Recognition of the group $G_2(5)$ by the prime graph

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Abstract. Let $G$ be a finite group. The prime graph of $G$ is a graph $\Gamma(G)$ with vertex set $\pi(G)$, the set of all prime divisors of $|G|$, and two distinct vertices $p$ and $q$ are adjacent by an edge if $G$ has an element of order $pq$. In this paper we prove that if $\Gamma(G) = \Gamma(G_2(5))$, then $G$ has a normal subgroup $N$ such that $\pi(N) \subseteq \{2, 3, 5\}$ and $G/N \cong G_2(5)$.

Keywords: prime graph, recognition, linear group

1. Introduction

Let $G$ be a finite group. The spectrum $\omega(G)$ of $G$ is the set of orders of elements in $G$, where each possible order element occurs once in $\omega(G)$ regardless of how many elements of that order $G$ has. This set is closed and partially ordered by divisibility, hence it is uniquely determined by its maximal elements. The set of maximal elements of $\omega(G)$ is denoted by $\mu(G)$. The number of isomorphic classes of finite groups $H$ such that $\omega(G) = \omega(H)$ is denoted by $h(G)$. If $h(G) = k \geq 1$ is finite then the group $G$ is called a k-recognizable group by spectrum. If $h(G)$ is not finite, $G$ is called non-recognizable. A 1-recognizable group is usually called a recognizable group. The recognizability of finite groups by spectrum was first considered by W.J.Shi et.al. in [16]. A list of finite simple groups which are known to be or not to be recognizable by spectrum is given in [11].

For $n \in N$, let $\pi(n)$ denote the set of all the prime divisors of $n$, and for a finite group $G$ let us set $\pi(G) = \pi(|G|)$. The prime graph $\Gamma(G)$ of a finite group $G$ is a simple graph with vertex set $\pi(G)$ in which two distinct vertices $p$ and $q$ are

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joined by an edge if and only if $G$ has an element of order $pq$. It is clear that a knowledge of $w(G)$ determines $\Gamma(G)$ completely but not vice-versa in general. Given a finite group $G$, the number of non-isomorphic classes of finite groups $H$ with $\Gamma(G) = \Gamma(H)$ is denoted by $h_\Gamma(G)$. If $h_\Gamma(G) = 1$, then $G$ is said to be recognizable by prime graph. If $h_\Gamma(G) = k < \infty$, then $G$ is called $k$-recognizable by prime graph, in case $h_\Gamma(G) = 1$ the group $G$ is called non-recognizable by prime graph. Obviously a group recognizable by spectra need not to be recognizable by prime graph, for example $A_5$ is recognizable by spectra but $\Gamma(A_5) = \Gamma(A_6)$.

The number of connected components of $\Gamma(G)$ is denoted by $s(G)$. As a consequence of the classification of the finite simple groups it is proved in [19] and [9], that for any finite simple group $G$ we have $s(G) \leq 6$. Let $\pi_i = \pi_i(G)$, $1 \leq i \leq s$, be the connected components of $G$. For a group of even order we let $2 \in \pi_1$. Recognizability of groups by prime graph was first studied in [5] where some sporadic simple groups were characterized by prime graph. As another concept we say that a non-abelian simple group $G$ is quasi-recognizable by graph if every finite group whose prime graph is $\Gamma(G)$ has a unique non-abelian composition factor isomorphic to $G$.

It is proved in [20] that the simple groups $G_2(7)$ and $^2G_2(q)$, $q = 3^{2m+1} > 3$, are recognizable by prime graph, where both groups have disconnected prime graphs. A series of interesting results concerning recognition of finite simple groups were obtained by B.Khosravi et.al. In particular they have established quasi-recognizability of the group $L_{16}(2)$ by graph and the recognizability of $L_{10}(2)$ by graph in [7] and [8], where both groups have connected prime graphs.

Next we introduce useful notation. Let $p$ be a prime number. The set of all non-abelian finite simple groups $G$ such that $p \in \pi(G) \subseteq \{2, 3, 5, \ldots, p\}$ is denoted by $S_p$. It is clear that the set of all non-abelian finite simple groups is the disjoint union of the finite sets $S_p$ for all primes $p$. The sets $S_p$, where $p$ is a prime less than 1000 is given in [21].

2. Preliminary results

Let $G$ be a finite group with disconnected prime graph. The structure of $G$ is given in [19] which is stated as a lemma here. Let $G$ be a finite group with disconnected prime graph. Then $G$ satisfies one of the following conditions: $s(G) = 2$, $G = KC$ is a Frobenius group with kernel $K$ and complement $C$, and the two connected components of $G$ are $\Gamma(K)$ and $\Gamma(C)$. Moreover $K$ is nilpotent, and here $\Gamma(K)$ is a complete graph. $s(G) = 2$ and $G$ is a 2-Frobenius group, i.e., $G = ABC$ where $A, AB \leq G$, $B \leq BC$, and $AB, BC$ are Frobenius groups.
There exists a non-abelian simple group $P$ such that $P \leq \overline{G} = G/N \leq \text{Aut}(P)$ for some nilpotent normal $\pi_1(G)$-subgroup $N$ of $G$ and $\overline{G}/P$ is a $\pi_1(G)$-group. Moreover, $\Gamma(P)$ is disconnected and $s(P) \geq s(G)$. If a group $G$ satisfies condition(c) of the above lemma we may write $P = B/N$, $B \leq G$, and $\overline{G}/P = G/B = A$, hence in terms of group extensions $G = N \cdot P \cdot A$, where $N$ is a nilpotent normal $\pi_1(G)$-subgroup of $G$ and $A$ is a $\pi_1(G)$-group.

The above structure lemma was extended to groups with connected prime graphs satisfying certain conditions [17]. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise nonadjacent in $\Gamma(G)$ and $t(2, G)$ the maximal number of primes in $\pi(G)$ nonadjacent to 2. Let $G$ be a finite group satisfying the following conditions:

There exist three pairwise distinct primes in $\pi(G)$ nonadjacent in $\Gamma(G)$, i.e., $t(G) \geq 3$.

There exists an odd prime in $\pi(G)$ nonadjacent in $\Gamma(G)$ to 2, i.e., $t(2, G) \geq 2$. Then, there is a finite non-abelian simple group $S$ such that $S \leq \overline{G} = G/K \leq \text{Aut}(S)$ for the maximal normal solvable subgroup $K$ of $G$. Furthermore $t(S) \geq t(G) - 1$ and one of the following statements holds:

1. $S \cong A_7$ or $L_2(q)$ for some odd $q$, and $t(S) = t(2, G) = 3$.
2. For every prime $p \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ a Sylow $p$-subgroups of $G$ is isomorphic to a Sylow $p$-subgroup of $S$. In particular $t(2, S) \geq t(2, G)$.

In the following we list some properties of the Frobenius group where some of its proof can be found in [15]. Let $G$ be a Frobenius group with kernel $K$ and complement $H$, then:

- $K$ is nilpotent and $|H| \mid (|K| - 1)$.
- The connected components of $G$ are $\Gamma(K)$ and $\Gamma(H)$.
- $|\mu(K)| = 1$ and $\Gamma(K)$ is a complete graph.
- If $|H|$ is even, then $K$ is abelian.
- Every subgroup of $H$ of order $pq$, $p$ and $q$ not necessary distinct primes, is cyclic.
- If $H$ is non-solvable, then there is a normal subgroup $H_0$ of $H$ such that $|H : H_0| \leq 2$ and $H_0 \cong SL_2(5) \times Z$, where every Sylow subgroup of $Z$ is cyclic and $|Z|$ is prime to 2, 3 and 5. A Frobenius group with cyclic kernel of order $m$ and cyclic complement of order $n$ is denoted by $m : n$.

The following result is also used in this paper whose proof is included in [3]. Every 2-Frobenius group is solvable. [6] Let $G$ be a finite solvable group all of whose elements are of prime power order, then the order of $G$ is divisible by at most two distinct primes. [12] Let $G$ be a finite group, $K \leq G$, and let $G/K$ be a Frobenius group with kernel $F$ and cyclic complement $C$. If $(|F|, |K|) = 1$ and $F$ does not lie in $(K \cdot C_G(K))/K$, then $r \cdot |C| \in w(G)$ for some prime divisor $r$ of $|K|$. [18] If there exists a primitive prime divisor $r$ of $q^n - 1$, then $L_n(q)$ has a Frobenius subgroup with kernel of order $r$ and cyclic complement of order $n$. 
$L_n(q)$ contains a Frobenius subgroup with kernel of order $q^{n-1}$ and cyclic complement of order $(q^{n-1} - 1)/(n, q - 1)$. Using [?], we can find $\mu(G_2(5)) = \{20, 21, 24, 25, 30, 31\}$. Therefore, the prime graph of $G_2(5)$ is as follows.

\[
\begin{array}{ccc}
7 & \circ & 31 \\
\circ & & \\
2 & \circ & 3 \\
\end{array}
\]

Figure 1: The prime graph of $G_2(5)$

Our main results are the following: If $G$ is a finite group such that $\Gamma(G) = \Gamma(G_2(5))$, then $G$ has a normal subgroup $N$ such that $\pi(N) \subseteq \{2, 3, 5\}$ and $G/N \cong G_2(5)$.

3. Proof of the theorem

We assume $G$ is a group with $\Gamma(G) = \Gamma(G_2(5))$. By Figure 1, we have $s(G) = 2$, hence, $G$ has disconnected prime graph and we can use Lemma 2.1 here: $G$ is non-solvable.

If $G$ is solvable, then consider a $\{5, 7, 31\}$-Hall subgroup of $G$ and call it $H$. By Figure 1, $H$ does not contain elements of order $5 \cdot 7$, $7 \cdot 31$, $5 \cdot 31$, and since it is solvable, by [5] we deduce $|t(H)| \leq 2$, a contradiction.

$G$ is neither a Frobenius nor a 2-Frobenius group.

By (a) and Lemma 2.4, $G$ is not a 2-Frobenius group. If $G$ is a Frobenius group, then by Lemma 2.1, $G = KC$ with Frobenius kernel $K$ and Frobenius complement $C$ with connected components $\Gamma(K)$ and $\Gamma(C)$. Obviously $\Gamma(K)$ is a graph with vertex $\{31\}$ and $\Gamma(C)$ with vertex set $\{2, 3, 5, 7\}$. Since $G$ is non-solvable, by Lemma 2.3(a) $C$ must be non-solvable. Therefore, by Lemma 2.3(f) $C$ has a subgroup isomorphic to $H_0$ and $[C : H_0] \leq 2$, where $H_0 \cong SL_2(5) \times Z$ with $Z$ cyclic of order prime to $2, 3, 5$. But $\mu(SL_2(5)) = \{4, 6, 10\}$ from which we can observe that $H_0$ has no element of order 15. This implies that $C$ has no element of order 15, contradicting Figure 1.

(a) and (b) imply that case (c) of Lemma 2.1 holds for $G$. Hence, there is a non-abelian simple group $P$ such that $P \leq G = G/N \leq Aut(P)$ where $N$ is a nilpotent normal $\pi_1(G)$-subgroup of $G$ and $\overline{G}/P$ is a $\pi_1(G)$-group and $s(P) \geq 2$. We have $\pi_1(G) = \{2, 3, 5, 7\}$ and $\pi(G) = \{2, 3, 5, 7, 31\}$. Therefore, $P$ is a simple group with $\pi(P) \subseteq \{2, 3, 5, 7, 31\}$, i.e., $P \in \mathfrak{S}_p$ where $p$ is a prime number satisfying $p \leq 31, p \neq 11, 13, 17, 19, 23, 29$. Using [21] we list the possibilities for $P$ in Table I.

Table I: Simple groups in $\mathfrak{S}_p, p \leq 31, p \neq 11, 13, 17, 19, 23, 29$. 

\{31\} \subseteq \pi(P)

By Table I, \(|Out(P)|\) is a number of the form \(2^a \cdot 3^b\), therefore, if \(G/N = P \cdot S\) where \(S \leq Out(P)\), then \(|P|_p = |G/N|_p/|S|_p\) for all \(p \in \pi(G)\), where \(n_p\) denotes the \(p\)-part of the integer \(n \in N\). Hence, \(|N|_p = \left\lceil \frac{|G|}{|P|_p \cdot |S|_p} \right\rceil\), from which the claim follows because \(\pi(N) \subseteq \{2, 3, 5, 7\}\).

Therefore only the following possibilities arise for \(P\): \(L_2(31), L_5(2), L_6(2), L_3(5), L_2(5^3)\) and \(G_2(5)\).

\(P \cong G_2(5)\)

By \([4]\), we have \(\mu(L_5(2)) = \{8, 12, 14, 15, 21, 31\}\) and \(\mu(L_6(2)) = \{8, 12, 28, 30, 31, 63\}\). Therefore, if \(P \cong L_5(2)\) or \(L_6(2)\), then, we have 2 \sim 7 in \(\Gamma(G)\), is a contradiction.

By \([10]\), we have \(\mu(L_5(5^3)) = \{5, 62, 63\}\). Therefore, if \(P \cong L_5(5^3)\), then, we have 2 \sim 31 in \(\Gamma(G)\), a contradiction.

By \([7]\), we have \(\mu(L_3(2)) = \{15, 16, 31\}\). Therefore, if \(P \cong L_3(31)\), then, 7 \in \(\pi(N)\). By Lemma 2.7, \(P\) has a Frobenius subgroup \(31 : 15\), then, by Lemma 2.6, \(G\) has an element of order 5 \cdot 7, a contradiction.

By \([7]\), we have \(\mu(L_3(5)) = \{20, 24, 31\}\). Therefore, if \(P \cong L_3(5)\), then, 7 \in \(\pi(N)\). By Lemma 2.7, \(P\) has a Frobenius subgroup \(25 : 24\), then, by Lemma 2.6, \(G\) has an element of order 2 \cdot 7, a contradiction. Therefore \(P \cong G_2(5)\).

\(G/N \cong G_2(5)\)

So far we proved that \(P \leq G/N \leq Aut(P)\) where \(P \cong G_2(5)\). But \(Aut(G_2(5)) = G_2(5)\), therefore, \(G/N \cong G_2(5)\).

\(\pi(N) \subseteq \{2, 3, 5\}\)

We Know that \(N\) is a nilpotent normal \(\{2, 3, 5\}\)-subgroup of \(G\). Regarding Figure 1 we obtain:
If $2, 5 \mid |N|$, then $\pi(N) \subseteq \{2, 3, 5\}$
If $3 \mid |N|$, then $\pi(N) \subseteq \{2, 3, 5, 7\}$
If $7 \mid |N|$, then $\pi(N) \subseteq \{3, 7\}$

Now we observe that the group $G_2(5)$ contains Frobenius subgroup $31 : 5$. We may assume $N$ is elementary abelian $p$-group for $p \in \{2, 3, 5, 7\}$. Now if $7 \mid |N|$, then by Lemma 2.6, $G$ has an element of order $5 \cdot 7$, a contradiction. Therefore, $\pi(N) \subseteq \{2, 3, 5\}$.

References