New fixed and periodic point results on cone metric spaces

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Abstract. In this paper, several fixed point theorems for T-contraction of two maps on cone metric spaces under normality condition are proved. Obtained results extend and generalize well-known comparable results in the literature.

Keywords: Cone metric space; Fixed point; Property P; Property Q; Normal cone.

1. Introduction

In 1922, Banach proved his famous fixed point theorem [3]. Afterward, other people consider some various definitions of contractive mappings and proved several fixed point theorems in [4, 7, 10, 11, 13, 15] and the references contained therein. In 2007, Huang and Zhang [8] introduced cone metric space and proved some fixed point theorems. Afterward, several fixed and common fixed point results on cone metric spaces proved in [1, 14, 16, 17] and the references contained therein.

Recently, Morales and Rajes [12] introduced T-Kannan and T-Chatterjea contractive mappings in cone metric space and proved some fixed point theorems. Then, Filipović et al. [5] defined T-Hardy-Rogers contraction in cone metric space and proved some fixed and periodic point theorems. In this work, we prove several fixed and periodic point theorems for T-contraction of two maps on normal cone metric spaces. Our results extend various comparable results of Filipović et al. [5], and Morales and Rajes [12].

2. Preliminaries

Let us start by defining some important definitions.

Definition 2.1 (See [6, 8]). Let E be a real Banach space and P a subset of E. Then P is called a cone if and only if
(a) P is closed, non-empty and P ≠ {0};
(b) a, b ∈ R, a, b ≥ 0, x, y ∈ P imply that ax + by ∈ P;
(c) if x ∈ P and −x ∈ P, then x = 0.
Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to $P$ by $x \leq y \iff y-x \in P$.

We shall write $x < y$ if $x \leq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y-x \in intP$ (where $intP$ is interior of $P$). If $intP \neq \emptyset$, the cone $P$ is called solid.

The cone $P$ is named normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \implies \|x\| \leq K\|y\|.$$ (1)

The least positive number satisfying the above is called the normal constant of $P$.

**Example 2.2** (See [14]). Let $E = C_{[0,1]}$ with the supremum norm and $P = \{f \in E : f \geq 0\}$. Then $P$ is a normal cone with normal constant $K = 1$.

**Definition 2.3** (See [8]). Let $X$ be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

1. $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then, $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space.

**Example 2.4** (See [8]). Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E|x, y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}$ and $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $d(x, y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then $(X, d)$ is a cone metric space.

**Definition 2.5** (See [5]). Let $(X, d)$ be a cone metric space, $\{x_n\}$ a sequence in $X$ and $x \in X$. Then

(i) $\{x_n\}$ converges to $x$ if for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$.

(ii) $\{x_n\}$ is called a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$.

Also, a cone metric space $X$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$. In the sequel, we always suppose that $E$ is a real Banach space, $P$ is a normal cone in $E$, and $\leq$ is partial ordering with respect to $P$.

**Definition 2.6** (See [5]). Let $(X, d)$ be a cone metric space, $P$ a solid cone and $S : X \to X$. Then

(i) $S$ is said to be sequentially convergent if we have for every sequence $\{x_n\}$, if $S(x_n)$ is convergent, then $\{x_n\}$ also is convergent.

(ii) $S$ is said to be subsequentially convergent if we have for every sequence $\{x_n\}$ that $S(x_n)$ is convergent, implies $\{x_n\}$ has a convergent subsequence.

(iii) $S$ is said to be continuous, if $\lim_{n \to \infty} x_n = x$ implies that $\lim_{n \to \infty} S(x_n) = S(x)$, for all $\{x_n\}$ in $X$.

**Definition 2.7** (See [5]). Let $(X, d)$ be a cone metric space and $T, f : X \to X$ two mappings. A mapping $f$ is said to be a $T$-Hardy-Rogers contraction, if there exist $\alpha_i \geq 0$, $i = 1, \cdots, 5$ with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$ such that for all $x, y \in X$,

$$d(Tfx, Tfy) \leq \alpha_1d(Tx, Ty) + \alpha_2d(Tx, Tfx) + \alpha_3d(Ty, Tfy) + \alpha_4d(Tx, Tfy) + \alpha_5d(Ty, Tfx).$$ (2)

In previous definition, suppose that $\alpha_1 = \alpha_4 = \alpha_5 = 0$ and $\alpha_2 = \alpha_3 \neq 0$ (resp. $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_4 = \alpha_5 \neq 0$). Then we obtain $T$-Kannan (resp. $T$-Chatterjea) contraction from [12].
3. Fixed point results

**Theorem 3.1** Suppose that \((X,d)\) be a complete cone metric space, \(P\) be a normal cone with normal constant \(K\), and \(T : X \rightarrow X\) be a continuous and one to one mapping. Moreover, let \(f\) and \(g\) be two maps of \(X\) satisfying

\[
d(Tfx, Tgy) \leq \alpha_1d(Tx, Ty) + \alpha_2[d(Tx, Tfx) + d(Ty, Tgy)]
\]

\[
+ \alpha_3[d(Tx, Tgy) + d(Ty, Tfx)],
\]

for all \(x, y \in X\), where

\[
\alpha_1 \geq 0 \quad \text{for} \quad i = 1, 2, 3 \quad \text{and} \quad \alpha_1 + 2\alpha_2 + 2\alpha_3 < 1.
\]

That is, \(f\) and \(g\) be a \(\alpha\)-contraction. Then

1. There exist \(u_x \in X\) such that \(\lim_{n \to \infty} Tfx_{2n} = \lim_{n \to \infty} Tgx_{2n+1} = u_x\).
2. If \(T\) is subsequentially convergent, then \(\{fx_{2n}\}\) and \(\{gx_{2n+1}\}\) have a convergent subsequence.
3. There exist a unique \(v_x \in X\) such that \(fv_x = gv_x = v_x\), that is, \(f\) and \(g\) have a unique common fixed point.
4. If \(T\) is sequentially convergent, then iterate sequences \(\{fx_{2n}\}\) and \(\{gx_{2n+1}\}\) converge to \(v_x\).

**Proof** Suppose \(x_0\) is an arbitrary point of \(X\), and define \(\{x_n\}\) by

\[
x_1 = fx_0, \quad x_2 = gx_1, \quad \cdots, \quad x_{2n+1} = fx_{2n}, \quad x_{2n+2} = gx_{2n+1} \quad \text{for} \quad n = 0, 1, 2, \ldots
\]

First, we prove that \(\{T x_n\}\) is a Cauchy sequence.

\[
d(Tx_{2n+1}, Tx_{2n+2}) = d(Tfx_{2n}, Tgx_{2n+1})
\]

\[
\leq \alpha_1d(Tx_{2n}, Tx_{2n+1})
\]

\[
+ \alpha_2[d(Tx_{2n}, Tfx_{2n}) + d(Tx_{2n+1}, Tgx_{2n+1})]
\]

\[
+ \alpha_3[d(Tx_{2n}, Tgx_{2n+1}) + d(Tx_{2n+1}, Tfx_{2n})]
\]

\[
= \alpha_1d(Tx_{2n}, Tx_{2n+1})
\]

\[
+ \alpha_2[d(Tx_{2n}, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n+2})]
\]

\[
+ \alpha_3[d(Tx_{2n}, Tx_{2n+2}) + d(Tx_{2n+1}, Tx_{2n+1})]
\]

\[
\leq (\alpha_1 + \alpha_2 + \alpha_3)d(Tx_{2n}, Tx_{2n+1})
\]

\[
+ (\alpha_2 + \alpha_3)d(Tx_{2n+1}, Tx_{2n+2}),
\]

which implies that

\[
d(Tx_{2n+1}, Tx_{2n+2}) \leq \gamma d(Tx_{2n}, Tx_{2n+1}),
\]

where \(\gamma = \frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_2 - \alpha_3} < 1\).

Similarly, we get

\[
d(Tx_{2n+3}, Tx_{2n+2}) \leq \gamma d(Tx_{2n+2}, Tx_{2n+1}),
\]

where \(\gamma = \frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_2 - \alpha_3} < 1\).
Thus, for all $n$

$$d(Tx_n, Tx_{n+1}) \leq \gamma d(Tx_{n-1}, Tx_n) \leq \gamma^2 d(Tx_{n-2}, Tx_{n-1})$$
$$\leq \cdots \leq \gamma^n d(Tx_0, Tx_1). \quad (5)$$

Now, for any $m > n$

$$d(Tx_n, Tx_m) \leq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \cdots + d(Tx_{m-1}, Tx_m)$$
$$\leq (\gamma^n + \gamma^{n+1} + \cdots + \gamma^{m-1}) d(Tx_0, Tx_1)$$
$$\leq \frac{\gamma^n}{1-\gamma} d(Tx_0, Tx_1).$$

From (1), we have

$$\|d(Tx_n, Tx_m)\| \leq K \frac{\gamma^n}{1-\gamma} \|d(Tx_0, Tx_1)\|.$$

It follows that $\{Tx_n\}$ is a Cauchy sequence by Definition 2.5.(ii). Since cone metric space $X$ is complete, there exist $u_x \in X$ such that $Tx_n \to u_x$ as $n \to \infty$. Thus,

$$\lim_{n \to \infty} Tf x_{2n} = u_x, \quad \lim_{n \to \infty} Tg x_{2n+1} = u_x. \quad (6)$$

Now, if $T$ is subsequentially convergent, $\{fx_{2n}\}$ (resp. $\{gx_{2n+1}\}$) has a convergent subsequence. Thus, there exist $v_{x_1} \in X$ and $\{fx_{2n}\}$ (resp. $v_{x_2} \in X$ and $\{gx_{2n+1}\}$) such that

$$\lim_{n \to \infty} fx_{2n} = v_{x_1}, \quad \lim_{n \to \infty} gx_{2n+1} = v_{x_2}. \quad (7)$$

Because of continuity $T$ and by (7), we have

$$\lim_{n \to \infty} Tf x_{2n} = T v_{x_1}, \quad \lim_{n \to \infty} Tg x_{2n+1} = T v_{x_2}. \quad (8)$$

Now, by (6) and (8) and because of injectivity of $T$, there exist $w_x \in X$ (set $v_x = v_{x_1} = v_{x_2}$) such that $Tv_x = u_x$.

On the other hand, by ($d_3$) and (3), we have

$$d(Tv_x, Tv_y) \leq d(Tv_x, Tg x_{2n+1}) + d(Tg x_{2n+1}, Tf x_{2n}) + d(Tf x_{2n}, Tg v_x)$$
$$\leq d(Tv_x, Tg x_{2n+2}) + d(Tg x_{2n+2}, Tx_{2n+1}) + \alpha_1 d(Tx_{2n}, Tv_x)$$
$$+ \alpha_2 \langle d(Tx_{2n}, Tx_{2n+1}) + d(Tv_x, Tg v_x) \rangle$$
$$+ \alpha_3 \langle d(Tx_{2n}, Tg v_x) + d(Tv_x, Tx_{2n+1}) \rangle$$
$$\leq d(Tv_x, Tg x_{2n+2}) + d(Tg x_{2n+2}, Tx_{2n+1}) + (\alpha_1 + \alpha_3) d(Tx_{2n}, Tv_x)$$
$$+ \alpha_2 d(Tx_{2n}, Tx_{2n+1}) + \alpha_3 d(Tv_x, Tx_{2n+1})$$
$$+ (\alpha_2 + \alpha_3) d(Tv_x, Tg v_x).$$
Now, by (4) and (5) we have
\[
d(Tv_x, Tgv_x) \leq \frac{1}{1 - \alpha_2 - \alpha_3} d(Tv_x, Tx_{2n}, + 2) + \frac{1}{1 - \alpha_2 - \alpha_3} d(Tx_{2n}, + 2, Tx_{2n+1}) \\
+ \frac{\alpha_1 + \alpha_3}{1 - \alpha_2 - \alpha_3} d(Tx_{2n}, Tv_x) + \frac{\alpha_2}{1 - \alpha_2 - \alpha_3} d(Tx_{2n}, Tx_{2n+1}) \\
+ \frac{\alpha_3}{1 - \alpha_2 - \alpha_3} d(Tv_x, Tx_{2n+1}) \\
= A_1 d(Tv_x, Tx_{2n}, + 2) + A_2 \gamma 2^n + A_3 d(Tx_{2n}, Tv_x) \\
+ A_4 d(Tv_x, Tx_{2n+1}),
\]
where
\[
A_1 = \frac{1}{1 - \alpha_2 - \alpha_3}, \quad A_2 = \frac{\alpha_2 + \gamma}{1 - \alpha_2 - \alpha_3} d(Tx_0, Tx_1) \\
A_3 = \frac{\alpha_1 + \alpha_3}{1 - \alpha_2 - \alpha_3}, \quad A_4 = \frac{\alpha_3}{1 - \alpha_2 - \alpha_3}.
\]

From (1), we have
\[
\|d(Tv_x, Tgv_x)\| \leq A_1 K\|d(Tv_x, Tx_{2n}, + 2)\| + A_2 K \gamma 2^n \|d(Tx_0, Tx_1)\| \\
+ A_3 K\|d(Tx_{2n}, Tv_x)\| + A_4 K\|d(Tv_x, Tx_{2n+1})\|
\]
Now the right hand side of the above inequality approaches zero as \(i \to \infty\). The convergence above give us that \(d(Tv_x, Tgv_x)\) = 0. Hence \(d(Tv_x, Tgv_x) = 0\), that is, \(Tv_x = Tgv_x\). Since \(T\) is one to one, then \(gv_x = v_x\). Now, we shall show that \(fv_x = v_x\).
\[
d(Tfv_x, Tv_x) = d(Tfv_x, Tgv_x) \\
\leq \alpha_1 d(Tv_x, Tv_x) + \alpha_2 d(Tv_x, Tfv_x) + d(Tv_x, Tgv_x) \\
+ \alpha_3 [d(Tv_x, Tgv_x) + d(Tv_x, Tfv_x)] \\
= (\alpha_2 + \alpha_3) d(Tv_x, Tfv_x)
\]
which, using the definition of partial ordering on \(E\) and properties of cone \(P\), gives \(d(Tfv_x, Tv_x) = 0\). Hence, \(Tfv_x = Tv_x\). Since \(T\) is one to one, then \(fv_x = v_x\). Thus, \(fv_x = gv_x = v_x\), that is, \(v_x\) is a common fixed point of \(f\) and \(g\). Now, we shall show that \(v_x\) is a unique common fixed point. Suppose that \(v'_x\) be another common fixed point of \(f\) and \(g\), then
\[
d(Tv_x, Tv'_x) = d(Tfv_x, Tgv'_x) \\
\leq \alpha_1 d(Tv_x, Tv'_x) + \alpha_2 [d(Tv_x, Tfv_x) + d(Tv'_x, Tgv'_x)] \\
+ \alpha_3 [d(Tv_x, Tgv'_x) + d(Tv'_x, Tfv_x)] \\
= (\alpha_1 + 2\alpha_3) d(Tv_x, Tv'_x)
\]
By the same arguments as above, we conclude that \(d(Tv_x, Tv'_x) = 0\), which implies the equality \(Tv_x = Tv'_x\). Since \(T\) is one to one, then \(v_x = v'_x\). Thus \(f\) and \(g\) have a unique common fixed point.
Ultimately, if \( T \) is sequentially convergent, then we replace \( n \) for \( n_1 \). Thus, we have

\[
\lim_{n \to \infty} f x_{2n} = v_x , \quad \lim_{n \to \infty} g x_{2n+1} = v_x.
\]

Therefore if \( T \) is sequentially convergent, then iterate sequences \( \{f x_{2n}\} \) and \( \{g x_{2n+1}\} \) converge to \( v_x \).

The following results is obtained from Theorem 3.1.

**Corollary 3.2** Let \((X, d)\) be a complete cone metric space, \( P \) be a normal cone and \( T : X \to X \) be a continuous and one to one mapping. Moreover, let mapping \( f \) be a map of \( X \) satisfying

\[
d(Tfx, Tf y) \leq \alpha_1 d(Tx, Ty) + \alpha_2 [d(Tx, Tf x) + d(Ty, Tf y)] + \alpha_3 [d(Tx, Tf y) + d(Ty, Tf x)],
\]

for all \( x, y \in X \), where

\[
\alpha_i \geq 0 \quad \text{for} \quad i = 1, 2, 3 \quad \text{and} \quad \alpha_1 + 2\alpha_2 + 2\alpha_3 < 1. \tag{10}
\]

That is, \( f \) be a \( T \)-contraction. Then,

1. For each \( x_0 \in X \), \( \{f^n x_0\} \) is a cauchy sequence.
2. There exist \( u_{x_0} \in X \) such that \( \lim_{n \to \infty} T^n x_0 = u_{x_0} \).
3. If \( T \) is subsequentially convergent, then \( \{f^n x_0\} \) has a convergent subsequence.
4. There exist a unique \( v_{x_0} \in X \) such that \( f v_{x_0} = v_{x_0} \), that is, \( f \) has a unique fixed point.
5. If \( T \) is sequentially convergent, then for each \( x_0 \in X \) the iterate sequence \( \{f^n x_0\} \) converges to \( v_{x_0} \).

Recently, Fillipović et al. prove that the Corollary 3.2 for a non-normal cone.

**Corollary 3.3** Let \((X, d)\) be a complete cone metric space, \( P \) be a solid cone and \( T : X \to X \) be a continuous and one to one mapping. Moreover, let mapping \( f \) be a \( T \)-Hardy-Rogers contraction. Then, the results of previous Corollary hold.

**Proof** See [5].

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4. **Periodic point results**

Obviously, if \( f \) is a map which has a fixed point \( z \), then \( z \) is also a fixed point of \( f^n \) for each \( n \in \mathbb{N} \). However the converse is not true [2]. If a map \( f : X \to X \) satisfies \( \text{Fix}(f) = \text{Fix}(f^n) \) for each \( n \in \mathbb{N} \), where \( \text{Fix}(f) \) stands for the set of fixed points of \( f \) [9], then \( f \) is said to have property \( P \). Recall also that two mappings \( f, g : X \to X \) is said to have property \( Q \) if \( \text{Fix}(f) \cap \text{Fix}(g) = \text{Fix}(f^n) \cap \text{Fix}(g^n) \).

The following results extend some theorems of [2].

**Theorem 4.1** Let \((X, d)\) be a cone metric space, \( P \) be a normal cone and \( T : X \to X \) be a one to one mapping. Moreover, let mapping \( f \) be a map of \( X \) satisfying

(i) \( d(fx, f^n x) \leq \lambda d(x, fx) \) for all \( x \in X \), where \( \lambda \in [0, 1) \) and or (ii) with strict inequality, \( \lambda = 1 \) for all \( x \in X \) with \( x \neq fx \). If \( \text{Fix}(f) \neq \emptyset \), then \( f \) has property \( P \).

**Proof** See [5].
THEOREM 4.2 Let \((X, d)\) be a complete cone metric space, and \(P\) a normal cone with normal constant \(K\). Suppose that mappings \(f, g : X \to X\) satisfy all the conditions of Theorem 3.1. Then \(f\) and \(g\) have property \(Q\).

Proof From Theorem 3.1, \(f\) and \(g\) have a unique common fixed point in \(X\). Suppose that \(z \in \text{Fix}(f^n) \cap \text{Fix}(g^n)\), thus we have

\[
d(Tz, Tgz) = d(Tf(f^{n-1}z), Tg(g^{n}z))
\]

\[
\leq \alpha_1 d(Tf^{n-1}z, Tg^n z) + \alpha_2[d(Tf^{n-1}z, Tf^n z) + d(Tg^n z, Tg^{n+1} z)]
+ \alpha_3[d(Tf^{n-1}z, Tg^{n+1} z) + d(Tg^n z, Tf^n z)]
\]

\[
= \alpha_1 d(Tf^{n-1}z, Tz) + \alpha_2[d(Tf^{n-1}z, Tz) + d(Tz, Tg)]
+ \alpha_3 d(Tf^{n-1}z, Tg z)
\]

\[
\leq (\alpha_1 + \alpha_2 + \alpha_3)d(Tf^{n-1}z, Tz) + (\alpha_2 + \alpha_3)d(Tz, Tg z),
\]

which implies that

\[
d(Tz, Tgz) \leq \gamma d(Tf^{n-1}z, Tz),
\]

where \(\gamma = \frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_2 - \alpha_3} < 1\) (by relation (4)). Now, we have

\[
d(Tz, Tgz) = d(Tf^n z, Tg^{n+1} z) \leq \gamma d(Tf^{n-1}z, Tz) \leq \cdots \leq \gamma^n d(Tf z, Tz).
\]

From (1), we have

\[
\|d(Tz, Tgz)\| \leq \gamma^n \|d(Tf z, Tz)\|.
\]

Now the right hand side of the above inequality approaches zero as \(n \to \infty\). Hence, \(\|d(Tz, Tgz)\| = 0\). It follows that \(d(Tz, Tgz) = 0\), that is, \(Tgz = Tz\). Since \(T\) is one to one, then \(gz = z\). Also, Theorem 3.1 implies that \(fz = z\) and \(z \in \text{Fix}(f) \cap \text{Fix}(g)\).

THEOREM 4.3 Let \((X, d)\) be a complete cone metric space, and \(P\) a solid cone. Suppose that mapping \(f : X \to X\) satisfies all the conditions of Corollary 3.2. Then \(f\) has property \(P\).

Proof From Corollary 3.2, \(f\) has a unique common fixed point in \(X\). Suppose that \(z \in \text{Fix}(f^n)\), we have

\[
d(Tz, Tf z) = d(Tf(f^{n-1}z), Tf^n z))
\]

\[
\leq \alpha_1 d(Tf^{n-1}z, Tf^n z) + \alpha_2[d(Tf^{n-1}z, Tf^n z) + d(Tf^n z, Tf^{n+1} z)]
+ \alpha_3[d(Tf^{n-1}z, Tf^{n+1} z) + d(Tf^n z, Tf^n z)]
\]

\[
\leq (\alpha_1 + \alpha_2 + \alpha_3)d(Tf^{n-1}z, Tz) + (\alpha_2 + \alpha_3)d(Tz, Tf z),
\]

which implies that

\[
d(Tz, Tf z) \leq \gamma d(Tf^{n-1}z, Tz)\)

where \(\gamma = \frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_2 - \alpha_3} < 1\) (by relation (10)). Hence,

\[
d(Tz, Tf z) = d(Tf^n z, Tf^{n+1} z) \leq \gamma d(Tf^{n-1}z, Tz) \leq \cdots \leq \gamma^n d(Tf z, Tz).
\]
Therefore, we have \( d(Tz, Tfz) \leq \gamma^nd(Tfz, Tz) \leq \gamma d(Tfz, Tz) \). By the same arguments as Theorem 4.2, we conclude that \( d(Tfz, Tz) = 0 \), that is, \( Tfz = Tz \). Since \( T \) is one to one, then \( fz = z \) and proof is complete.

**Corollary 4.4** Let \((X, d)\) be a complete cone metric space, and \(P\) be a solid cone. Suppose that mapping \( f : X \to X \) satisfies all the conditions of Corollary 3.3. Then \( f \) has property \( P \).

**Proof** See [5].

**References**


