Free Vibration of Simply Supported Rectangular Composite Plates with Patch Mass

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Abstract: The effect of a distributed patch mass on the natural frequency of vibration of a laminated rectangular plate with simply supported boundaries is investigated. The third order displacement field of a composite laminated rectangular plate is defined using the two-variable refined plate theory. Equations of motion of the plate are obtained with the help of the calculus of variation. Parametric study of non-dimensional natural frequencies of vibration is carried out and the effects of geometrical parameters such as the aspect ratio of the plate, size and location of the patch mass on these frequencies are studied. The results are then compared with those reported using the third order shear deformation theory. The findings are found to be in a very good agreement.

Keywords: Free Vibration, Patch Mass, Laminated Composite Plate, Refined Plate Theory

1. Introduction

Composite materials have found various engineering applications due to their high strength to weight ratio, high degrees of anisotropy and low rigidity in transverse shear compared with other materials, and are widely used in the automotive, aerospace and other application in structures. These materials are practically subjected to various loading conditions such as distributed patch mass, transverse and in-plane loadings, hence it is necessary to study their response to these loading conditions. Withney and Pagano [1] investigated free vibration response of a composite plate using the first order shear deformation theory (FSDT). This theory needs shear correction factor to rectify the unrealistic variation of the shear strain-stress through the thickness. To overcome the limitations of FSDT, higher order shear deformation theory (HSDT) were developed by Levinson [2], Reddy [3], Ren [4], Kant and Pandta [5] and Mohan et al. [6].


Seung-Eock et al. [16,17] employed the two-variable refined plate theory for vibration and buckling analysis of cross-ply and angle-ply laminated composite plates under the action of the transverse and in-plane force. While there are many papers on...
plate vibrations with added point masses, very few reports on plate vibration with patch mass can be found in this case such as Kompaz and Telli [18], and Wong [11]. The aim of this paper is to develop RPT2 for laminated composite plates with mass. The most important feature of this theory is that it does not require shear correction factor which is essential for making an equilibrium condition at the top and bottom faces of the plates, and has considerable similarities with the classical plate theory in boundary conditions, moment expressions and constitutive equations.

2. Basic formulation

A rectangular plate with length, width and thickness equal to \( a, b \) and \( h \) respectively is considered. The plate supports an arbitrary patch mass, \( M_{\text{mass}} \), with dimensions equal to \( c \) and \( d \) in the \( x \) and \( y \)-direction respectively, with its centre positioned at \((x', y')\) on the upper surface of the plate, as shown in Fig. 1. The global Cartesian coordinate system is chosen with the origin at the corner and on the middle plane of the plate, \( z=0 \). Therefore, the domain is defined as \( 0 \leq x \leq a, 0 \leq y \leq b \) and \(-h/2 \leq z \leq h/2 \). In order to proceed with the formulation of the problem using the two-variable refined plate theory (RPT2), it is assumed that the displacement components \((u, v, w)\) of the plate are small in comparison with the thickness of the plate.

Also, the transverse normal stress in the \( z \)-direction, \( \sigma_z \), is very small in comparison with the inplane stresses, \( \sigma_x \) and \( \sigma_y \). In view of the above assumptions, the stress-strain relations can be reduced from a \( 6 \times 6 \) matrix to a \( 5 \times 5 \) matrix that reduces the complexity of the problem. The total displacements of the plate in the \( x \) and \( y \)-direction \((U, V)\) are inclusive of three components, \( u, u_x, u_y \) and the total displacement in \( z \)-direction \((W)\) is assumed to be consisting of three components, \( w_a, w_b \) and \( w_s \) which are the functions of \( x, y \) and time \([16]\).

\[
U = u - z \partial w_a / \partial x + \left( \frac{1}{4} - \frac{5}{3} \frac{z}{h} \right) \frac{\partial w_s}{\partial x}
\]

Where subscripts, \( a, b \) and \( s \) denote extension, bending and shearing components, respectively. In order to define the stress-strain relations in the geometrical coordinate system of the plate, that is the global Cartesian coordinate system, the components of the reduced stiffness tensor should be transformed according to the transformation law of the fourth order tensors. The strain energy and the kinetic energy of the plate are defined as:

\[
U_{\text{strain}} = \frac{1}{2} \int \sigma_{ij} \varepsilon_{ij} \, dV
\]

\[
T = \frac{1}{2} \int \rho \left( \dot{u}^2 + \dot{v}^2 + \dot{w}_a + \dot{w}_b + \dot{w}_s \right) \, dx dy dz
\]

As a matter of fact, the stress resultants of the total \( N \) layers of the plate are found in terms of the mid-plane strain and curvatures of the plate.

3. Equations of motion

The total kinetic energy of the plate with the patch mass is the summation of the kinetic energies of the plate and the uniformly localized patch mass acting on the top surface of the plate:

\[
T_{\text{total}} = T_{\text{plate}} + T_{\text{mass}}
\]

\[
T_{\text{total}} = \frac{1}{2} \int \int \left[ \dot{u}^2 + \dot{v}^2 + \left( \dot{w}_a + \dot{w}_b + \dot{w}_s \right)^2 \right] \, dx dy + \frac{1}{2} \int \int \left[ \frac{\partial \varepsilon_{xx}}{\partial x} \right]^2 + \frac{\partial \varepsilon_{yy}}{\partial y} \right] \, dx dy + \frac{1}{2} \int \int \left[ \frac{\partial \varepsilon_{xy}}{\partial x} \right]^2 + \frac{\partial \varepsilon_{yx}}{\partial y} \right] \, dx dy + \frac{1}{2} \int \int \int \left( \dot{w}_a + \dot{w}_b + \dot{w}_s \right)^2 \rho_a h_a \, dy dx dz
\]

\[
V = \nu - z \left( \frac{\partial w_a}{\partial x} \right) + \left( \frac{1}{4} - \frac{5}{3} \frac{z}{h} \right) \frac{\partial w_s}{\partial x}
\]
Where $I_x, I_z = \int_{x=a}^{x=b} \rho(z,z) dz$ are the inertia terms, $\rho_m$ and $h_m$ are the density of the patch mass and its thickness in the $z$-direction respectively. The first variation of the Lagrangian, (i.e., Hamilton's principle [19]) is employed to obtain the coefficients of mass and stiffness matrix.

$$\int_{v_0} \delta (U_{strain} - T_{mech}) = 0$$  \hspace{1cm} \text{(5)}$$

Where $\delta$ presents a variation with respect to $x$ and $y$. Here $U_{strain}$ denotes the strain energy. Since this study here is in the free vibration analysis, the potential energy due to the applied force leads to zero. Substituting the displacements field in the relevant strain energy and kinetic energy terms, integrating the equations by parts and collecting the coefficients of $\delta u, \delta v, \delta w_x, \delta w_y$ and $\delta w_z$, the equations of motion for homogeneous laminates are derived as follows:

1. $\delta u \rightarrow \frac{d}{dt} \left( \int_{v_0} \left[ \frac{\partial N_{u_x}}{\partial x} \right] dydx + \int_{s^{z=2}} \frac{\partial N_{u_y}}{\partial y} dydx + \int_{s^{z=2}} \frac{\partial N_{u_z}}{\partial z} dydx \right) dydx + \left( \int_{s^{z=2}} \int_{s^{z=2}} 3.125I_s \frac{d^2u}{d^2z} dydx \right) = 0,$  \hspace{1cm} \text{(6)}$$

2. $\delta v \rightarrow \frac{d}{dt} \left( \int_{v_0} \left[ \frac{\partial N_{v_x}}{\partial x} \right] dydx + \int_{s^{z=2}} \frac{\partial N_{v_y}}{\partial y} dydx + \int_{s^{z=2}} \frac{\partial N_{v_z}}{\partial z} dydx \right) dydx + \left( \int_{s^{z=2}} \int_{s^{z=2}} 3.125I_s \frac{d^2v}{d^2z} dydx \right) = 0,$  \hspace{1cm} \text{(6)}$$

3. $\delta w_x \rightarrow \frac{d}{dt} \left( \int_{v_0} \left[ \frac{\partial N_{w_{x_x}}}{\partial x} \right] dydx + \int_{s^{z=2}} \frac{\partial N_{w_{x_y}}}{\partial y} dydx + \int_{s^{z=2}} \frac{\partial N_{w_{x_z}}}{\partial z} dydx \right) dydx + \left( \int_{s^{z=2}} \int_{s^{z=2}} 3.125I_s \frac{d^2w_x}{d^2z} dydx \right) = 0,$  \hspace{1cm} \text{(6)}$$

4. $\delta w_y \rightarrow \frac{d}{dt} \left( \int_{v_0} \left[ \frac{\partial N_{w_{y_x}}}{\partial x} \right] dydx + \int_{s^{z=2}} \frac{\partial N_{w_{y_y}}}{\partial y} dydx + \int_{s^{z=2}} \frac{\partial N_{w_{y_z}}}{\partial z} dydx \right) dydx + \left( \int_{s^{z=2}} \int_{s^{z=2}} 3.125I_s \frac{d^2w_y}{d^2z} dydx \right) = 0,$  \hspace{1cm} \text{(6)}$$

5. $\delta w_z \rightarrow \frac{d}{dt} \left( \int_{v_0} \left[ \frac{\partial N_{w_{z_x}}}{\partial x} \right] dydx + \int_{s^{z=2}} \frac{\partial N_{w_{z_y}}}{\partial y} dydx + \int_{s^{z=2}} \frac{\partial N_{w_{z_z}}}{\partial z} dydx \right) dydx + \left( \int_{s^{z=2}} \int_{s^{z=2}} 3.125I_s \frac{d^2w_z}{d^2z} dydx \right) = 0,$  \hspace{1cm} \text{(6)}$$

Finally, the sets of governing equations of plate vibration are obtained in the form of $([K] - \omega^2 [S])[\lambda] = 0$, where $[K]$ and $[S]$ are the stiffness and mass matrices respectively, $\omega$ is the natural frequency of vibration of the plate and $\lambda$ is a vector of unknown coefficients. For convenience, the non-dimensional natural frequency of plate is defined as: $\tilde{\omega} = \frac{\omega}{\sqrt{\rho h^2/E}}$.

Here, two sets of boundary conditions are considered. The first set of boundary condition is called the SS-1 boundary condition that is applied for an anti-symmetric cross-ply laminate and the second one is called SS-2 boundary condition for an anti-symmetric angle-ply laminate.

**SS-1 boundary condition:**

For $i = a, b, s$, $m = 0, a$ and $n = 0, b$

\[ u(x,n) = v(m,y) = 0, \quad w_i(m,y) = w_i(x,n) = 0, \]

\[ = 0, \quad \frac{\partial w_i}{\partial y}(m,y) = \frac{\partial w_i}{\partial x}(m,y) \]

\[ N_i(m,y) = N_i(x,n) = M^*_{ij}(m,y) = M^*_{ij}(x,n) = 0 \]

**SS-2 boundary condition:**

\[ w_x + w_y + w_z = 0, \]

\[ \frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z} = 0. \]
For $i = a, b, s$, $m = 0, a$ and $n = 0, b$

$u(m, n) = v(x, n) = w(m, n) = 0$,

$\frac{\partial u}{\partial m} \bigg|_{m=0} = \frac{\partial u}{\partial n} \bigg|_{n=0} = 0,$

$N_m(0, y) = N_n(x, 0) = N_{x,y}(0, 0, 0) = M_{x}(0, 0) = M_{y}(0, 0) = 0$

$N_{x}(a, y) = N_{y}(x, b) = M_{x}(a, y) = M_{y}(x, b)$

In order to satisfy the boundary conditions, the following displacement fields are assumed:

\[
\begin{align*}
 u & = U_{mn} \cos \alpha_m x \sin \beta_n y \\
 v & = V_{mn} \sin \alpha_m x \cos \beta_n y \\
 w_x & = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin \alpha_m x \sin \beta_n y \\
 w_y & = W_{mn} \sin \alpha_m x \sin \beta_n y \\
 w & = W_{mn} \sin \alpha_m x \sin \beta_n y
\end{align*}
\]

(7)

Where $\alpha = m \pi/a$, $\beta = n \pi/b$, and $(U_{mn}, V_{mn}, W_{mn}, W_{mn})$ are coefficients.

4. Numerical results and discussion

Two sets of dimensionless material properties are considered:

**MAT1:**

$E_1/E_2 = 0.6, G_{12}/E_2 = 0.5, v_{12} = 0.25$

**MAT2:**

$E_1/E_2 = 0.4, G_{12}/E_2 = 0.6, v_{12} = 0.3$

At first, the vibration response of a plate without the patch mass is studied. The first non-dimensional natural frequencies of vibration for a four-layer anti-symmetric cross-ply laminate made of MAT1 with $G_{13} = G_{23} = 0.6E_2$ and $a/h = 5$ are found with different values of $E_2/E_1$ as a parameter. The results are compared with those obtained using the first order shear deformation theory [1], the 3D-elasticity solution [8] and higher order shear deformation theory [9], as shown in Table 1. As it is expected, the results obtained by the present theory are in a very good agreement with the results of the higher shear deformation theory.

Now, a distributed patch mass at the centre of the plate with the following properties is considered.

$M_{max}/M_{plate} = 0.5, c/a = d/b = 0.4 \rightarrow$

$\rho_{section} h_{cd} = 0.5 \rightarrow \rho_{section} h = 3.125 \rho h = 3.125I_o$

The effect of the distributed patch mass on the first non-dimensional natural frequency of angle-ply laminated plates made of MAT1, $\{45/-45\}_2$ with different ratios of $a/h$ and $a/b$ are studied and the results obtained by the present method (RPT2) are compared with those obtained by (TSDT) method [15], as shown in Table 2. It is observed that, as the $a/b$ ratio increases (the $a/h$ ratio being constant), the first natural frequency increases. Comparison of the results of the first natural frequency between RPT2 and TSDT methods with distributed patch mass is shown in Fig. 2.

In the next step, the distributed patch mass is considered in three different positions on the plate.

These three positions of the distributed patch mass are:

1) $x'_1 = a/4, y'_1 = b/2$
2) $x'_2 = a/2, y'_2 = b/4$
3) $x'_3 = a/6, y'_3 = b/5$

and other parameters related to the distributed patch mass are:

$M_{max}/M_{plate} = 0.3, c/a = d/b = 0.2$

The effect of position of the distributed patch mass on the first natural frequency of an angle-ply laminated plate made of MAT2 with $\{30/-30\}_2$ lamina is studied and the results are shown in Table 3. It is observed from Table 3, that the natural frequency of the plate with distributed patch mass at position 3 is higher than the other two positions under study, that is due to its larger stiffness induced comparing with the other two positions. The non-dimensional first natural frequency of a rectangular plate made of MAT2 for $\{45/-45\}_2$ lamination for four different positions of the patch mass is shown in Fig. 3. Due to the symmetry imposed by the boundary conditions of the plate, it is observed that the patch mass in position 1 and position 2 would have similar natural frequencies of vibration.
Table 1. The first non-dimensional natural frequency $\tilde{\omega}$, anti-symmetric cross-ply square plate, $(0/90)_2$, MAT1 with $G_{13} = G_{33} = 0.6E_z$, $a/h = 5$

<table>
<thead>
<tr>
<th>$E_z/E_z$</th>
<th>Theory</th>
<th>3</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
</table>

Table 2. The non-dimensional first natural frequency $\tilde{\omega}$, MAT1, $[45/-45]_2$

<table>
<thead>
<tr>
<th>$a/b$</th>
<th>$a/h$</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>0.2</td>
<td>5.3728</td>
<td>5.8127</td>
</tr>
<tr>
<td>0.8</td>
<td>9.5102</td>
<td>11.1744</td>
</tr>
<tr>
<td>1.2</td>
<td>13.0434</td>
<td>16.1777</td>
</tr>
<tr>
<td>1.6</td>
<td>16.3026</td>
<td>21.9344</td>
</tr>
<tr>
<td>2.0</td>
<td>21.4663</td>
<td>28.7637</td>
</tr>
</tbody>
</table>

Table 3. The effect of $a/b$ and $a/h$ on $\tilde{\omega}$ of a plate with patch mass, MAT2, $[30/-30]_2$

<table>
<thead>
<tr>
<th>$a/b$</th>
<th>$a/h$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10</td>
</tr>
<tr>
<td>Pose 1</td>
<td>7.4084</td>
</tr>
<tr>
<td></td>
<td>10.4399</td>
</tr>
<tr>
<td></td>
<td>15.3371</td>
</tr>
<tr>
<td></td>
<td>21.3574</td>
</tr>
<tr>
<td>Pose 2</td>
<td>7.4322</td>
</tr>
<tr>
<td></td>
<td>10.4516</td>
</tr>
<tr>
<td></td>
<td>15.3048</td>
</tr>
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<td></td>
<td>21.2429</td>
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<td></td>
<td>17.5157</td>
</tr>
<tr>
<td></td>
<td>24.5518</td>
</tr>
</tbody>
</table>
5. Conclusions

In this paper, the free vibration of a rectangular composite plate carrying a distributed patch mass was presented. The governing equations have been obtained by the two-variable refined plate theory and the effect of aspect ratios, size and location of the patch mass on the free vibration of the plate studied. The main conclusions are as:

- The first natural frequency of vibration of a plate increases with an increase in the ratio of $a/b$ (with $a/h$ being constant).

- The results obtained by the two-variable refined plate theory is in a very good agreement with the results obtained by the higher order shear deformation theory.

- The lowest natural frequency of plates with symmetric boundary conditions occurs with the patch mass at the centre of the plate. The natural frequency increases with the patch mass moving towards the corner of the plate.

Nomenclature

- $w_a$: displacement term in extension in the $z$-direction
- $w_b$: displacement term in bending in the $z$-direction
- $w_s$: displacement term in shear in the $z$-direction
- $\sigma_{ij}$: Stress in the local coordinate
- $e_{ij}$: Strain in the local coordinate
- $G_{ij}$: Modulus of elasticity in shear in the $ij$-direction
- $E_i$: Modulus of elasticity in tension and compression in the $i$-direction
- $v_{ij}$: Poisson's ratio

References


[7] Singh, B. N.; Yadav, D.; Iyengar, N. G. R., "Natural frequencies of composite plates with random material properties using higher-order shear deforma-


