Common fixed point theorems of contractive mappings sequence in partially ordered G-metric spaces

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Abstract

We consider the concept of Ω-distance on a complete partially ordered G-metric space and prove some common fixed point theorems.

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1 Introduction

The Banach fixed point theorem for contraction mapping has been generalized and extended in many direction [1-15]. Nieto and Rodriguez-

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Lopez [16], Ran and Reurings [17] and Petrusel and Rus [18] presented some new results for contractions in partially ordered metric spaces. The main idea in [12,16,17] involves combining the ideas of an iterative technique in the contraction mapping principle with those in the monotone technique. Also, Mustafa and Sims [19] introduced the concept of G-metric. Some authors [20-24] have proved some fixed point theorems in these spaces. In [25] Gajić proved a common fixed point theorem for a sequence of mappings on this space. Recently, Saadati et al. [26], using the concept of G-metric, defined an Ω-distance on complete G-metric space and generalized the concept of ω-distance due to Kada et al. [27]. In [28,29] some fixed point theorems proved and generalized under this concept.

In this paper, we extend some fixed point theorems by using this concept in partially ordered G-metric spaces.

At first we recall some definitions and lemmas. For more information see [19,26].

**Definition 1.1** [19] Let $X$ be a non-empty set. A function $G : X \times X \times X \rightarrow [0, \infty)$ is called a G-metric if the following conditions are satisfied:

(i) $G(x, y, z) = 0$ if $x = y = z$ (coincidence),
(ii) $G(x, x, y) > 0$ for all $x, y \in X$, where $x \neq y$,
(iii) $G(x, x, z) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$,
(iv) $G(x, y, z) = G(p\{x, y, z\})$, where $p$ is a permutation of $x, y, z$ (symmetry),
(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

A G-metric is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

**Definition 1.2** [19] Let $(X, G)$ be a G-metric space,

(1) a sequence $\{x_n\}$ in $X$ is said to be G-Cauchy sequence if, for each $\varepsilon > 0$, there exists a positive integer $n_0$ such that for all
\(m, n, l \geq n_0, G(x_n, x_m, x_l) < \varepsilon\).

(2) A sequence \(\{x_n\}\) in \(X\) is said to be \(G\)-convergent to a point \(x \in X\) if, for each \(\varepsilon > 0\), there exists a positive integer \(n_0\) such that for all \(m, n \geq n_0\), \(G(x_m, x_n, x) < \varepsilon\).

**Definition 1.3** [19] Let \((X, G)\) be a \(G\)-metric space. Then a function \(\Omega : X \times X \times X \rightarrow [0, \infty)\) is called an \(\Omega\)-distance on \(X\) if the following conditions are satisfied:

(a) \(\Omega(x, y, z) \leq \Omega(x, a, a) + \Omega(a, y, z)\) for all \(x, y, z, a \in X\),
(b) for any \(x, y \in X\), \(\Omega(x, y, \cdot) : X \rightarrow [0, \infty)\) are lower semi-continuous,
(c) for each \(\varepsilon > 0\), there exists a \(\delta > 0\) such that \(\Omega(a, x, a) \leq \delta\) and \(\Omega(a, y, z) \leq \delta\) imply \(G(x, y, z) \leq \varepsilon\).

**Example 1.1** [26] Let \((X, d)\) be a metric space and \(G : X^2 \rightarrow [0, \infty)\) defined by

\[G(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\},\]

for all \(x, y, z \in X\). Then \(\Omega = G\) is an \(\Omega\)-distance on \(X\).

**Example 1.2** [26] In \(X = \mathbb{R}\) we consider the \(G\)-metric \(G : \mathbb{R}^3 \rightarrow [0, \infty)\) defined by

\[G(x, y, z) = \frac{1}{3}(\mid x - y \mid + \mid y - z \mid + \mid x - z \mid),\]

for all \(x, y, z \in \mathbb{R}\). Then \(\Omega : \mathbb{R}^3 \rightarrow [0, \infty)\) defined by

\[\Omega(x, y, z) = \frac{1}{3}(\mid x - y \mid)+ \mid x - z \mid,\]

for all \(x, y, z \in \mathbb{R}\) is an \(\Omega\)-distance on \(\mathbb{R}\).

For more examples see [26].

**Lemma 1.1** [26] Let \(X\) be a metric space with metric \(G\) and \(\Omega\) be
an $\Omega$-distance on $X$. Let $\{x_n\}, \{y_n\}$ be sequences in $X$, $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0, \infty)$ converging to zero and let $x, y, z, a \in X$. Then we have the following:

1. If $\Omega(y, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y, z) \leq \beta_n$ for $n \in \mathbb{N}$, then $G(y, y, z) < \varepsilon$ and hence $y = z$.
2. If $\Omega(y_n, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y_m, z) \leq \beta_n$ for $m > n$, then $G(y_n, y_m, z) \to 0$ and hence $y_n \to z$.
3. If $\Omega(x_n, x_m, x_l) \leq \alpha_n$ for any $l, m, n \in \mathbb{N}$ with $n \leq m \leq l$, then $\{x_n\}$ is a $G$-Cauchy sequence.
4. If $\Omega(x_n, a, a) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a $G$-Cauchy sequence.

**Definition 1.4** [26] G-metric space $X$ is said to be $\Omega$-bounded if there is a constant $M > 0$ such that $\Omega(x, y, z) \leq M$ for all $x, y, z \in X$.

2 Conclusion

In this section, we obtain common fixed point theorems for sequence of mappings satisfying contractiv and expansive conditions on partially ordered complete G-metric spaces.

**Definition 2.1** Suppose $(X, \leq)$ is a partially ordered space and $T : X \to X$ is a mapping of $X$ into itself. We say that $T$ is non-decreasing if for $x, y \in X$,

$$x \leq y \implies T(x) \leq T(y).$$

**Theorem 2.1** Let $(X, \leq)$ and $(Y, \leq)$ be a partially ordered space. Suppose that there exists a G-metric on $X$ and $Y$ such that $(X, G)$ and $(Y, G)$ are complete G-metric space and $\Omega_1$ is an $\Omega$-distance on $X$, $\Omega_2$ is $\Omega$-distance on $Y$ such that $X$ be $\Omega_1$-bounded and $Y$ be $\Omega_2$-bounded. Let $T_n : X \to Y$, $n \in \mathbb{N}$ and $S_n : Y \to X$, $n \in \mathbb{N} \cup \{0\}$ be a non-decreasing and continuous sequence of mappings with following properties:
(a) for all \(x, y, z \in X, x', y', z' \in Y\) and \(i, j, k \in \mathbb{N}\) such that \(0 \leq r < 1\),

\[
\Omega_1(S_iT_ix, S_jT_jy, S_kT_kz) \leq r \max \left\{ \Omega_1(y, S_jT_jy, S_kT_kz), \Omega_1(x, y, z), \Omega_2(T_ix, T_jy, T_kz) \right\},
\]

\[
\Omega_2(T_iS_{i-1}x', T_jS_{j-1}y', T_kS_{k-1}z') \leq r \max \left\{ \Omega_2(y', T_jS_{j-1}y', T_kS_{k-1}z'), \Omega_2(x', y', z'), \Omega_1(S_{i-1}x', S_{j-1}y', S_{k-1}z') \right\};
\]

(b) for every \(x, y, z \in X\) with \(y \neq S_nT_ny, n \in \mathbb{N}\),

\[
\inf \left\{ \Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z \right\} > 0;
\]

(c) for every \(x', y', z' \in Y\) with \(y' \neq T_nS_{n-1}y', n \in \mathbb{N}\),

\[
\inf \left\{ \Omega(x', y', x') + \Omega(x', y', z') + \Omega(x', z', y') : x' \leq z' \right\} > 0;
\]

(d) \(\Omega_2(T_ix, T_jy, T_kz) = 0\) for each \(x, y, z \in X\) and \(\Omega_1(S_ix', S_jy', S_kz') = 0\) for each \(x', y', z' \in Y\).

Then \(\{S_nT_n\}\) has a unique common fixed point \(u\) in \(X\) and \(\{T_nS_{n-1}\}\) has a unique common fixed point \(w\) in \(Y\). Furthermore, \(\lim_{n \to \infty} T_nu = w\) and \(\lim_{n \to \infty} S_nw = u\).

**Proof:** Let \(x_0 \in X\) such that \(S_nT_n(x_{n-1}) = x_n\) and \(T_n(x_{n-1}) = y_n\) and \(x_n \leq x_{n+1}\) for any \(n \in \mathbb{N}\). For all \(n \in \mathbb{N}\) and \(t \geq 0\),

\[
\Omega_1(x_n, x_{n+1}, x_{n+t}) = \Omega_1(S_nT_n(x_{n-1}), S_{n+1}T_{n+1}(x_n), S_{n+t}T_{n+t}(x_{n+t-1}))
\]

\[
\leq r \max \left\{ \Omega_1(x_{n-1}, x_n, x_{n+t-1}), \Omega_1(x_n, x_{n+1}, x_{n+t}), \Omega_2(T_n(x_{n-1}), T_{n+1}(x_n), T_{n+t}(x_{n+t-1})) \right\}
\]

\[
= r \max \left\{ \Omega_1(x_{n-1}, x_n, x_{n+t-1}), \Omega_1(x_n, x_{n+1}, x_{n+t}), \Omega_2(y_n, y_{n+1}, y_{n+t}) \right\}.
\]

Then,

\[
\Omega_1(x_n, x_{n+1}, x_{n+t}) \leq r \max \left\{ \Omega_1(x_{n-1}, x_n, x_{n+t-1}), \Omega_2(y_n, y_{n+1}, y_{n+t}) \right\}.
\]
Similarly,
\[ \Omega_2(y_n, y_{n+1}, y_{n+t}) \leq r \max\{\Omega_2(y_{n-1}, y_n, y_{n+t-1}), \Omega_1(x_{n-1}, x_n, x_{n+t-1})\}. \]
So,
\[ \Omega_1(x_n, x_{n+1}, x_{n+t}) \leq r^n \max\{\Omega_1(x_0, x_1, x_t), \Omega_2(y_0, y_1, y_{t+1})\}, \]
and
\[ \Omega_2(y_n, y_{n+1}, y_{n+t}) \leq r^n \max\{\Omega_1(x_0, x_1, x_t), \Omega_2(y_0, y_1, y_{t+1})\}. \]
Now, for any \( l > m > n \) with \( m = n + k \) and \( l = m + t \) \((k, t \in \mathbb{N})\), we have
\[ \lim_{n,m,l \to \infty} \Omega_1(x_n, x_m, x_l) = 0. \]
Since \( X \) is \( \Omega_1 \)-bounded and,
\[ \Omega_1(x_n, x_m, x_l) \leq \Omega_1(x_n, x_{n+1}, x_{n+1}) + \Omega_1(x_{n+1}, x_m, x_l) \]
\[ \leq \Omega_1(x_n, x_{n+1}, x_{n+1}) + \Omega_1(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + \Omega_1(x_{m-1}, x_m, x_l) \]
\[ \leq r^n M + r^{n+1} M + \cdots + r^{m-1} M \]
\[ \leq \sum_{j=0}^{n-m+1} r^{n-j} M \]
\[ \leq \frac{r^n}{1-r} M. \]
So, by \( 0 \leq r < 1 \) and Part (3) of Lemma (1.6), \( \{x_n\} \) is a G-Cauchy sequence. Since \( X \) is G-complete, \( \{x_n\} \) converges to a point \( u \in X \). Similarly, \( \{y_n\} \) is a G-Cauchy sequence such that has a limit \( w \) in \( Y \). Fixed \( n \in \mathbb{N} \) and by the lower semi-continuity of \( \Omega \), we have
\[ \Omega_1(x_n, x_m, u) \leq \liminf_{p \to \infty} \Omega_1(x_n, x_m, x_p) \leq \frac{r^n}{1-r} M, \quad m \geq n \]
\[ \Omega_1(x_n, u, x_l) \leq \liminf_{p \to \infty} \Omega_1(x_n, x_p, x_l) \leq \frac{r^n}{1-r} M, \quad l \geq n. \]
Assume that \( u \neq S_nTn u \). Since \( x_n \leq x_{n+1} \), we have
0 < \inf \{ \Omega_1(x_n, u, x_n) + \Omega_1(x_n, u, x_{n+1}) + \Omega_1(x_n, x_{n+1}, u) \}
\leq 3 \inf \{ \frac{r^n}{1 - r} M : n \in \mathbb{N} \}
= 0,

which is a contraction. Therefore, \( u = S_n T_n u \) and consequently \( u \) is a common fixed point \( \{S_n T_n\} \). Similarly, \( w \) is a common fixed point \( \{T_n S_{n-1}\} \).

To prove the uniqueness, suppose \( \{S_n T_n\} \) has another fixed point \( u' \). Then,

\[
\Omega_1(u, u', u') = \Omega_1(S_n T_n u, S_n T_n u', S_n T_n u')
\leq r \max \{ \Omega_1(u, u', u'), \Omega_1(u', S_n T_n u', S_n T_n u'), \\
\Omega_2(T_n u, T_n u', T_n u') \}
= r \max \{ \Omega_1(u, u', u'), \Omega_1(u', u', u'), \\
\Omega_2(T_n u, T_n u', T_n u') \}.
\]

By (d) either \( \Omega_1(u, u', u') = 0 \) or \( \Omega_1(u, u', u') \leq r \Omega_1(u', u', u') \).

Since,

\[
\Omega_1(u', u', u') = \Omega_1(S_n T_n u', S_n T_n u', S_n T_n u')
\leq r \max \{ \Omega_1(u', u', u'), \Omega_1(u', S_n T_n u', S_n T_n u'), \\
\Omega_2(T_n u', T_n u', T_n u') \},
\]
then, \( \Omega_1(u', u', u') = 0 \) and consequently \( \Omega_1(u, u', u') = 0 \). By Part (c) of Definition (1.3) fixed point of \( \{S_n T_n\} \) is unique. Similarly, \( w \) is a unique fixed point of \( \{T_n S_{n-1}\} \). By continuity of \( \{T_n\} \), we have

\[
\lim_{n \to \infty} T_n u = \lim_{n \to \infty} T_n (x_{n-1}) = \lim_{n \to \infty} y_n = w.
\]

Similarly, \( \lim_{n \to \infty} S_n w = u. \square \)

**Corollary 2.1** Let \((X, \leq)\) be a partially ordered space. Suppose that there exists a G-metric on \( X \) such that \((X, G)\) is a complete G-metric space and \( \Omega \) is an \( \Omega \)-distance on \( X \) such that \( X \) is \( \Omega \)-
bounded. Let $T_n : X \rightarrow X$, $n \in \mathbb{N}$ be a non-decreasing sequence of mappings with property that for any $i, j, k \in \mathbb{N}$, we have:

(a) for all $x, y, z \in X$ and $0 \leq r < 1$,

$$\Omega(T_i x, T_j y, T_k z) \leq r \max \{\Omega(x, y, z), \Omega(y, T_j y, T_k z)\};$$

(b) for every $x, y, z \in X$ with $y \neq T_n y$, $n \in \mathbb{N}$,

$$\inf \{\Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z\} > 0.$$  

Then $\{T_n\}$ has a unique common fixed point $u$ in $X$ and $\Omega(u, u, u) = 0$. 

**Proof:** It is sufficient that put $\Omega = \Omega_1 = \Omega_2$, $X = Y$ and $S_n = I_n$ that $I_n$ is identity mapping on $X$ in Theorem (2.2).

**Theorem 2.2** Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a $G$-metric on $X$ such that $(X, G)$ is a complete $G$-metric space and $\Omega$ is an $\Omega$-distance on $X$ such that $X$ is $\Omega$-bounded. Let $T_n : X \rightarrow X$, $n \in \mathbb{N}$ be a non-decreasing sequence of mappings with property that for any $i, j, k \in \mathbb{N}$, we have:

(a) for all $x, y, z \in X$ and $0 \leq r < 1$,

$$\Omega(T_i x, T_j y, T_k z) \leq r \max \{\Omega(x, y, z), \Omega(y, T_j y, T_k z)\};$$

(b) for every $x, y, z \in X$ with $y \neq T_n y$, $n \in \mathbb{N}$,

$$\inf \{\Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z\} > 0.$$  

Then $\{T_n\}$ has a unique common fixed point $u$ in $X$ and $\Omega(u, u, u) = 0$. 

**Proof:** Theorem is proved by similar proof of Theorem 2.1.

**Corollary 2.2** Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a $G$-metric on $X$ such that $(X, G)$ is a complete $G$-metric space and $\Omega$ is an $\Omega$-distance on $X$ such that $X$ is $\Omega$-
bounded. Let \( T_n : X \rightarrow X, n \in \mathbb{N} \) be a non-decreasing sequence of mappings with property that for some \( m \in \mathbb{N} \) and each \( i, j, k \in \mathbb{N} \), we have:

(a) for all \( x, y, z \in X \) and \( 0 \leq r < 1 \), \( \Omega(T^m_i x, T^m_j y, T^m_k z) \leq r \Omega(x, y, z) \);

(b) for every \( x, y, z \in X \) with \( y \neq T_n y, n \in \mathbb{N} \),

\[
\inf \{ \Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z \} > 0.
\]

Then \( \{ T_n \} \) has a unique common fixed point \( u \) in \( X \) and \( \Omega(u, u, u) = 0 \).

**Proof:** By Theorem 2.2, the sequence \( \{ T^m_n \} \) has the unique common fixed point \( u \). But,

\[
T_n u = T_n(T^m_n u) = T^{m+1}_n u = T^m_n(T_n u).
\]

So, \( T_n u \) is the fixed point \( \{ T^m_n \} \). Now, by uniqueness of the fixed point, \( T_n u = u \). \( \square \)

**Definition 2.2** Let \( (X, G) \) be a \( G \)-metric space, \( \Omega \) be an \( \Omega \)-distance on \( X \) and \( \Omega \) be a selfmapping on \( X \). Then \( \Omega \) is called expansive mapping with respect \( \Omega \) if there exists a constant \( a > 1 \) such that for all \( x, y, z \in X \), we have:

\[
\Omega(Tx, Ty, Tz) \geq a \Omega(x, y, z).
\]

**Theorem 2.3** Let \( (X, \leq) \) be a partially ordered space. Suppose that there exists a \( G \)-metric on \( X \) such that \( (X, G) \) is a complete \( G \)-metric space and \( \Omega \) is an \( \Omega \)-distance on \( X \) such that \( X \) is \( \Omega \)-bounded. Let \( T_n : X \rightarrow X, n \in \mathbb{N} \) be a non-decreasing sequence of surjective mappings and \( S_n : X \rightarrow X, n \in \mathbb{N} \) be a non-decreasing sequence of injective mappings with property that for any \( i, j, k \in \mathbb{N} \), we have:

(a) for all \( x, y, z \in X \) and \( a > 1 \), \( \Omega(T_i x, T_j y, T_k z) \geq a \Omega(S_i x, S_j y, S_k z) \);
(b) for all \( n \in \mathbb{N} \), \( T_n \) and \( S_n \) commute;

(c) for every \( x, y, z \in X \) with \( y \neq T_n y, n \in \mathbb{N} \),

\[
\inf \{ \Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z \} > 0
\]

Then \( \{ T_n \} \) and \( \{ S_n \} \) have a unique common fixed point \( u \) in \( X \) and \( \Omega(u, u, u) = 0 \).

**Proof:** If \( T_i x = T_j y \) for any \( i \in \mathbb{N} \) and \( x, y \in X \), then,

\[
\Omega(T_i x, T_j y, T_j y) \geq a \Omega(S_i x, S_j y, S_j y);
\]

\[
\Omega(T_j y, T_i x, T_i y) \geq a \Omega(S_j y, S_i x, S_i y);
\]

thus,

\[
\Omega(S_i x, S_j y, S_j y) \leq \frac{1}{a} \Omega(T_i x, T_j y, T_j y);
\]

\[
\Omega(S_j y, S_i x, S_i y) \leq \frac{1}{a} \Omega(T_j y, T_i x, T_i y).
\]

Now, since \( a > 1 \) and \( X \) is \( \Omega \)-bounded then, for any \( \varepsilon > 0 \), we choose \( \delta = \frac{1}{a} M \), which implies, \( \Omega(S_i x, S_j y, S_j y) \leq \delta \) and \( \Omega(S_j y, S_i x, S_i y) \leq \delta \). By Part (c) of Definition (1.3), \( G(S_i x, S_j x, S_j y) \leq \varepsilon \). Since \( \varepsilon \) is arbitrary, hence \( S_i x = S_i y \). Now, by injectivity \( S_i \) for every \( i \in \mathbb{N} \), we imply that \( x = y \). So, \( T_n \) is injective and consequently invertible.

Let \( H_n \) be the inverse mapping of \( T_n \) for any \( n \in \mathbb{N} \). Then,

\[
\Omega(x, y, z) = \Omega(T_i(H_i x), T_j(H_j y), T_k(H_k z))
\]

\[
\geq a \Omega(S_i(H_i x), S_j(H_j y), S_k(H_k z)).
\]

So, for each \( x, y, z \in X \) and any \( i, j, k \in \mathbb{N} \), we obtain

\[
\Omega(S_i o H_i x, S_j o H_j y, S_k o H_k z) \leq r \Omega(x, y, z),
\]

where \( r = \frac{1}{a} \). Then \( \Omega(G_i x, G_j y, G_k z) \leq r \Omega(x, y, z) \), where \( G_n = S_n o H_n \). By Theorem 2.1, \( G_n \) or \( S_n o H_n \) have a unique common fixed point \( u \) in \( X \),i.e. \( G_n u = u = S_n o H_n u \). It follows that \( T_n(S_n(H_n u) = \).
Since $T_n$ and $S_n$ commute, we obtain
\[ S_n(T_n(H_nu)) = T_nu \implies S_nu = T_nu, \]
for any $n \in \mathbb{N}$. If we put $x = u$, $y = H_ju$ and $z = H_ku$, we have
\[ \Omega(T_iu, T_j(H_ju), T_k(H_ku)) \geq a\Omega(S_iu, S_j(H_ju), S_k(H_ku)). \]
So,
\[ \Omega(T_iu, u, u) \geq a\Omega(S_iu, u, u) = a\Omega(T_iu, u, u). \]
Since $a > 1$, then $\Omega(T_iu, u, u) = 0$. By putting $x = H_iu$, $y = H_ju$, $z = H_ku$ and similar proof $\Omega(u, u, u) = 0$. Now by Part (3) of Definition (1.3), $T_iu = u$. Hence $T_nu = S_nu = u$ and $u$ is a unique common fixed point of $T_n$ and $S_n$.

The following corollary is a generalization of [18, theorem 2.1].

**Corollary 2.3** Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a $G$-metric on $X$ such that $(X, G)$ is a complete $G$-metric space and $\Omega$ is an $\Omega$-distance on $X$ such that $X$ is $\Omega$-bounded. Let $T_n : X \longrightarrow X$, $n \in \mathbb{N}$ be a non-decreasing sequence of surjective mappings with property that for any $i, j, k \in \mathbb{N}$, we have:

(a) for all $x, y, z \in X$ and $a > 1$, $\Omega(T_ix, T_jy, T_kz) \geq a\Omega(x, y, z)$;

(b) for every $x, y, z \in X$ with $y \neq T_ny$, $n \in \mathbb{N}$,
\[ \inf \{\Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z\} > 0. \]

Then $\{T_n\}$ has a unique common fixed point $u$ in $X$ and $\Omega(u, u, u) = 0$.

**Proof:** Follows from Theorem 2.3, by taking $S_n = I_n$ for any $n \in \mathbb{N}$ such that $I_n$ is identity mapping on $X$. □

**Corollary 2.4** Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a $G$-metric on $X$ such that $(X, G)$ is a complete
$G$-metric space and $\Omega$ is an $\Omega$-distance on $X$ such that $X$ is $\Omega$-bounded. Let $T_n : X \rightarrow X$, $n \in \mathbb{N}$ be a non-decreasing sequence of surjective mappings with property that for each $i, j, k \in \mathbb{N}$, we have:

(a) for all $x, y, z \in X$ and $a > 1$,

$$\Omega(T_i x, T_j y, T_k z) \geq a \max \{\Omega(x, y, y) + \Omega(y, y, z), \Omega(x, z, z) + \Omega(z, y, z)\},$$

(b) for every $x, y, z \in X$ with $y \neq T_n y$, $n \in \mathbb{N}$,

$$\inf \{\Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z\} > 0.$$

Then $\{T_n\}$ has a unique common fixed point $u$ in $X$ and $\Omega(u, u, u) = 0$.

**Proof:** Since by Part (a) of Definition (1.3),

$$a \max \{\Omega(x, y, y) + \Omega(y, y, z), \Omega(x, z, z) + \Omega(z, y, z)\} \geq a \Omega(x, y, z).$$

So, Theorem 2.3 implies that $\{T_n\}$ has a unique common fixed point $u$ in $X$ and $\Omega(u, u, u) = 0$. □

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**References**


