Commutativity degree of $\mathbb{Z}_p \wr \mathbb{Z}_p^n$

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Abstract. For a finite group $G$ the commutativity degree denote by $d(G)$ and defind:

$$d(G) = \frac{|\{(x, y)|x, y \in G, xy = yx\}|}{|G|^2}.$$  

In [2] authors found commutativity degree for some groups, in this paper we find commutativity degree for a class of groups that have high nilpotencies.

Keywords: Presentation of groups, Finite groups, commutativity degree.

1. Introduction

For a finite group $G$ the commutativity degree

$$d(G) = \frac{|\{(x, y)|x, y \in G, xy = yx\}|}{|G|^2}.$$  

is defined and studied by several authors (see for example [2, 3, 7]). When $d(G) \geq \frac{1}{2}$, it is proved by P.Lescot in 1995 that $G$ is abelian, or $\frac{G}{Z(G)}$ is elementary abelian with $|\hat{G}| = 2$, or $G$ is isoclinic with $S_3$ and $d(G) = 1$.

Throughout this paper $n$ is positive integer and $p$ is odd prime number. We consider the wreath product $G_n = \mathbb{Z}_p \wr \mathbb{Z}_p^n$, where the standard wreath product $G \wr H$ of the finite groups $G$ and $H$ is defined to be semidirect product of $G$ by direct product $B$ of $|G|$ copies of $H$.

In [1] it is proved that $G_n$ has efficient presentation as follows:

$$G_n = \langle x, y | y^p = x^{p^n} = 1, [x, x^y] = 1, 1 \leq i \leq \frac{p - 1}{2} \rangle.$$  

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Main theorems in this paper are:

**Theorem 1.1**

\[ d(G_n) = \frac{p^{(p-1)n} + (p^2 - 1)}{p^{(p-1)n+2}}. \]

**Theorem 1.2**

\[ \lim_{n \to \infty} d(G_n) = \frac{1}{p^2}. \]

**Theorem 1.3**

\[ \frac{1}{p^2} < d(G_n) < \frac{1}{p}. \]

2. Proofs

We need some lemmas for proving Theorems 1.1, 1.2 and 1.3.

**Lemma 2.1** In group \( G_n \) every element \( z \) has an unique presentations as follows:

\[ z = y^\alpha (x)^{\beta_0} (x^y)^{\beta_1} ... (x^{y^{p-1}})^{\beta_{p-1}} \]

where \( \alpha \in \{0, 1, 2, ..., p-1\} \) and \( \beta_i \in \{0, 1, 2, ..., p^n - 1\} \) \( (0 \leq i \leq p-1) \).

**Proof** By presentation of \( G_n \), it is clearly.

**Lemma 2.2** Let \( z_1, z_2 \in G_n \) and \( z_1 = y^{\alpha_1} (x)^{\beta_0} (x^y)^{\beta_1} ... (x^{y^{p-1}})^{\beta_{p-1}} \) and \( z_2 = y^{\alpha_2} (x)^{\gamma_0} (x^y)^{\gamma_1} ... (x^{y^{p-1}})^{\gamma_{p-1}} \). Then \( z_1 z_2 = z_2 z_1 \) if and only if:

\[ \beta_i + \gamma_{\alpha_2+i} \equiv \beta_{\alpha_2+i} + \gamma_{\alpha_2-\alpha_1+i} \pmod{p^n}, \quad (i = 0, 1, 2, ..., p-1) \]

where indices are reduced module of \( p \).

**Proof** We have:

\[ z_1 z_2 =
\]

\[ y^{\alpha_1 + \alpha_2} (x^{y^{\alpha_1+1}})^{\gamma_0} ... (x^{y^{\alpha_1+p-1}})^{\gamma_{p-1}} (x)^{\beta_0} (x^y)^{\beta_1} ... (x^{y^{p-1}})^{\beta_{p-1}} \]

and

\[ z_2 z_1 =
\]

\[ y^{\alpha_1 + \alpha_2} (x^{y^{\alpha_2+1}})^{\beta_0} ... (x^{y^{\alpha_2+p-1}})^{\beta_{p-1}} (x)^{\gamma_0} (x^y)^{\gamma_1} ... (x^{y^{p-1}})^{\gamma_{p-1}}. \]

By lemma 2.1 every element in \( G_n \) has unique presentation, so we have:
So we have:

\[
\begin{align*}
\beta_0 + \gamma_{\alpha_2} & \equiv \beta_{\alpha_2} + \gamma_{2-\alpha_1} \pmod{p^n} \\
\beta_1 + \gamma_{\alpha_2+1} & \equiv \beta_{\alpha_2+1} + \gamma_{2-\alpha_1+1} \pmod{p^n} \\
\vdots & \\
\beta_{p-1} + \gamma_{\alpha_2+p-1} & \equiv \beta_{\alpha_2+p-1} + \gamma_{2-\alpha_1+p-1} \pmod{p^n}.
\end{align*}
\]

Then we have:

\[
\beta_i + \gamma_{\alpha_2+i} \equiv \beta_{\alpha_2+i} + \gamma_{2-\alpha_1+i} \pmod{p^n}, (i = 0, 1, 2, ..., p-1).
\]

**Remark:** On set \(G_n \times G_n\), we consider:

\[
\zeta(G_n) = \{(z_1, z_2) | z_1, z_2 \in G_n, z_1 z_2 = z_2 z_1\}.
\]

**Lemma 2.3**

\[
|\zeta(G_n)| = p(p+1)n(p(p-1)n + p^2 - 1).
\]

**Proof** Let \(z \in G_n\) and \(z = y^\alpha(x)\beta_1(x^2)^\beta_2(...(x^{p-1})^\beta_{p-1}.

We consider \(\psi(z) = \alpha\). Now let

\[
\zeta_{\alpha_1, \alpha_2}(G_n) = \{(z_1, z_2) | z_1, z_2 \in G_n, z_1 z_2 = z_2 z_1, \psi(z_1) = \alpha_1, \psi(z_2) = \alpha_2\}.
\]

So we have:

\[
\bigcup_{\alpha_1=0}^{p-1} \bigcup_{\alpha_2=0}^{p-1} \zeta_{\alpha_1, \alpha_2}(G_n) = \zeta(G_n).
\]

More over:

\[
|\zeta(G_n)| = \sum_{\alpha_1=0}^{p-1} \sum_{\alpha_2=0}^{p-1} |\zeta_{\alpha_1, \alpha_2}(G_n)|.
\]

Now we have two cases.

**Case I:** \(\alpha_1 = 0, \alpha_2 = 0\)

Let \(z_1 = x^{\beta_1}(x^2)^{\beta_2}(...(x^{p-1})^{\beta_{p-1}}\) and \(z_2 = x^{\gamma_1}(x^2)^{\gamma_2}(...(x^{p-1})^{\gamma_{p-1}}\) where \(\beta_i, \gamma_j \in \{0, 1, ..., p^n - 1\}\) and \(0 \leq i, j \leq p - 1\).

Since \(z_1 z_2 = z_2 z_1\) then:

\[
|\zeta_{0,0}(G_n)| = \frac{p^n \times p^n \times \cdots \times p^n}{2^p} = p^{2p^n}.
\]

**Case II:** \(\alpha_1 \neq 0\) or \(\alpha_2 \neq 0\),

Let \(z_1 = y^{\alpha_1}(x)^{\beta_1}(x^2)^{\beta_2}(...(x^{p-1})^{\beta_{p-1}}\) and \(z_2 = y^{\alpha_2}(x)^{\gamma_1}(x^2)^{\gamma_2}(...(x^{p-1})^{\gamma_{p-1}}\). If \(z_1 z_2 = z_2 z_1\) by lemma 2.2 we have:

\[
\beta_i + \gamma_{\alpha_2+i} \equiv \beta_{\alpha_2+i} + \gamma_{2-\alpha_1+i} \pmod{p^n}, (i = 0, 1, 2, ..., p-1) \quad (*)
\]
where indices are reduced module of \( p \).
Now we can choose \( \beta_0, \beta_1, ..., \beta_{p-1}, \gamma_0 \) and find \( \gamma_1, \gamma_2, ..., \gamma_{p-1} \) uniquely by (\(*\)), then
\[
|\zeta_{\alpha_1, \alpha_2}(G_n)| = \frac{p^n \times p^n \times ... \times p^n}{p+1} = p^{n(p+1)}.
\]
Finally we have
\[
|\zeta(G_n)| = \sum_{\alpha_1=0}^{p-1} \sum_{\alpha_2=0}^{p-1} |\zeta_{\alpha_1, \alpha_2}(G_n)| = p^{2np} + (p^2 - 1)p^{n(p+1)} = p^{(p+1)n}(p^{(p-1)n} + p^2 - 1).
\]

**Proof theorems 1.1, 1.2 and 1.3:**
For 1.1 since \( d(G_n) = \frac{|\zeta(G_n)|}{|G_n|^2} \) so by lemma 2.3 we find \( d(G_n) = \frac{p^{(p-1)n} + (p^2 - 1)}{p^{(p-1)n} + 2} \).
For 1.2 and 1.3 we have \( d(G_n) = \frac{1}{p^r} + \frac{p^2 - 1}{p^{2r-1} + 2} \), so
\[
\lim_{n \to \infty} d(G_n) = \frac{1}{p^2}
\]
and \( d(G_n) > \frac{1}{p^r}, \quad d(G_n) < \frac{1}{p} \) is simple. \( \square \)

**References**

[2] H. Doostie, M. Maghasedi, *Certain classes of groups with commutativity degree \( d(G) < \frac{1}{2} \)*, Ars combinatorial, 89 (2008) ,263–270 .