OD-characterization of almost simple groups related to $U_3(11)$

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**Abstract.** Let $L := U_3(11)$. In this article, we classify groups with the same order and degree pattern as an almost simple group related to $L$. In fact, we prove that $L$, $L.2$ and $L.3$ are OD-characterizable, and $L.S_3$ is 5-fold OD-characterizable.

**Keywords:** prime graph, recognition, linear group, finite simple group, degree pattern

1. Introduction

Let $G$ be a finite group. Denote by $\pi(G)$ the set of all prime divisors of the order of $G$. The prime graph $\Gamma(G)$ of a finite group $G$ is a simple graph with vertex set $\pi(G)$ in which two distinct vertices $p$ and $q$ are joined by an edge if and only if $G$ has an element of order $pq$.

**Definition 1.1** Let $G$ be a finite group and $|G| = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1 < p_2 < \ldots < p_k$. For $p \in \pi(G)$, let $\text{deg}(p) = |\{q \in \pi(G) | p \sim q\}|$ be the degree of $p$ in the graph $\Gamma(G)$, we define $D(G) = (\text{deg}(p_1), \text{deg}(p_2), \ldots, \text{deg}(p_k))$, which is called the degree pattern of $G$.

Given a finite group $G$, denote by $h_{OD}(G)$ the number of isomorphism classes of finite groups $S$ such that $|G| = |S|$ and $D(G) = D(S)$. In terms of the function $h_{OD}$, groups $G$ are classified as follows:

**Definition 1.2** A group $G$ is called $k$-fold OD-characterizable if there exist exactly $k$ non-isomorphic group $S$ such that $|G| = |S|$ and $D(G) = D(S)$. Moreover, a 1-fold OD-characterizable group is simply called an OD-characterizable.

**Definition 1.3** A group $G$ is said to be an almost simple related to $S$ if and only if $S \trianglelefteq G \trianglelefteq \text{Aut}(S)$ for some non-abelian simple group $S$.

**Definition 1.4** Let $p$ be a prime number. The set of all non-abelian finite simple groups $G$ such that $p \in \Pi(G) \subseteq \{2, 3, 5, \ldots, p\}$ is denoted by $\mathcal{G}_p$. It is clear that the set of all non-abelian finite simple groups is the disjoint union of the finite sets $\mathcal{G}_p$ for all primes $p$.

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2. Preliminaries

For any group $G$, let $w(G)$ be the set of orders of elements in $G$, where each possible order element occurs once in $w(G)$ regardless of how many elements of that order $G$ has. This set is closed and partially ordered by divisibility, hence it is uniquely determined by its maximal elements. The set of maximal elements of $w(G)$ is denoted by $\mu(G)$. The number of connected components of $\Gamma(G)$ is denoted by $t(G)$. Let $\pi_i = \pi_i(G)$, $1 \leq i \leq t(G)$, be the $i$th connected components of $\Gamma(G)$.

For a group of even order we let $2 \in \pi_i(G)$. We denote by $\pi(n)$ the set of all primes divisors of $n$, where $n$ is a natural number. Then $|G|$ can be expressed as a product of $m_1, m_2, \ldots, m_{t(G)}$, where $m_i$'s are positive integers with $\pi(m_i) = \pi_i$. These $m_i$'s are called the order components of $G$. We write $OC(G) = \{m_1, m_2, \ldots, m_{t(G)}\}$ and call it the set of order components of $G$. The set of prime graph components of $G$ is denoted by $T(G) = \{\pi_i(G) | i = 1, 2, \ldots, t(G)\}$.

**Definition 2.1** Let $n$ be a natural number. We say that a finite simple group $G$ is a simple $K_n$-group if $|\pi(G)| = n$.

**Definition 2.2** Suppose that $K \unlhd G$ and $G/K \cong H$. Then we shall call $G$ an extension of $K$ by $H$.

3. Elementary Results

**Lemma 3.1** [5] Let $G$ be a finite group and $|\pi(G)| \geq 3$. If there exist prime numbers $r, s, t \in \pi(G)$ such that $\{tr, ts, rs\} \cap \omega(G) = \emptyset$, then $G$ is non-solvable.

**Definition 3.2** A group $G$ is called a 2-Frobenius group, if there exists a normal series $1 < H < K < G$, such that $K$ and $\frac{G}{H}$ are Frobenius groups with kernels $H$ and $\frac{K}{H}$, respectively.

**Lemma 3.3** [1] Let $G$ be a 2-Frobenius group of even order which has a normal series $1 < H < K < G$, such that $K$ and $\frac{G}{H}$ are Frobenius groups with kernels $H$ and $\frac{K}{H}$, respectively. Then

1. $t(G) = 2$ and $T(G) = \{\pi_1(G) = \pi(H) \cup \pi(\frac{G}{H}), \pi_2(G) = \pi(\frac{K}{H})\}$.
2. $\frac{G}{H}$ and $\frac{K}{H}$ are cyclic groups, $|\frac{G}{H}| | \text{Aut}(\frac{K}{H})|$, and $|\frac{G}{H}| | \frac{K}{H}| = 1$.
3. $H$ is a nilpotent group and $G$ is a solvable group.

The following lemmas are useful when dealing with a Frobenius group.

**Lemma 3.4** [3], [8] Let $G$ be a Frobenius group with complement $H$ and kernel $K$. Then the following assertions hold:

1. $K$ is a nilpotent group;
2. $|K| \equiv 1 \pmod{|H|}$;
3. Every subgroup of $H$ of order $pq$, with $p, q$ (not necessarily distinct)primes, is cyclic. In particular, every Sylow Subgroup of $H$ of odd order is cyclic and a 2-Sylow subgroup of $H$ is either cyclic or a generalized quaternion group. If $H$ is a non-solvable group, then $H$ has a subgroup of index at most 2 isomorphic to $Z \times \text{SL}(2, 5)$, where $Z$ has cyclic $p$-subgroups and $\pi(Z) \cap \{2, 3, 5\} = \emptyset$. In particular, $15, 20 \notin \omega(H)$.

**Lemma 3.5** [1] Let $G$ be a Frobenius group of even order where $H$ and $K$ are Frobenius complement and Frobenius kernel of $G$, respectively. Then $t(G) = 2$ and $T(G) = \{\pi(H), \pi(K)\}$.
The structure of a finite group with non-connected prime graph is described in the following lemma.

**Lemma 3.6** [4], [9] Let $G$ be a finite group with $t(G) \geq 2$. Then $G$ is one of the following groups:

1. $G$ is a Frobenius or a 2-Frobenius group;
2. $G$ has a normal series $1 \leq H < K \leq G$, such that $H$ and $G/K$ are $\pi_1$-groups and $K/H$ is a non-abelian simple group, where $\pi_1$ is the prime graph component containing 2, $H$ is a nilpotent group, and $|G/H| \mid |Aut(K/H)|$. Moreover, any odd order component of $G$ is also an odd order component of $K/H$.

The following lemma is taken from [10].

**Lemma 3.7** Let $S = P_1 \times P_2 \times \ldots \times P_r$, where $P_i$’s are isomorphic non-abelian simple groups. Then $\text{Aut}(S) \cong (\text{Aut}(P_1) \times \text{Aut}(P_2) \times \ldots \times \text{Aut}(P_r)) \cdot S_r$.

## 4. Main Results

**Theorem 4.1** If $G$ is a finite group such that $D(G) = D(M)$ and $|G| = |M|$, where $M$ is an almost simple group related to $L := U_3(11)$, then the following assertions hold:

1. If $M = L$, then, $G \cong L$,
2. If $M = L.2$, then, $G \cong L.2$,
3. If $M = L.3$, then, $G \cong L.3$,
4. If $M = L.S_3$, then, $G \cong L.S_3$, $Z_3 \times (L.2)$ or $Z_3.L.S_3$.

In particular, $L_2$ and $L_3$ are OD-characterizable; and $L.S_3$ is 5-fold OD-characterizable.

**Proof** We break the proof into a number of separate cases:

Case 1: If $M = L$, then, $G \cong L$ by [7].
Case 2: If $M = L.2$, then, $G \cong L.2$.

If $M = L.2$, by [2], we have $\mu(L.2) = \{24, 37, 40, 44\}$ from which we deduce that $D(L.2) = (3, 1, 1, 1, 0)$. The prime graph of $L.2$ has the following form:

![Prime Graph of L.2](image)

Figure 1: The prime graph of $L.2$

As $|G| = |L.2| = 2^6 \cdot 3^2 \cdot 5 \cdot 11^3 \cdot 37$ and $D(G) = D(L.2) = (3, 1, 1, 1, 0)$, then $\Gamma(G) = \Gamma(M) = \{2 \sim 3, 2 \sim 5, 2 \sim 11; 37\}$.

$G$ is non-solvable. Since $\{3 \cdot 37, 5 \cdot 37, 3 \cdot 5\} \cap \omega(G) = \emptyset$, therefore by lemma 3.1, $G$ is not solvable. Therefore, by lemma 3.2(iii), $G$ is not a 2-Frobenius group.

Suppose that $G$ is a non-solvable Frobenius group with $H$ and $K$ as its Frobenius complement and Frobenius kernel, respectively. Using the same notations as in lemma 3.3(iii), we obtain $11 \in \pi(Z)$, it follows that $H_0$ has an element of order $11 \cdot 5$, a contradiction.

By lemma 3.5(ii), $G$ has a normal series $1 \leq H < K \leq G$, such that $H$ is a nilpotent $\pi_1$-group, $K/H$ is a non-abelian simple group and $G/K$ is a solvable $\pi_1$-group. Therefore, $K/H \leq G/H \leq Aut(K/H)$. Since $37 \nmid |H|$, we have $37 \in \pi(K/H)$.
Therefore, $K/H \in S_{37}$ and $\{7, 13, 17, 19, 23, 29, 31\} \not\subseteq \pi(K/H)$. Using [11] we listed the possibilities for $K/H$ in Table 1.

By Table 1, we obtain that $K/H$ isomorphic to $A_5$, $A_6$, $L_2(11)$, $M_{11}$ or $L$.
If $K/H \cong A_5$ we get $A_5 \leq G/H \leq \text{Aut}(A_5)$, because $G/H \leq \text{Aut}(K/H)$. It follows that $|H| = 2^4 \cdot 3 \cdot 11^3 \cdot 37$ or $|H| = 2^5 \cdot 3 \cdot 11^3 \cdot 37$. By nilpotency of $H$, $11 \sim 37$ in $\Gamma(G)$, a contradiction. Similarly, we can prove that $K/H \not\cong A_6$, $L_2(11)$ or $M_{11}$.
Therefore, $K/H \cong L$. As $|G| = 2|L|$, we deduce $|H| = 1$ or 2.
If $|H| = 1$, then, $G \cong L.2$.
If $|H| = 2$, then, $G/C_G(H) \leq \text{Aut}(H) \cong Z^*_{11} = 1$, so $G = C_G(H)$. Therefore, $H \leq Z(G)$. It follows that $2 \sim 37$ in $\Gamma(G)$, a contradiction.

| Table 1: Non-abelian simple group $S \in S_{37}$ with $\pi(S) \subseteq \{2, 3, 5, 11, 37\}$ |
|---|---|---|---|
| $S$ | $|S|$ | $|\text{out}(S)|$ | $S$ | $|S|$ | $|\text{out}(S)|$ |
| $A_5$ | $2^4 \cdot 3 \cdot 5$ | 2 | $M_{11}$ | $2^4 \cdot 3^2 \cdot 5 \cdot 11$ | 1 |
| $A_6$ | $2^3 \cdot 3^2 \cdot 5$ | 4 | $M_{12}$ | $2^6 \cdot 3^3 \cdot 5 \cdot 11$ | 2 |
| $U_3(2) \cong S_4(3)$ | $2^6 \cdot 3^4 \cdot 5$ | 2 | $U_3(2)$ | $2^3 \cdot 3 \cdot 5 \cdot 11$ | 2 |
| $L_2(11)$ | $2^2 \cdot 3 \cdot 5 \cdot 11$ | 2 | $U_3(11)$ | $2^5 \cdot 3^2 \cdot 5 \cdot 11^3 \cdot 37$ | 6 |

Case 3: If $M = L.3$, then $G \cong L.3$.
If $M = L.3$, by [2], we have $\mu(L.3) = \{111, 120, 132\}$ from which we deduce that $D(L.3) = (3, 3, 2, 2, 1)$. The prime graph of $L.3$ has the following form:

![Prime Graph of L.3](image)

As $|G| = |L.3| = 2^5 \cdot 3^3 \cdot 5 \cdot 11^3 \cdot 37$ and $D(G) = D(L.3) = (3, 4, 2, 2, 1)$, then, $\Gamma(G) = \Gamma(M) = \{2 \sim 3, 2 \sim 5, 2 \sim 11, 3 \sim 5, 3 \sim 11, 3 \sim 37\}$.

**Lemma 4.2** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3\}$-group. In particular, $G$ is non-solvable.

**Proof** First assume that $\{5, 11\} \subseteq \pi(K)$. Let $H$ be a Hall $\{5, 11\}$-subgroup of $K$. It is easy to see that $H$ is a subgroup of order $5 \cdot 11^3$. $H$ is nilpotent, since $H = H_5 \cdot H_{11}$, $5 \sim 11$, therefore $H_5 \cap H_{11} = \{1\}$. We have $H_5 \leq H$ and $N_{11} = 11k+1 | |H| = 5.11^3$, where $N_{11}$ is the number of 11- Sylow subgroups from $H$, and $(N_{11}, 11) = 1$ then $11k+1 | 5$, hence $k = 0$ and, by Sylow’s Lemma, $H_{11} \leq H$. Therefore $H \cong H_5 \times H_{11}$ and by Tampson’s Lemma, we have $H_{11}$ is nilpotent, hence $H$ is nilpotent.
Since $H$ is nilpotent, which implies that $5 \cdot 11 \in \omega(K) \subseteq \omega(G)$, a contradiction.
Thus $\{5\} \subseteq \pi(K) \subseteq \{2, 3, 5, 37\}$. Let $K_5 \in Syl_5(K)$, by Frattini argument $G = KN_G(K_5)$. Therefore, the normalizer $N_G(K_5)$ contains an element of order 11, say $x$. Similar to $H$ we can prove that $< x > K_5$ is a nilpotent subgroup of $G$ of order $5 \cdot 11$. Hence $5 \cdot 11 \in \omega(G)$, a contradiction. Similarly, we can prove that $\{11, 37\} \cap \pi(K) = \emptyset$. Therefore, $K$ is a $\{2, 3\}$-group. In addition, since $K \neq G$, it follows that $G$ is non-solvable. This completes the proof.

**Lemma 4.3** The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \leq \text{Aut}(S)$, where $S \cong L$. 

Proof Let $\overline{G} := G/K$, $S := Soc(\overline{G})$, where $Soc(\overline{G})$ denotes the socle of the group $\overline{G}$, i.e., the subgroup of $\overline{G}$ generated by the set of all the minimal normal subgroups of $\overline{G}$. Then, $S \cong P_1 \times P_2 \times \cdots \times P_r$, where $P_i$'s are non-abelian simple groups and $S \leq \overline{G} \leq Aut(S)$. In what follows, we will show that $r = 1$ and $P_1 \cong L$.

Suppose that $r > 3$, then, there exists distinct $P_i$ and $P_j$ such that $\pi(P_i) \neq \pi(P_j)$, because $|G|_5 = 5$, $|G|_{11} = 11^3$ and $|G|_{37} = 37$, where $n_p$ denotes the $p$-part of the integer $n \in N$. If $|\pi(P_i)| = 5$ or $|\pi(P_j)| = 5$, then, $37 \in \pi(P_i)$ or $37 \in \pi(P_j)$. It follows that $2.37 \in \omega(G)$, a contradiction. Hence, without loss of generality, by Table 1, we can suppose that $\{2, 3\} \subseteq \pi(P_i) \subseteq \{2, 3, p, q\}$ and $\{2, 3\} \subseteq \pi(P_j) \subseteq \{2, 3, r, s\}$, where $\{r, s\}$, $\{p, q\} \subseteq \{\{5, 11\}, \{5, 37\}, \{11, 37\}\}$ and $\{r, s\} \neq \{p, q\}$. As $S \cong P_1 \times \cdots \times P_i \times \cdots \times P_j \times \cdots \times P_r$, we have $\{pr, ps, qr, qs\} \subseteq \omega(S)$. Thus, $\{pr, ps, qr, qs\} \subseteq \omega(G)$, which is a contradiction because there exists no edge between 5, 11 and 37 in $\Gamma(G)$.

Hence, $r = 2$ if $r > 1$. Recall that $|G| = 2^{5 \cdot 3^3 \cdot 5 \cdot 11^3 \cdot 37}$ and $S \cong P_1 \times P_2 \times \cdots \times P_r$, where $P_i$'s are finite non-abelian simple groups. By Table 1, we have $5 \in \pi(P_i)$, therefore, if $S \cong P_1 \times P_2$, then, $5^2 | |S|$, a contradiction. Thus, $r = 1$ and $S = P_1$.

By Table 1, $\{2, 3\} \subseteq \pi(S)$ and $\pi(Out(S)) \subseteq \{2, 3\}$. Therefore, by Lemma 4.7, it is evident that $|S| = 2^a \cdot 3^b \cdot 5 \cdot 11^3 \cdot 37$, where $2 \leq a \leq 5$ and $1 \leq b \leq 3$. Now, using collected results contained in Table 1, we deduce that $S \cong U_3(11)$ and the proof is completed.

Lemma 4.4 $G \cong L_3$.

Proof By Lemma 4.8, $L \leq G/K \leq Aut(L)$. Hence, $|K| = 1$ or 3.

If $|K| = 1$, then, $G \cong L_3$.

If $|K| = 3$, then, $G/K \cong L$. In this case we have $G/C_G(K) \leq Aut(K) \cong Z_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$. It follows that $3 \sim 37$ in $\Gamma(G)$, a contradiction. If $|G/C_G(K)| = 2$, then $K \subset C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L$. Thus, we obtain $G = C_G(K)$ because $L$ is simple, which is a contradiction.

Case 4: If $M = L.S_3$, then, $G \cong L.S_3$, $Z_3 \times (L.2)$, $Z_3 \cdot (L.2)$, $(Z_3 \times L).Z_2$, $(Z_3 \cdot L).Z_2$.

If $M = L.S_3$, by [2], we have $\mu(L.S_3) = \{111, 120, 132\}$ from which we deduce that $D(L.S_3) = (3, 3, 2, 2, 1)$. The prime graph of $L.S_3$ has the following form:

Figure 3: The prime graph of $L.S_3$

As $|G| = |L.S_3| = 2^6 \cdot 3^3 \cdot 5 \cdot 11^3 \cdot 37$ and $D(G) = D(L.S_3) = (3, 4, 2, 2, 1)$, then, $\Gamma(G) = \Gamma(M) = \{2 \sim 3, 2 \sim 5, 2 \sim 11, 3 \sim 5, 3 \sim 11, 3 \sim 37\}$.

Similarly to Lemma 4.7 in Case 3, we can prove that, if $K$ be the maximal normal solvable subgroup of $G$, then $K$ is a $\{2, 3\}$-group and $G$ is non-solvable. Also, similarly to Lemma 4.8 in case 3, we can prove that, the quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \leq Aut(S)$, where $S \cong L$.

Now, we proof that $G \cong L.S_3$, $Z_3 \times (L.2)$, $Z_3 \cdot (L.2)$, $(Z_3 \times L).Z_2$, $(Z_3 \cdot L).Z_2$.

Since $L \leq G/K \leq Aut(L)$, then, $|K| = 1, 2, 3$ or 6.

If $|K| = 1$, then, $G \cong L.S_3$.

If $|K| = 2$, then, $K \leq Z(G)$. It follows that $2 \sim 37$ in $\Gamma(G)$, a contradiction.

If $|K| = 3$, then, $G/K \cong L.2$. In this case we have $G/C_G(K) \leq Aut(K) \cong Z_2$.

Thus, $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then, $K \leq Z(G)$, i.e., $G$ is a
central extension of $Z_3$ by $L.2$. If $G$ splits over $K$ we obtain $G \cong Z_3 \times (L.2)$, otherwise, we have $G \cong Z_3 \cdot (L.2)$. If $|G/C_G(K)| = 2$, then, $K \subset C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L.2$, and we obtain that $C_G(K)/K \cong L$. Because $K \leq Z(C_G(K)), C_G(K)$ is a central extension of $K$ by $L$. If $G$ splits over $K$, we obtain that $C_G(K) \cong Z_3 \times L$. Otherwise, we have $C_G(K) = Z_3 \cdot L$. Thus, $G \cong (Z_3 \times L).Z_2$ or $G \cong (Z_3 \cdot L).Z_2$.

If $|K| = 6$, then, $G/K \cong L$ and $K \cong Z_6$ or $S_3$.

Subcase 1: If $K \cong Z_6$, then, $G/C_G(K) \leq Aut(Z_6) = Z_6^\times \cong Z_2$ and so $|G/C_G(K)| = 1$ or $2$. If $|G/C_G(K)| = 1$, then, $Z_6 \cong K \leq Z(G)$. It follows that $2 \sim 37$ in $\Gamma(G)$, a contradiction. If $|G/C_G(K)| = 2$, then, $K \subset C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L$, which is a contradiction since $L$ is simple.

Subcase 2: If $K \cong S_3$, then, $K \cap C_G(K) = 1$ and $G/C_G(K) \leq S_3$. Thus, $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)/K \leq G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because $L$ is simple. Therefore, $G \cong S_3 \times L$, Which implies that $2 \sim 37$ in $\Gamma(G)$, a contradiction. 

\section*{References}


