A Comparative Study of Least-Squares and the Weak-Form Galerkin Finite Element Models for the Nonlinear Analysis of Timoshenko Beams

W. Kim¹, J.N. Reddy²,*

¹Department of Mechanical Engineering, Korea Army Academy at Yeong cheon, Yeong cheon, 770-849 South Korea
²Department of Mechanical Engineering, Texas A&M University, College Station, TX 77840, USA

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ABSTRACT

In this paper, a comparison of weak-form Galerkin and least-squares finite element models of Timoshenko beam theory with the von Kármán strains is presented. Computational characteristics of the two models and the influence of the polynomial orders used on the relative accuracies of the two models are discussed. The degree of approximation functions used varied from linear to the 5th order. In the linear analysis, numerical results of beam bending under different types of boundary conditions are presented along with exact solutions to investigate the degree of shear locking in the newly developed mixed finite element models. In the nonlinear analysis, convergences of nonlinear finite element solutions of newly developed mixed finite element models are presented along with those of existing traditional model to compare the performance.

Keywords: Finite element models; Least-squares model; Weak-form Galerkin model; Geometric nonlinearity; Timoshenko beam theory

1 INTRODUCTION

FOR several decades, finite element models based on the weak-form Galerkin or Ritz formulation [1] have dominated both commercial finite element software and academic research. An advantage of the weak-form Galerkin model as applied to structural mechanics problems is that it is the most natural approach resulting from the application of the principle of virtual displacements. For problems outside of solid and structural mechanics, such principles are not available. It is possible, however, to develop the so-called weak forms or weighted-residual integral statements from the governing differential equations of any physical problem. The weak forms, in most cases, are not equivalent to any principle or minimization of error introduced in the approximation of the variables. Thus, the weak forms of problems for which there is no underlying physical or mathematical principle, are merely a means to computing solutions of the integral statements represented by the weak forms [2]. Consequently, it is possible that such solutions may degenerate for certain combinations of physical characteristics of the problem and finite element approximations used. It is well known that the weak formulations lower the differentiability requirements on the variables of the formulation. The traditional displacement based (i.e., principle of virtual displacements) finite element models can provide high level of accuracy only for the generalized displacements, increasing errors in the post-computed variables involving the derivatives of the generalized displacements. To remedy this, the so-called mixed formulations [3] are adopted. The mixed models include secondary variables as independent degrees of freedom and they yield increased accuracy of the secondary variables, sometimes at the expense of the primary variables and increased computational effort. Thus weak-form, mixed, Galerkin finite element models can provide higher level of accuracy for secondary variables included in the model. But to establish
mixed Galerkin finite element models, each of the governing equation should be multiplied by proper weight functions, and then should be integrated over the computational domain. The Lagrange multiplier method can be used to find proper weight functions, which is still burdensome.

In recent times, the least-squares method is considered as a good alternative method because of its simplicity in applications to a variety of problems. It can also overcome many drawbacks of the weak-form Galerkin formulations. A brief idea of the least-squares method is to minimize sums of squares of the residuals over the finite element space. In the least-squares method, procedures of finding proper weight function for each residual is unnecessary. Because variations of residuals can be used as weight function from the weighted integral viewpoint. Despite of the computational advantage and simplicity that the least squares method can provide, applying this method to the analysis of practical structural problems with mixed formulation is not easy, especially in the case of the structural problems which include geometric nonlinearity. For example, in structural mechanics problems, the displacements are numerically very small compared to forces or stresses. Thus, squares of the residuals of equations involving force like variables are very large compared to those containing displacements in the least-squares functional. To handle these magnitude differences, variables can be properly nondimensionalized or residuals themselves should be multiplied by proper scale factors. The latter approach was adopted in the present paper.

Both the least-squares and weak-form Galerkin methods may have their own merits and demerits. The purpose of this paper is to provide some insights and proper strategies on the development of nonlinear finite element models in the structural problems using the two different formulations and to compare the computational aspects of various finite element models of Timoshenko beam theory with the von Kármán nonlinearity.

2 GOVERNING EQUATIONS

To develop new mixed finite models and verify their performance, the Timoshenko beam theory (TBT) with the von Kármán nonlinearity was considered for. First, the governing equations are reviewed briefly (see Reddy [1-4] for details). In the TBT, the displacement field is adopted as to include the transverse shear deformation (strain) in the simplest way, as shown in Fig. 1. The components $u_1$, $u_2$ and $u_3$ of the displacement field are

$$
    u_1 = u_0(x) + z\phi(x), \quad u_2 = 0, \quad u_3 = w_0(x)
$$

(1)

where the $u_0$ is axial displacement of centroid line of cross section along the $x$-axis, the $w_0$ is vertical displacement of the center line along the $z$-axis, and $z$ is a vertical distance between the centroid line and arbitrary point of the beam.

The von Kármán nonlinear strains of the TBT are

![Fig. 1](image-url)

Fig. 1

Description of undeformed and deformed edges of the TBT beam, source from.
The constitutive relations are

\[ \sigma_{xx} = E\varepsilon_{xx}, \quad \sigma_{z} = 2G\varepsilon_{z} \]

where \( E \) is Young’s modulus and \( G \) is shear modulus. The stress resultants are

\[ N_{xx} = \int_A \sigma_{xx} \, dy \, dz = EA [u_{0}' + 0.5 (w_{0}')^2] \]
\[ Q_x = \int_A \sigma_z \, dy \, dz = K \, G \, A \, (\phi_{x} + w_{0}') \]
\[ M_{xx} = \int_A z \sigma_{xx} \, dy \, dz = EI\phi_{x}' \]

where the \( K \) (5/6) is shear correction factor, the \( N_{xx} \) is axial force, the \( V_x \) is the transverse shear force, and \( M_{xx} \) is bending moment at any cross section of the beam. The equilibrium equations of the TBT are

\[ N_{xx}' + f(x) = 0, \quad (Q_x + w_0'N_{xx})' + q(x) = 0, \quad M_{xx}' - Q_x = 0 \]

where \( f(x) \) and \( q(x) \) are distributed axial force and transverse loads, respectively. The resultant forces and bending moment can be included in the finite element models to develop mixed finite element models, which will be discussed in the next section.

### 3 FINITE ELEMENT MODELS

Traditionally, the three equilibrium equations in (5) are written in terms of the generalized displacements \((u_0, w_0, \phi_x)\) to develop the displacement finite element model of the TBT. Here, we discuss weighted-residual finite element models of these equations. Since we approximate the variables of the formulation, the differential equations in (4) and (5) are not satisfied exactly everywhere in the domain. Thus, there is error introduced into the differential equations, called residuals. All approximation methods try to minimize the residuals in some suitable sense. For example, consider the three equations in Eq. (5), expressed in terms of the displacements. The residuals in the three equations are

\[ R_1 = -[EA [u_{0}' + 0.5 (w_{0}')^2]] - f(x) \neq 0 \]
\[ R_2 = -[K \, G \, A \, (\phi_{x} + w_{0}') + w_{0}'EA [u_{0}' + 0.5 (w_{0}')^2]]' - q(x) \neq 0 \]
\[ R_3 = -EI\phi_{x}'' + K \, G \, A \, (\phi_{x} + w_{0}') \neq 0 \]

The resulting weighted-residual or weak-form model will contain only the displacement variables. On the other hand, one may also consider Eqs. (4) and (5) to be six independent equations and treat three displacements \((u_0, w_0, \phi_x)\) and there stress resultants \((N_{xx}, V_x, M_{xx})\) as independent variables. The resulting finite element model is termed a mixed model of the TBT because it contains displacements as well as stress resultants:

\[ R_1 = N_{xx}' + f(x) \neq 0, \]
\[ R_2 = (Q_x + w_0'N_{xx})' + q(x) \neq 0, \]
\[ R_3 = M_{xx}' - Q_x \neq 0, \]
\[ R_4 = (K \, G \, A)^{-1}Q_x - (\phi_{x} + w_{0}') \neq 0, \]
\[ R_5 = (EA)^{-1}N_{xx} - (u_{0}' + 0.5w_0'w_{0}') \neq 0, \]
\[ R_6 = (EI)^{-1}M_{xx} - \phi_{x}' \neq 0. \]
An advantage of mixed formulations is that they yield increased accuracy of the resultants and they can be computed at the nodes as opposed to the Gauss points in displacement finite element models. As a special case of the weighted-residual method, one may construct least-squares finite element models. In Eq. (7), one can find that the differentiability requirements on \( u_0, w_0 \) and \( \phi_x \) are lowered by inclusion of the stress resultants \( N_{xx}, V_x \) and \( M_{xx} \).

### 3.1 Traditional displacement based Galerkin weak-form model

In this section, a traditional displacement based Galerkin model is reviewed to provide better comparison of the characteristics and behavior of the newly developed mixed models herein. The traditional displacement based model includes only \( u_0, w_0 \) and \( \phi_x \) as nodal variables, yields the smallest stiffness matrix. The weak formulation of the traditional displacement model is based on the statements \([1, 3]\)

\[
\begin{align*}
\int_0^L [E A \delta u_0' [u_0' + 0.5(w_0')^2] - f(x) \delta u_0] \, dx - [\delta u_0 Q_1]\bigg|_0^L & = 0 \\
\int_0^L [K_c G A \delta w_0' \phi_x + w_0' E A \delta w_0' [u_0' + 0.5(w_0')^2] - q(x) \delta w_0] \, dx - [\delta w_0 Q_2]\bigg|_0^L & = 0 \\
\int_0^L [E I \delta \phi_x' \phi_x' + K_c G A \delta \phi_x' (\phi_x + w_0')] \, dx - [\delta \phi_x Q_3]\bigg|_0^L & = 0
\end{align*}
\]

where \( Q_1 = N_{xx}, Q_2 = Q_3 + w_0' N_{xx} = V_x \), and \( Q_3 = M_{xx} \) at the boundaries. In the traditional displacement models, boundary conditions can be imposed exactly only on the displacements and in integral sense on the stress resultants. However, mixed models allow exact imposition of boundary conditions even on the stress resultants that appear in the model. The displacements appearing in Eq. (8) can be approximated by

\[
\begin{align*}
u_0 & \approx \sum_{j=1}^p \psi_j u_j \\
w_0 & \approx \sum_{j=1}^m \psi_j w_j \\
\phi_x & \approx \sum_{j=1}^n \psi_j \phi_j
\end{align*}
\]

and substitution into Eq. (8) yields the following finite element equations:

\[
[K_T ([U_T]))[U_T] = [F_T] \text{ or } \begin{bmatrix}
[K^{11}] & [K^{12}] & [K^{13}] \\
[K^{21}] & [K^{22}] & [K^{23}] \\
[K^{31}] & [K^{32}] & [K^{33}]
\end{bmatrix}
\begin{bmatrix}
[u_j] \\
[w_j] \\
[\phi_j]
\end{bmatrix} =
\begin{bmatrix}
[F^1] \\
[F^2] \\
[F^3]
\end{bmatrix}
\]

\[
(K^{ij}) = \int_0^L (E A \psi'_i \psi'_j) \, dx, \quad K^{23} = \int_0^L (K_c G A \psi'_i \psi'_j) \, dx, \quad K^{32} = \int_0^L (E I \psi'_i \psi'_j + K_c G A \psi'_j) \, dx, \quad K^{33} = K^{23}
\]

\[
f_i = \int_0^L [f(x) \psi_i] \, dx, \quad f^i = \int_0^L [q(x) \psi_i] \, dx + [\psi_i Q_1]\bigg|_0^L \quad \text{and} \quad f^3 = [\psi_i Q_2]\bigg|_0^L
\]
3.2 Mixed Galerkin model

For a mixed finite element model, the principle of minimum potential energy and the Lagrange multiplier method can be employed simultaneously to minimize the total potential energy and include the stress resultant-displacement relations as constrains through the Lagrange multiplier method. For the problem at hand, the total potential energy functional of the TBT beam can be expressed as

\[
L_c = \int_0^{L_e} \left[ 0.5(\sigma_x e_x + \sigma_z e_z) - fu_0 - qw_0 \right] dx
\]

where the \( L_c \) is the potential energy functional. Then minimizing \( L_c \) subject to the constraints can be expressed as

\[
L_l(u_0, w_0, \phi_s, N_x, Q_z, M_s) = L_c + \sum_{i=1}^{3} \left( \int_0^{L_e} \lambda_i G_i \ dx \right)
\]

\[
= \int_0^{L_e} \left[ 0.5(EA)^{-1} N_x^2 + 0.5(EI)^{-1} M_s^2 + 0.5(K_s GA)^{-1} Q_z^2 \right. \\
+ \left. \int_0^{L_e} \lambda_1 \left( (EA)^{-1} N_x - (u'_0 + 0.5w'_0 w_0) \right) dx + \int_0^{L_e} \lambda_2 \left( (K_s GA)^{-1} Q_z - (\phi_s + w'_0) \right) dx \right. \\
+ \left. \int_0^{L_e} \lambda_3 \left( (EI)^{-1} M_s - \phi'_s \right) dx \right)
\]

where \( L_l \) is the mixed functional and \( \lambda_i \) are the Lagrange multipliers. The condition for the minimum of \( L_l \) is \( \delta L_l = 0 \), where the \( \delta \) is variational operator. Then we can obtain the following relations, which can be used to develop the mixed Galerkin finite element model. Since \( L_l \) contains 6 variables and their first derivatives only, coefficients of variations of the variables can be computed from the Euler-Lagrange equations [2] as follows:

\[
\delta u_0 : \frac{\partial L_l}{\partial u_0} - \frac{\partial}{\partial x} \left( \frac{\partial L_l}{\partial u'_0} \right) = \int_0^{L_e} \left( -f + \lambda'_0 \right) dx = 0
\]

\[
\delta w_0 : \frac{\partial L_l}{\partial w_0} - \frac{\partial}{\partial x} \left( \frac{\partial L_l}{\partial w'_0} \right) = \int_0^{L_e} \left[ -q - \lambda'_2 - (\lambda'_1 w'_0) \right] dx = 0
\]

\[
\delta \phi_s : \frac{\partial L_l}{\partial \phi'_s} - \frac{\partial}{\partial x} \left( \frac{\partial L_l}{\partial \phi''_s} \right) = \int_0^{L_e} \left( \lambda_3 - \lambda'_{1s} \right) dx = 0
\]

\[
\delta N_{xs} : \frac{\partial L_l}{\partial N_{xs}} = \int_0^{L_e} \left[ (EA)^{-1} N_{xs} - (EA)^{-1} \lambda_1 \right] dx = 0
\]

\[
\delta Q_z : \frac{\partial L_l}{\partial Q_z} = \int_0^{L_e} \left[ (K_s GA)^{-1} Q_z - (K_s GA)^{-1} \lambda_2 \right] dx = 0
\]

\[
\delta M_{sx} : \frac{\partial L_l}{\partial M_{sx}} = \int_0^{L_e} \left[ (EI)^{-1} M_{sx} - (EI)^{-1} \lambda_3 \right] dx = 0
\]

The Lagrange multipliers can be chosen to be \( \lambda_1 = N_s \), \( \lambda_2 = Q_s \), and \( \lambda_3 = M_s \) satisfying all conditions given in Eq. (7). By using the Lagrange multiplier method, all governing equations of the TBT beam can be fully recovered. Thus, we can develop mixed Galerkin finite element model with properly chosen weight functions for the residuals. We have

\[
\delta L_l(u_0, w_0, \phi'_s, N_x, Q_z, M_s) = \int_0^{L_e} \left[ \delta u_0 (-f - N'_s) + \delta w_0 \left[ -q - (Q_z + N_s w'_0) \right] \right. \\\n+ \left. \delta N_{xs} \left[ (EA)^{-1} N_{xs} - (u'_0 + 0.5w'_0 w_0) \right] \right. \\\n+ \left. \delta Q_z \left[ (K_s GA)^{-1} Q_z - (\phi_s + w'_0) \right] \right. \\\n+ \left. \delta M_{sx} \left[ (EI) \phi'_s - M'_s \right] \right] dx = 0
\]
Note that following relation [3] can be used to lower differentiability requirement for the \( w_0 \) simplifying force vector terms in the matrix form of finite element equations.

\[
V_s = Q_s + w_0^* N_{as}, \quad Q_s = V_s - w^* N_{as}
\]  

(15)

In the least squares method, which will be discussed in the next section, use of the relation Eq. (15) will allow us to have only first-order differential equation, and to construct simple element-wise force vectors that contain linear terms only. In Eq. (15), \( V_s \) is the shear resultant on the undeformed edge, and \( Q_s \) is the shear force on the deformed edge. We write

\[
\delta L_i(u_0, w_0, \phi, N_s, V_s, M_s)
\]

\[
= \int_0^L [\delta u_0 [N' + f(x)] + \delta w_0 [V' + q(x)] + \delta \phi (M' - V_s + w_0^* N_{as})] + \delta N_{as} [(EA)^{-1} N_{as} - (u_0' + 0.5w_0^* w_0^*)] + (\delta V_s - w_0^* \delta N_{as}) [(K, GA)^{-1} (V_s - w_0^* N_{as}) - (\phi + w_0^*)] + \delta M_{as} [(EI)^{-1} M_{as} - \phi'] \, dx = 0
\]  

(16)

Then the variables and their variations in Eq. (16) can be approximated as linear combinations of nodal values and known interpolation functions, as

\[
u_i \approx \sum_{j=1}^n \psi_i \psi_j, \quad w_0 \approx \sum_{j=1}^n \psi_i \phi_j, \quad \phi \approx \sum_{j=1}^n \psi_i \phi_j, \quad N_{as} \approx \sum_{j=1}^n \psi_i N_{ij}, \quad \approx \sum_{j=1}^n \psi_i V_j,
\]

\[
V_s \approx \sum_{i=1}^n \psi_i V_i, \quad M_{as} \approx \sum_{j=1}^n \psi_i M_j
\]  

(17)

where \( u, w, \phi, N, Q \) and \( M \) are nodal unknowns of the finite element model, the \( \psi_i \) and the \( \psi_j \) are \( i^{th} \) and \( j^{th} \) are Lagrange type interpolation functions. By substituting all the approximations given in Eq. (17) into the function \( \delta L_i \) of Eq. (16) and collecting coefficients of the variation of the nodal unknowns (i.e. \( \delta u, \delta w, \delta \phi, \delta N, \delta Q \) and \( \delta M \)), the following finite element equations can be obtained:

\[
[K_G \{U_G\}] \{U_G\} = \{F_G\}
\]

where \( K_G \) is element stiffness matrix of the mixed Galerkin model, \( U_G \) is the column vector of unknowns, and \( F_G \) is the force vector. Definitions of all nonzero stiffness and force coefficients are given below.

\[
K_{ij}^{41} = \int_0^L (\psi_i \psi_j') \, dx, \quad K_{ij}^{25} = \int_0^L (\psi_i \psi_j) \, dx, \quad K_{ij}^{36} = \int_0^L (\psi_i \psi_j') \, dx,
\]

\[
K_{ij}^{25} = \int_0^L (-\psi_i \psi_j) \, dx, \quad K_{ij}^{44} = \int_0^L \left( (E)A^{-1} \psi_i \psi_j \right) \, dx,
\]

\[
K_{ij}^{31} = \int_0^L (-\psi_i \psi_j') \, dx, \quad K_{ij}^{32} = \int_0^L (-0.5w_0^* \psi_i \psi_j') \, dx, \quad K_{ij}^{43} = K_{ji}^{34},
\]

\[
K_{ij}^{34} = \int_0^L [(K, GA)^{-1} w_0^* \psi_i \psi_j] \, dx, \quad K_{ij}^{45} = \int_0^L \left( -(K, GA)^{-1} w_0^* \psi_i \psi_j \right) \, dx,
\]
A Comparative Study of Least-Squares and the Weak-Form Galerkin Finite Element Models for ...
We can minimize the above least-squares equation, which includes nonlinear terms, by taking first variation of it and setting it to zero:

\[
0 = \delta I(u, w, \phi, N, V, M) = \int_{\Omega} (\delta L + \delta N) \, dx
\]

\[
= \int_{\Omega} (\delta L + \sigma_1^2 \delta R_1 + \sigma_2^2 \delta R_2 + \sigma_3^2 \delta R_3) \, dx
\]

\[
= \int_{\Omega} (\delta L + \sigma_1^2 (\delta M'x - \delta V_x + \delta w_0'N_x) + \delta w_0'N_x)((M'x - V_x + \delta w_0'N_x)(E) - V_x + \delta w_0'N_x) - (u_0' + 0.5w_0')[(E) - V_x + \delta w_0'][(E) - V_x + \delta w_0'])
\]

\[
+ \sigma_1^2 [(K, GA)^{-1} (\delta V_x - \delta w_0'N_x) - (\delta \phi + \delta w_0')] [(K, GA)^{-1} (V_x - \delta w_0'N_x) - (\delta \phi + \delta w_0')] \, dx
\]

But if we linearize \( I \) (i.e., treat \( w_0' \) as known from the previous iteration) before taking its variation, then we have

\[
\tilde{I}(u, w, \phi, N, V, M) = \int_{\Omega} (\delta L + \tilde{\delta} N) \, dx
\]

\[
= \int_{\Omega} (\delta L + + \sigma_1^2 (\delta M'x - \delta V_x + \delta w_0'N_x) + \delta w_0'N_x)((M'x - V_x + \delta w_0'N_x) - (u_0' + 0.5w_0'))
\]

\[
+ \sigma_1^2 [(K, GA)^{-1} (\delta V_x - \delta w_0'N_x) - (\delta \phi + \delta w_0')] [(K, GA)^{-1} (V_x - \delta w_0'N_x) - (\delta \phi + \delta w_0')] \, dx
\]

where quantities with bars indicate that they are linearized values. Then \( \delta \tilde{I} = 0 \) is not the same condition as \( \delta I = 0 \) unless

\[
\sigma_1^2 \delta w_0'N_x \left( M'x - V_x + \delta w_0'N_x \right) + 0.5\sigma_1^2 \delta w_0'[(E) - V_x + \delta w_0']
\]

\[
= \sigma_1^2 (K, GA)^{-1} \delta w_0'N_x [(K, GA)^{-1} (V_x - \delta w_0'N_x) - (\delta \phi + \delta w_0')] = 0
\]

Since the sufficient condition for the minimum of \( I \) is \( \delta^2 I > 0 \),

\[
\delta^2 I = \int_{\Omega} \left( \delta L + \sigma_1^2 (\delta M'x - \delta V_x + \delta w_0'N_x) + \delta w_0'N_x)(\delta M'x - \delta V_x + \delta w_0'N_x) - (u_0' + 0.5w_0')[(E) - V_x + \delta w_0']
\]

\[
+ \sigma_1^2 [(K, GA)^{-1} (\delta V_x - \delta w_0'N_x) - (\delta \phi + \delta w_0')] [(K, GA)^{-1} (V_x - \delta w_0'N_x) - (\delta \phi + \delta w_0')] \, dx > 0
\]

Thus, in order for \( \delta^2 I > 0 \), we must have

\[
\sigma_1^2 \delta w_0'N_x \left( M'x - V_x + \delta w_0'N_x \right) + 0.5\sigma_1^2 \delta w_0'[(E) - V_x + \delta w_0']
\]

\[
- \sigma_1^2 (K, GA)^{-1} \delta w_0'N_x [(K, GA)^{-1} (V_x - \delta w_0'N_x) - (\delta \phi + \delta w_0')] \geq 0
\]
\[ 0 = \int_{0}^{L} \left[ \sigma_{v}^{2}(\delta N_{s}) N_{s} f(x) + f(x) \right] + \sigma_{v}^{2}(\delta V_{s}) V_{s} + q(x) \right] + \sigma_{v}^{2}(\delta M_{s}) M_{s} + w_{0}^{2}(\delta N_{s}) (M_{s} - V_{s} + w_{0} N_{s}) + \sigma_{v}^{2}(\delta V_{s}) (\delta V_{s}) + w_{0}^{2}(\delta N_{s}) (M_{s} - V_{s} + w_{0} N_{s}) \]
\[ + \sigma_{v}^{2}(\delta M_{s}) (\delta M_{s}) + w_{0}^{2}(\delta N_{s}) (M_{s} - V_{s} + w_{0} N_{s}) \]
\[ + \sigma_{v}^{2}[(EA)^{(s)}(\delta V_{s}) + w_{0}^{2}(\delta N_{s})] + (EA)^{-1} N_{s} - (w_{0} + 0.5 w_{0} N_{s}) \]
\[ + \sigma_{v}^{2}[(KGA)^{(s)}(\delta V_{s}) - w_{0}^{2}(\delta N_{s})] + (KGA)^{-1} (V_{s} - w_{0} N_{s}) - (\phi_{s} + w_{0}) \]
\[ + \sigma_{v}^{2}[(EI)^{(s)}(\delta M_{s}) + (\delta M_{s})] + (EI)^{-1} M_{s} - \phi_{s} \]  
\[ \text{(29)} \]

Note that all bars are omitted for consistency with the Galerkin mixed model. Then the variables involved in Eq. (29) can be approximated and replaced by using Eq. (17) which are the same functions used in the Galerkin model. By using matrix notation, above equations can be rewritten simply as

\[ [K_{L}]([U_{L}]) = [F_{L}] \]

or

\[ \begin{bmatrix} [K_{11}] & [K_{12}] & [K_{13}] & [K_{14}] & [K_{15}] & [K_{16}] \\ [K_{21}] & [K_{22}] & [K_{23}] & [K_{24}] & [K_{25}] & [K_{26}] \\ [K_{31}] & [K_{32}] & [K_{33}] & [K_{34}] & [K_{35}] & [K_{36}] \\ [K_{41}] & [K_{42}] & [K_{43}] & [K_{44}] & [K_{45}] & [K_{46}] \\ [K_{51}] & [K_{52}] & [K_{53}] & [K_{54}] & [K_{55}] & [K_{56}] \\ [K_{61}] & [K_{62}] & [K_{63}] & [K_{64}] & [K_{65}] & [K_{66}] \end{bmatrix} \begin{bmatrix} [u_{1}] \\ [u_{2}] \\ [u_{3}] \\ [u_{4}] \\ [u_{5}] \\ [u_{6}] \end{bmatrix} = \begin{bmatrix} [F_{1}] \\ [F_{2}] \\ [F_{3}] \\ [F_{4}] \\ [F_{5}] \\ [F_{6}] \end{bmatrix} \]

(30)

where \( K_{L} \) is element stiffness matrix of the mixed least-squares model, \( U_{L} \) is the vector of unknowns, and \( F_{L} \) is the force vector. The nonzero coefficients of Eq. (30) are defined as

\[ K_{ij}^{1} = \int_{0}^{L} \sigma_{v}^{2}(\psi_{i}\psi_{j}) \, dx, \quad K_{ij}^{12} = \int_{0}^{L} 0.5 \sigma_{v}^{2}(w_{o}^{2}(\psi_{i}\psi_{j}) + \sigma_{v}^{2}(\psi_{i}\psi_{j})) \, dx, \quad K_{ij}^{14} = \int_{0}^{L} -\sigma_{v}^{2}(EA)^{-1}(\psi_{i}\psi_{j}) \, dx, \]
\[ K_{ij}^{22} = \int_{0}^{L} 0.25 \sigma_{v}^{2}(w_{o}^{2}(\psi_{i}\psi_{j}) + \sigma_{v}^{2}(\psi_{i}\psi_{j})) \, dx, \quad K_{ij}^{23} = \int_{0}^{L} \sigma_{v}^{2}(\psi_{i}\psi_{j}) \, dx, \]
\[ K_{ij}^{24} = \int_{0}^{L} -0.5 \sigma_{v}^{2}(EA)^{-1}(w_{o}(\psi_{i}\psi_{j}) + \sigma_{v}^{2}(KGA)^{-1}(w_{o}(\psi_{i}\psi_{j})) \, dx, \quad K_{ij}^{25} = \int_{0}^{L} -\sigma_{v}^{2}(KGA)^{-1}(\psi_{i}\psi_{j}) \, dx, \]
\[ K_{ij}^{26} = \int_{0}^{L} -\sigma_{v}^{2}(KGA)^{-1}(\psi_{i}\psi_{j}) \, dx, \quad K_{ij}^{36} = \int_{0}^{L} -\sigma_{v}^{2}(EI)^{-1}(\psi_{i}\psi_{j}) \, dx, \]
\[ K_{ij}^{31} = \int_{0}^{L} -0.5 \sigma_{v}^{2}(w_{o}(\psi_{i}\psi_{j}) + \sigma_{v}^{2}(KGA)^{-1}(w_{o}(\psi_{i}\psi_{j})) \, dx, \quad K_{ij}^{34} = \int_{0}^{L} -\sigma_{v}^{2}(KGA)^{-1}(w_{o}(\psi_{i}\psi_{j}) \, dx, \quad K_{ij}^{35} = \int_{0}^{L} \sigma_{v}^{2}(KGA)^{-2}(w_{o}(\psi_{i}\psi_{j}) \, dx, \quad K_{ij}^{36} = \int_{0}^{L} \sigma_{v}^{2}(KGA)^{-2}(\psi_{i}\psi_{j}) \, dx, \]
\[ K_{ij}^{43} = \int_{0}^{L} \sigma_{v}^{2}(\psi_{i}\psi_{j}) + \sigma_{v}^{2}(w_{o}^{2}(\psi_{i}\psi_{j}) + \sigma_{v}^{2}(\psi_{i}\psi_{j})) \, dx, \quad K_{ij}^{45} = \int_{0}^{L} \sigma_{v}^{2}(\psi_{i}\psi_{j}) + \sigma_{v}^{2}(\psi_{i}\psi_{j})) \, dx, \quad K_{ij}^{46} = \int_{0}^{L} \sigma_{v}^{2}(\psi_{i}\psi_{j}) \, dx, \]
\[ K_{ij}^{53} = \int_{0}^{L} \sigma_{v}^{2}(\psi_{i}\psi_{j}) + \sigma_{v}^{2}(\psi_{i}\psi_{j}) + \sigma_{v}^{2}(KGA)^{-2}(\psi_{i}\psi_{j}) \, dx, \quad K_{ij}^{56} = \int_{0}^{L} \sigma_{v}^{2}(\psi_{i}\psi_{j}) \, dx, \quad K_{ij}^{54} = \int_{0}^{L} \sigma_{v}^{2}(\psi_{i}\psi_{j}) \, dx, \]
\[ K_{ij}^{63} = \int_{0}^{L} \sigma_{v}^{2}(\psi_{i}\psi_{j}) + \sigma_{v}^{2}(\psi_{i}\psi_{j}) + \sigma_{v}^{2}(KGA)^{-2}(\psi_{i}\psi_{j}) \, dx, \quad K_{ij}^{66} = \int_{0}^{L} \sigma_{v}^{2}(\psi_{i}\psi_{j}) \, dx, \quad K_{ij}^{65} = \int_{0}^{L} \sigma_{v}^{2}(\psi_{i}\psi_{j}) \, dx, \]
\[ f_{i}^{s} = \int_{0}^{L} \sigma_{v}^{2}(f(x)\psi_{i}) \, dx \quad \text{and} \quad f_{i}^{s} = \int_{0}^{L} \sigma_{v}^{2}(q(x)\psi_{i}) \, dx \]

(31)

Comparing Eqs. (30) and (31) with Eqs. (18) and (19), it can be seen that the \( K_{L} \) is symmetric while the \( K_{G} \) is not. In Eqs. (30) and (31), force vector contains only linear terms, by replacing \( Q_{x} \), with \( V_{s} + w_{0} N_{s} \).
4 NUMERICAL RESULTS

4.1 Description of problem

A steel beam with the geometry shown in Fig. 2 was chosen for the study. The properties given in Eq. (32) were used, with uniform applied distributed load \( q(x) \), which was set to vary from 1.0 to 10.0 lb/in with ten steps to archive proper nonlinear convergence.

\[
E = 30 \times 10^6 \text{ psi, } \nu = 0.25, \quad K = 5/6, \quad f(x) = 0, \quad q(x) = 1.0 \sim 10.0 \text{ lb/in}
\] (32)

Three different types of boundary conditions shown in Fig. 3, i.e. H-H (hinged-hinged), P-P (Pined-Pined) and C-C (clamped-clamped), were considered to evaluate the performance of the elements. All boundary conditions allow modeling half of the beam.

The C-C and P-P boundary conditions can be used to test the nonlinear behavior of newly developed beam finite elements, and the H-H boundary condition can be used to determine membrane locking is experienced [10]. Since the beam has no horizontal support under the H-H boundary condition, \( u_0' + 0.5(w_0')^2 = 0 \) should be satisfied to have \( N_{xx} = 0 \). In the traditional displacement model, this condition is barely satisfied because of the inconsistency in polynomial orders between the \( u_0' \) and the \( (w_0')^2 \) terms. Also, the inconsistency in polynomial orders between the \( \phi_1 \) and the \( w_1^i \) in \( KGA(\phi_1 + w_1^i) \) may cause shear locking [3, 10] which results in inaccurate linear solution. In displacement based models, performance of beam element is mostly depends on the relations of the displacements.

![Fig. 2](image1.png)

**Fig. 2**
Geometry of beam and chosen coordinate system for the numerical analysis [source from 3].

![Fig. 3](image2.png)

**Fig. 3**
Description of symmetry boundary conditions.
and their derivatives, thus they experience severe locking compared with mixed models. Some techniques like reduced integration and use of consistency approximations of the displacement field may be employed, but it is not easy (but possible) to implement them into computers. On the other hand, the locking can be weakened or even eliminated by the inclusion of resultants which can be done by mixed formulations, because certain conditions which cause locking are not solely depend on the relations of displacement but also on the resultant in mixed models. The numerical results that show the effects of mixed formulation are presented in the next sections.

4.2 Linear analysis

In this section linear solutions of the newly developed mixed models are compared with known exact solutions and solutions of traditional displacement based model. By taking all nonlinear terms to be zero, we can eliminate geometric nonlinearity in the beam element to study linear behavior of beam element. Linear study is important because finding nonlinear solutions can be started from linear solutions in most of structural problems. Linear study of the TBT beam bending includes interesting phenomena like shear locking. For shear locking, accuracy level of the solutions can be possibly calculated by using known exact solutions. As mentioned in previous section, displacement based models experience severe shear locking even with the use of higher interpolations. But present models developed with mixed formulations showed less locking, especially for the mixed Galerkin model, if higher order of interpolation functions were used. The solutions of the newly developed models are presented in the Table 1, to investigate shear locking. With the results presented in the Table 1, we can find that accuracy level of the solution does depend on methods and formulations adopted. The mixed Galerkin model showed the best accuracy in finite element solutions obtained under both C-C and P-P boundary conditions, while traditional displacement models showed the most inaccurate results. With the results presented in the Table 1, errors which mean degree of shear locking are compared in the Fig. 3.

The degree of shear locking is measured using the definition

\[
\text{Degree of shear locking} = \frac{|w_{\text{exact}} - w_{\text{ld}}|}{w_{\text{exact}}} \tag{33}
\]

| Table 1 | Linear results of beam obtained with 4-element mesh under C-C boundary condition, \(q(x) = 1.0 \text{ lb/in}, \) with full integration |
| Interpolation order | Center deflection of the beam (\(w_0\)), in C-C boundary condition (exact: \(0.104291667\)) | P-P boundary condition (exact: \(0.520958333\)) |
| | Mixed | Mixed | Traditional | Mixed | Mixed | Traditional |
| | Galerkin | Least-squares | displacement | Galerkin | Least-squares | displacement |
| 1st | 0.112972222 | 0.096004271 | 0.001964678 | 0.526745370 | 0.497652359 | 0.009691326 |
| 2nd | 0.105593750 | 0.104283529 | 0.098352041 | 0.522000000 | 0.520938133 | 0.515034294 |
| 3rd | 0.104291667 | 0.104296375 | 0.104291651 | 0.520958335 | 0.521078238 | 0.520958540 |
| 4th | 0.104291667 | 0.104273233 | 0.104291612 | 0.520958333 | 0.520958178 | 0.520958178 |
| 5th | 0.104291667 | 0.104270593 | 0.104291670 | 0.520958333 | 0.520958178 | 0.520959431 |
| 6th | 0.104291668 | 0.104363783 | 0.104291730 | 0.520958337 | 0.522800451 | 0.520958056 |

| Table 2 | Linear results of beam obtained with 1-element mesh under C-C boundary condition, \(q(x) = 1.0 \text{ lb/in}, \) with full integration |
| Interpolation order | Center deflection of the beam (\(w_0\)), in C-C boundary condition (exact: \(0.104291667\)) | P-P boundary condition (exact: \(0.520958333\)) |
| | Mixed | Mixed | Traditional | Mixed | Mixed | Traditional |
| | Galerkin | Least-squares | displacement | Galerkin | Least-squares | Displacement |
| 1st | 0.139013889 | 0.072895696 | 0.000498209 | 0.544106482 | 0.433308428 | 0.002364252 |
| 2nd | 0.109500000 | 0.104161909 | 0.078860218 | 0.525125000 | 0.520637115 | 0.495521997 |
| 3rd | 0.104291667 | 0.104291651 | 0.104291468 | 0.520958334 | 0.520958051 | 0.520951347 |
| 4th | 0.104291667 | 0.104290570 | 0.104291646 | 0.520958333 | 0.520958341 | 0.520926382 |
| 5th | 0.104291665 | 0.104283217 | 0.104291663 | 0.520958326 | 0.520958009 | 0.520744177 |
| 6th | 0.104291671 | 0.104291639 | 0.104299478 | 0.520958357 | 0.520957624 | 0.521167093 |

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where \( w_{sl} \) is center vertical deflection of beam which is obtained by linear finite element analysis with full integration, and \( w_{exact} \) is mathematically exact solution. Since \( w_{sl} \) is obtained with full integration, it will contain certain degree of shear locking caused by inconsistent approximation of the variables. Thus, degree of shear locking can be determined from Eq. (33).

### 4.3 Nonlinear analysis

Under C-C boundary condition, each model showed similar convergence, as shown in the Fig. 4a, while the converged solutions of the current mixed Galerkin and the least-squares models are more accurate than those of the displacement model (see Fig. 4b). The converged solutions of the displacement model close to the solutions predicted by the current mixed models when proper reduced integration techniques or consistent approximations of variables are used.

Converged solutions are presented in Table 3. The direct iterative method was used to get the solutions. Two mixed models showed good result with full integrations, while the traditional displacement model showed some degree of locking. Normally, locking of element is sever in finite element solution of lower interpolations, while it is likely to disappear in higher interpolations. Two mixed models showed closest converged solutions, while the
displacement based model did not as presented in the Fig. 5b. As mentioned before, membrane locking [3] is caused by the use of inconsistent order of interpolation of the terms like \( u'_0 + 0.5(w'_0)^2 = (EA)^{-1} N_{xx} \). In particular, when the beam undergoes no extensional deformation, it should produce \( u'_0 + 0.5(w'_0)^2 = (EA)^{-1} N_{xx} \simeq 0 \).

(a) Load verses center deflection under C-C boundary condition  
(b) Normalized difference of converged solutions respect to that of the mixed Galerkin model

Fig. 5  
Nonlinear behavior of beams under C-C boundary condition, with 4 cubic elements.

(a) Load verses center deflection  
(b) Estimated degree of membrane locking

Fig. 6  
Nonlinear behavior of beams under H-H boundary condition, with 4 cubic elements.
This is satisfied only when the axial displacement and transverse displacements are interpolated such a way that the membrane strain has the possibility of becoming zero. Thus, the beam element may behave in a manner physically unrealistic. Traditionally, this phenomenon is overcome using reduced integration of the nonlinear stiffness coefficients. In the case of the mixed model presented herein, membrane locking completely disappears. As in Eq. (33), the degree of membrane locking can be measured as

\[
\text{Degree of membrane locking} = \frac{|w_{\text{exact}} - w_{\text{ml}}|}{w_{\text{exact}}}
\]  

(34)

where \(w_{\text{ml}}\) is converged center vertical deflection of the beam obtained with full integration, and \(w_{\text{exact}}\) is mathematically exact solution of the same. Judging from the nonlinear results obtained, both models showed good nonlinear convergence while the mixed Galerkin model showed least degree of membrane locking.

5 CONCLUSIONS

Developing procedures of nonlinear beam bending finite element models were presented with two different methods. All element wise coefficient matrices and force vectors are also presented. Mixed formulation provided superior accuracy in linear solutions and better performance in nonlinear analysis with the use of same order interpolations and full integrations. Two types of locking phenomena were discussed and current mixed models showed less locking compared with traditional displacement based model.

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