Solving the linear quadratic differential equations with constant coefficients using Taylor series with step size $h$

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Abstract. In this study we produced a new method for solving regular differential equations with step size $h$ and Taylor series. This method analyzes a regular differential equation with initial values and step size $h$. This type of equations include quadratic and cubic homogeneous equations with constant coefficients and cubic and second-level equations.

Keywords: Differential equation; initial value; step length; numerical methods; Taylor series.

1. Introduction

In the first and second sections of this paper, the numerical solution of a linear differential equations of cubic and quadratic homogeneous with constant coefficients is calculated by using Taylor series with length $h$. In this part the series $\sum_{n=0}^{\infty} a_n x^n$ will be replaced in Taylor expansion of $y(x_i), y'(x_i)$ and $y''(x_i)$, and then the obtained series replace in the given original differential equation, and we obtain $a_0, a_1, a_2, \ldots, a_n$. In Section 3 we will solve a quadratic homogeneous differential equation

$$y'' + (A_0 x + B_0)y' + (A_1 x + B_1)y = 0,$$

using the mentioned method. Details are thoroughly discusses in the books [3] and [4].

An introduction to differential and their application of Zill et. al. [2], and numerical methods for partial differential equation of Ames [2].

2. Method of Solution

2.1 Case 1.

We consider the following initial value problem

$$y'' + Ay' + By = 0, \quad y(x_0) = y_0, \quad y'(x_1) = y_1. \quad (1)$$
We assume that the solution of Equation (1) has the following form:

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n. \]

According to Taylor series respect to \( x_0 \), we have

\[ y(x_i) = y(x_0 + ih) = y(x_0) + ith'(x_0) + \frac{(ih)^2}{2!} y''(x_0) + \frac{(ih)^3}{3!} y^{(3)}(x_0) + ... \quad (2) \]

and so

\[ y'(x_i) = y'(x_0) + ith''(x_0) + \frac{(ih)^2}{2!} y^{(3)}(x_0) + \frac{(ih)^3}{3!} y^{(4)}(x_0) + ... \quad (3) \]

therefore we have

\[ y''(x_i) = y''(x_0) + ith^{(3)}(x_0) + \frac{(ih)^2}{2!} y^{(4)}(x_0) + \frac{(ih)^3}{3!} y^{(5)}(x_0) + ... \quad (4) \]

If we set \( y(x_0) = \sum_{n=0}^{\infty} a_n x^n \),

then we have

\[ y^{(k)}(x_0) = \sum_{n=k}^{\infty} n(n-1)...(n-k+1)a_n x_0^{n-k}, \quad k = 1, 2, 3, ... \quad (5) \]

Now, by substituting the Equation (5) in Equation (2) we have

\[
y(x_i) = \sum_{n=0}^{\infty} a_n x_0^n + ih \sum_{n=1}^{\infty} n a_n x_0^{n-1} + \frac{(ih)^2}{2!} \sum_{n=2}^{\infty} n(n-1)a_n x_0^{n-2} + \frac{(ih)^3}{3!} \sum_{n=3}^{\infty} (n-1)(n-2)a_n x_0^{n-3} + \frac{(ih)^4}{4!} \sum_{n=4}^{\infty} (n-1)(n-2)(n-3)a_n x_0^{n-4} + ... \quad (6)
\]

or

\[
y(x_i) = \sum_{n=0}^{\infty} x_0^n \left( a_n + ih(n+1)a_{n+1} + \frac{(ih)^2}{2!} (n+1)(n+2)a_{n+2} + \frac{(ih)^3}{3!} (n+1)(n+2)(n+3)a_{n+3} + ... \right). \quad (7)
\]
If we substitute Equation (5) in Equation (3), we have

$$y'(x_i) = \sum_{n=0}^{\infty} x_0^n \left( (n+1)a_{n+1} + ih(n+1)(n+2)a_{n+2} + \frac{(ih)^2}{2!}(n+1)(n+2)(n+3)a_{n+3} + \frac{(ih)^3}{3!}(n+1)(n+2)(n+3)(n+4)a_{n+4} + \ldots \right).$$

Finally, by substituting the Equation (5) in Equation (4) we have

$$y''(x_i) = \sum_{n=0}^{\infty} x_0^n \left( (n+1)(n+2)a_{n+2} + ih(n+1)(n+2)(n+3)a_{n+3} + \frac{(ih)^2}{2!}(n+1)(n+2)(n+3)(n+4)a_{n+4} + \frac{(ih)^3}{3!}(n+1)(n+2)(n+3)(n+4)(n+5)a_{n+5} + \ldots \right).$$

By substituting Equations (7)-(9) in Equation (1), we have

$$\sum_{n=0}^{\infty} (x_0^n)^{((n+1)a_{n+1}(B(ikh) + 1) + \sum_{k=2}^{m} \left( a_{n+k} \left( B\frac{(ikh)^k}{(k)!} + A\frac{(ikh)^{k-1}}{(k-1)!} + \frac{(ikh)^{k-2}}{(k-2)!}\prod_{i=1}^{k}(n+i) \right) \right) = 0, \quad (10)$$

so

$$a_{n+k} = \frac{-(n+1)a_{n+1}(B(ikh) + 1) + \sum_{k=2}^{m-1} (a_{n+k}B\frac{(ikh)^k}{(k)!} + A\frac{(ikh)^{k-1}}{(k-1)!} + \frac{(ikh)^{k-2}}{(k-2)!}\alpha_{n,m})}{\left( B\frac{(ikh)^m}{(m)!} + A\frac{(ikh)^{m-1}}{(m-1)!} + \frac{(ikh)^{m-2}}{(m-2)!}\right)\alpha_{n,m}}. \quad (11)$$

where $\alpha_{n,k} = \prod_{i=1}^{k}(n+i)$.

**Example 1.** Consider the following initial value problem

$$y''(x) + 2y'(x) + y(x) = 0, \quad y(0) = 0, \ y'(0) = 1,$$

according to the above algorithm for $y(0.1)$ and $h = 0.1$, if we set $m = 2$, then

$$a_0 = 0, \ a_1 = 1, \ a_{n+2} = \frac{-(a_{n+1}(n+1)(Bh + A))}{(B\frac{(ikh)^2}{2!} + A(ikh) + 1)\prod_{i=1}^{n}(n+i)}$$

so $a_2 = -0.8714$. Also if we set $m = 3$, then $a_3 = 0.00067$ and $y(0.1) = 0.0913$. We know that the exact solution is $y(0.1) = 0.0905$ and absolute error is $8 \times 10^{-4}$.

### 2.2 Case 2.

In this case we consider the following problem

$$y'''' + Ay'''' + By' + Cy = 0, \quad y(x_0) = y_0, \ y'(x_1) = y_1, \ y''(x_2) = y_2, \ A, B, C \in R. \quad (12)$$
According the Case 1. we have

\[
\sum_{n=0}^{\infty} (x_0)^n \left\{ ((n+1)a_{n+1} (B(\cdot)) + 1 + \right. \\
\sum_{k=3}^{m} a_{n+k} \left( B \frac{(ih)^{k-1}}{(k-1)!} + A \frac{(ih)^{k-2}}{(k-2)!} + \frac{(ih)^{k-3}}{(k-3)!} \prod_{i=1}^{k} (n+i) \right) \right\} = 0,
\]

and then

\[
a_{n+k} = \frac{-(Cn + B(n+1)a_{n+1} + \sum_{k=3}^{m-1} a_{n+k}(B \frac{(ih)^{k-1}}{(m-1)!} + A \frac{(ih)^{k-2}}{(m-2)!} + \frac{(ih)^{k-3}}{(m-3)!}) \prod_{i=2}^{m} (n+i)}{B \frac{(ih)^{m-1}}{(m-1)!} + A \frac{(ih)^{m-2}}{(m-2)!} + \frac{(ih)^{m-3}}{(m-3)!}) \prod_{i=2}^{m} (n+i)} \prod_{i=2}^{m} (n+i) \prod_{i=2}^{m} (n+i). \tag{13}
\]

**Example 2.** Consider the following initial value problem

\[
y'''(x) - 6y''(x) + 11y'(x) - 6y(x) = 0, \quad y(0) = 0, y'(0) = 1, y''(0) = 0.
\]

According to the above algorithm, we have

\[
y(x) = x + x^2 - 29.01099x^3 + 918.9361x^4,
\]

so \(y(0.1) = 0.1728\), and the absolute error is 0.0566.

3. **Case 3.**

Finally, we consider the following problem

\[
y''(x) + (A_0x + B_0)y'(x) + (A_1x + B_1)y(x) = 0, \quad y(x_0) = y_0, \quad y'(x_1) = y_1 \tag{14}
\]

According the above algorithm in Case 1., we have

\[
a_{n+k} = \frac{-(a_nA_1 + a_{n+1}(B_1 + A_1(ih)(n+1) + A_0(ih)(n+1))) + E}{(B_0 \frac{(ih)^{m-2}}{(m-2)!} + B_1 \frac{(ih)^{m-1}}{(m-1)!}) \prod_{i=2}^{m} (n+i)} + F \tag{15}
\]

where

\[
E = \sum_{k=3}^{m} a_{n+k} \left[ B_0 \frac{(ih)^{k-2}}{(k-2)!} + B_1 \frac{(ih)^{k-1}}{(k-1)!} \prod_{i=2}^{k} (n+i) \right] + \left[ A_0 \frac{(ih)^{k-1}}{(k-1)!} + B_1 \frac{(ih)^{k}}{(k)!} \prod_{i=1}^{k} (n+i) + \frac{(ih)^{k-3}}{(k-3)!} \prod_{i=2}^{k} (n+i) \right] \prod_{i=2}^{k} (n+i),
\]

and

\[
F = A_0 \frac{(ih)^{m-1}}{(m-1)!} + A_1 \frac{(ih)^{m}}{(m)!} \prod_{i=1}^{m} (n+i) + \frac{(ih)^{m-3}}{(m-3)!} \prod_{i=2}^{m} (n+i).
\]
4. Conclusions

The results obtained from the method introduced in this paper, can be used to numerical solution of \( n \)th order differential equation with constant coefficient and with initial value and with step size \( h \) by series \( \sum a_n x^n \), and thus for obtaining the answer of homogenous linear differential equation of \( n \)th order, \( a_n(x)y^{(n)} + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y' + a_0y = 0 \) can get with initial values and with step length.

References