \frac{(ih)^2}{2!} y''(x_0) + \frac{(ih)^3}{3!} y^{(3)}(x_0) + \ldots \]  

(2)

and so

\[ y'(x_i) = y'(x_0) + ih y''(x_0) + \frac{(ih)^2}{2!} y^{(3)}(x_0) + \frac{(ih)^3}{3!} y^{(4)}(x_0) + \ldots \]  

(3)

therefore we have

\[ y''(x_i) = y''(x_0) + ih y^{(3)}(x_0) + \frac{(ih)^2}{2!} y^{(4)}(x_0) + \frac{(ih)^3}{3!} y^{(5)}(x_0) + \ldots \]  

(4)

If we set

\[ y(x_0) = \sum_{n=0}^{\infty} a_n x_0^n, \]

then we have

\[ y^{(k)}(x_0) = \sum_{n=k}^{\infty} n(n-1)...(n-k+1)a_n x_0^{n-k}, \quad k = 1, 2, 3, \ldots \]

(5)

Now, by substituting the Equation (5) in Equation (2) we have

\[ y(x_i) = \sum_{n=0}^{\infty} a_n x_i^n + ih \sum_{n=1}^{\infty} n a_n x_0^{n-1} + \frac{(ih)^2}{2!} \sum_{n=2}^{\infty} n(n-1)a_n x_0^{n-2} \]

\[ + \frac{(ih)^3}{3!} \sum_{n=3}^{\infty} (n-1)(n-2)a_n x_0^{n-3} \]

\[ + \frac{(ih)^4}{4!} \sum_{n=4}^{\infty} (n-1)(n-2)(n-3)a_n x_0^{n-4} + \ldots \]

(6)

or

\[ y(x_i) = \sum_{n=0}^{\infty} x_0^n \left( a_n + ih(n+1)a_{n+1} + \frac{(ih)^2}{2!}(n+1)(n+2)a_{n+2} \right. \]

\[ + \left. \frac{(ih)^3}{3!}(n+1)(n+2)(n+3)a_{n+3} + \ldots \right). \]

(7)
If we substitute Equation (5) in Equation (3), we have

\[
y'(x_i) = \sum_{n=0}^{\infty} x_0^n ((n + 1)(n + 2) + (ih) + 2(n + 1)(n + 2) + \frac{(ih)^2}{2!} (n + 1)(n + 2)(n + 3) + \frac{(ih)^3}{3!} (n + 1)(n + 2)(n + 3)(n + 4) + ...).
\]  

(8)

Finally, by substituting the Equation (5) in Equation (4) we have

\[
y''(x_i) = \sum_{n=0}^{\infty} x_0^n ((n + 1)(n + 2) + (ih) + 2(n + 1)(n + 2) + \frac{(ih)^2}{2!} (n + 1)(n + 2)(n + 3) + \frac{(ih)^3}{3!} (n + 1)(n + 2)(n + 3)(n + 4) + ...).
\]  

(9)

By substituting Equations (7)-(9) in Equation (1), we have

\[
\sum_{n=0}^{\infty} (x_0)^n \{ ((n + 1)a_{n+1}(B(h) + 1)
+ \sum_{k=2}^{m} a_{n+k} \left( B \left( \frac{(ih)^k}{(k)!} + \frac{A(\frac{(ih)^{k-1}}{(k-1)!})}{(k-1)!} + \frac{(ih)^{k-2}}{(k-2)!} \Pi_{i=1}^{k}(n + i) \right) \right) \} = 0,
\]

(10)

so

\[
a_{n+k} = \frac{-(n + 1)a_{n+1}(B(h) + 1) + \sum_{k=2}^{m-1} a_{n+k} \left( B \left( \frac{(ih)^k}{(k)!} + \frac{A(\frac{(ih)^{k-1}}{(k-1)!})}{(k-1)!} + \frac{(ih)^{k-2}}{(k-2)!} \Pi_{i=1}^{k-1}(n + i) \right) \right) \alpha_{n,m}}{B \left( \frac{(ih)^m}{(m)!} + \frac{A(\frac{(ih)^{m-1}}{(m-1)!})}{(m-1)!} + \frac{(ih)^{m-2}}{(m-2)!} \Pi_{i=1}^{m-2}(n + i) \right) \alpha_{n,m}}.
\]

(11)

where \(\alpha_{n,k} = \Pi_{i=1}^{k}(n + i)\).

**Example 1.** Consider the following initial value problem

\[
y''(x) + 2y'(x) + y(x) = 0, \quad y(0) = 0, y'(0) = 1,
\]

according to the above algorithm for \(y(0.1)\) and \(h = 0.1\), if we set \(m = 2\), then

\[
a_0 = 0, a_1 = 1, a_{n+2} = \frac{-(a_{n+1}(n + 1)(Bh + A))}{B(\frac{(ih)^2}{2!}) + A(\frac{(ih)^1}{1!}) + B(\frac{(ih)^1}{1!}) + 2(\frac{(ih)^0}{0!}) \Pi_{i=1}^{2}(n + i)}.
\]

so \(a_2 = -0.8714\). Also if we set \(m = 3\), then \(a_3 = 0.00067\) and \(y(0.1) = 0.0913\). We know that the exact solution is \(y(0.1) = 0.0905\) and absolute error is \(8 \times 10^{-4}\).

### 2.2 Case 2.

In this case we consider the following problem

\[
y'' + Ay'' + By' + Cy = 0, \quad y(x_0) = y_0, y'(x_1) = y_1, y''(x_2) = y_2, A, B, C \in R.
\]  

(12)
According the above algorithm, we have
\[
\sum_{n=0}^{\infty} (x_0)^n \{((n + 1)a_{n+1}(B(ih) + 1) + 
\sum_{k=3}^{m} \left( a_{n+k} \left( B \frac{(ih)^{k-1}}{(k-1)!} + A \frac{(ih)^{k-2}}{(k-2)!} + \frac{(ih)^{k-3}}{(k-3)!} \prod_{i=1}^{k} \frac{(n+i)}{(n+i)} \right) \right) \} = 0,
\]
and then
\[
a_{n+k} = \frac{-(Ca_n + B(n + 1)a_{n+1} + \sum_{k=3}^{m-1} a_{n+k}(B \frac{(ih)^{k-1}}{(k-1)!} + A \frac{(ih)^{k-2}}{(k-2)!} + \frac{(ih)^{k-3}}{(k-3)!} \prod_{i=1}^{k} \frac{(n+i)}{(n+i)} \right) \alpha_{n,m}}{B \frac{(ih)^{m-1}}{(m-1)!} + A \frac{(ih)^{m-2}}{(m-2)!} + \frac{(ih)^{m-3}}{(m-3)!} \prod_{i=2}^{m} \frac{(n+i)}{(n+i)}} \tag{13}
\]

**Example 2.** Consider the following initial value problem
\[
y''(x) - 6y'' + 11y'(x) - 6y(x) = 0, \quad y(0) = 0, y'(0) = 1, y''(0) = 0.
\]
According to the above algorithm, we have
\[
y(x) = x + x^2 - 29.01099x^3 + 918.9361x^4,
\]
so \(y(0.1) = 0.1728\), and the absolute error is 0.0566.

3. Case 3.

Finally, we consider the following problem
\[
y''(x) + (A_0 x + B_0)y'' + (A_1 x + B_1)y(x) = 0, \quad y(x_0) = y_0, \quad y'(x_1) = y_1 \tag{14}
\]
According the above algorithm in Case 1., we have
\[
a_{n+k} = \frac{-(a_nA_1 + a_{n+1}(B_1 + A_1(ih)(n + 1) + A_0(ih)(n + 1)) + E}{\left( \left( B_0 \frac{(ih)^{m-2}}{(m-2)!} + B_1 \frac{(ih)^{m-1}}{(m-1)!} \prod_{i=2}^{k} \frac{(n+i)}{(n+i)} \right) \right) + F \tag{15}
\]
where
\[
E = \sum_{k=3}^{m} a_{n+k} \left( B_0 \frac{(ih)^{k-2}}{(k-2)!} + B_1 \frac{(ih)^{k-1}}{(k-1)!} \prod_{i=2}^{k} \frac{(n+i)}{(n+i)} \right) + \left( A_0 (ih)^{k-1} \prod_{i=1}^{k} \frac{(n+i)}{(n+i)} + \frac{(ih)^{k-3}}{(k-3)!} \prod_{i=2}^{k} \frac{(n+i)}{(n+i)} \right) \prod_{i=2}^{k} \frac{(n+i)}{(n+i)},
\]
and
\[
F = A_0 \frac{(ih)^{m-1}}{(m-1)!} + A_1 \frac{(ih)^{m}}{(m)!} \prod_{i=1}^{m} \frac{(n+i)}{(n+i)} + \frac{(ih)^{m-3}}{(m-3)!} \prod_{i=2}^{m} \frac{(n+i)}{(n+i)}.
\]
4. Conclusions

The results obtained from the method introduced in this paper, can be used to numerical solution of \( n \)th order differential equation with constant coefficient and with initial value and with step size \( h \) by series \( \sum a_n x^n \), and thus for obtaining the answer of homogenous linear differential equation of \( n \)th order, \( a_n(x)y^{(n)} + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y' + a_0y = 0 \) can get with initial values and with step length.

References