Some Results for $CAT(0)$ Spaces

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Abstract

We shall generalize the concept of $z = (1-t)x \oplus ty$ to $n$ times which contains to verify some their properties and inequalities in $CAT(0)$ spaces. In the sequel with introducing of $\alpha$-nonexpansive mappings, we obtain some fixed points and approximate fixed points theorems.

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1 Introduction

Let $(X,d)$ be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$) is a map $c$ from a closed interval $[0,l] \subseteq R$ to $X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t_0)) = |t - t_0|$ for all $t, t_0 \in [0,l]$. In particular, $c$ is an isometry and $d(x, y) = l$. The image $\alpha$ of $c$ is called a geodesic (or metric) segment joining $x$ and $y$. When it is unique, this geodesic is denoted by $[x, y]$. The space $(X,d)$ is said to be a geodesic space if every two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if $Y$ includes every

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geodesic segment joining any two of its points.
A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space $(X, d)$ consists of three points in $X$ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for a geodesic triangle $\triangle(x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, y_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic metric space is said to be a $\text{CAT}(0)$ space if all geodesic triangles of appropriate size satisfy the following comparison axiom:

"Let $\triangle$ be a geodesic triangle in $X$ and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $\text{CAT}(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle},$

$$d(x, y) \leq d_{E^2}(\overline{x}, \overline{y}).$$

**Definition 1.1.** ([1]) A hyperbolic space is a triple $(X, d, W)$ where $(X, d)$ is a metric space and $W : X \times X \times [0, 1] \to X$ is such that

1. $d(z, W(x, y, t)) \leq (1 - t)d(x, z) + td(z, y)$
2. $d(W(x, y, t), W(x, y, s)) = |t - s|d(x, y)$
3. $W(x, y, t) = W(y, x, 1 - t)$
4. $d(W(x, z, t), W(y, w, t)) \leq (1 - t)d(x, y) + td(z, w)$

for all $x, y, z, w \in X$ and $t, s \in [0, 1]$.

If $x, y \in X$ and $t \in [0, 1]$ then we use the notation $(1 - t)x \oplus ty$ for $W(x, y, t)$. We shall denote by $[x, y]$ the set $\{(1 - t)x \oplus ty : t \in [0, 1]\}$. A nonempty subset $C \subseteq X$ is convex if $[x, y] \subseteq C$ for all $x, y \in C$.

We remark that any normed space $(X, \|\|)$ is a hyperbolic space, with

$$(1 - t)x \oplus ty := (1 - t)x + ty.$$

Here we recall a couple of lemmas which will be used next.

**Lemma 1.2.** ([2, Lemma 2.4]) Let $(X, d)$ be a $\text{CAT}(0)$ space. Then

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z) \leq \max\{d(x, z), d(y, z)\},$$

for $x, y, z \in X$ and $t \in [0, 1]$.

**Lemma 1.3.** ([2, Lemma 2.5]) Let $(X, d)$ be a $\text{CAT}(0)$ space. Then

$$d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2,$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

In particular by Lemma 1.3 we have

$$d(z, \frac{1}{2}x \oplus \frac{1}{2}y)^2 \leq \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2,$$

for all $x, y, z \in X$, which is called (CN) inequality of Bruhat-Tits, as it was shown in [3]. In fact (cf. [4], p. 163), a geodesic space is a $\text{CAT}(0)$ space if and only if it satisfies the (CN) inequality.
2 Main results

Throughout this section we let \( n \in \mathbb{N} \), \( z_1 = x \) and \( z_n = y \) until Definition 3.2.

**Lemma 2.1.** Let \((X, d)\) be a CAT(0) space. Then

1. Let \( x, y \in X, x \neq y \) and \( z_i, z'_i \in [x, y] \) such that \( d(x, z_i) = d(x, z'_i) \) for all \( 1 \leq i \leq n \). Then \( z_i = z'_i \) for \( 1 \leq i \leq n \).

2. Let \( x, y \in X \), then for each \( \alpha = (\alpha_1, \ldots, \alpha_n) \in [0, 1]^n \) with \( \sum_{i=1}^{n} \alpha_i = 1 \) there exist points \( z_1, \ldots, z_n \in [x, y] \) and unique point \( z \in [x, y] \) such that \( d(z, z_i) = \alpha_i d(x, y) \) for \( 1 \leq i \leq n \).

**Proof.** Since \( z_i, z'_i \in [x, y] \), there exist \( t_i, t'_i \in [0, t] \) such that \( c(t_i) = z_i \) and \( c(t'_i) = z'_i \). Thus \( d(x, z_i) = d(c(0), c(t_i)) = t_i \) and similarly \( d(x, z'_i) = t'_i \). Since \( d(x, z_i) = d(x, z'_i) \), we have \( t_i = t'_i \), and consequently \( z_i = z'_i \) for \( 1 \leq i \leq n \), which proves (1).

To prove (2), by [2, Lemma 2.1(iv)], this is true for \( n = 2 \), because for \( \alpha = (\alpha_1, \alpha_2) \) with \( \alpha_1 + \alpha_2 = 1 \) there exists unique point \( z \in [x, y] \) such that \( d(x, z) = \alpha_1 l, d(z, y) = \alpha_2 l \) that for convention we had shown with \( z = \alpha_1 x + \alpha_2 y \).

Now by induction let it holds for \( n - 1 \) and choose \( \alpha = (\alpha_1, \ldots, \alpha_n) \in [0, 1]^n \) such that \( \sum_{i=1}^{n-1} \alpha_i = 1 \). Put \( \beta_i := \frac{\alpha_i}{\alpha_n} \) for \( 1 \leq i \leq n - 1 \). Thus \( \sum_{i=1}^{n-1} \beta_i = 1 \) and by hypothesis of induction there exists unique point \( z' \in [z_1, z_{n-1}] \) such that \( d(z', z_i) = \beta_i l \) for \( 1 \leq i \leq n - 1 \), now there exists unique point \( z \in [z', z_n] \) such that \( d(z, z_i) = \alpha_i l, d(z, z') = (1 - \alpha_n)l \).

To prove (2) directly, let \( t_i = 1 - \alpha_n - \alpha_i, t = 1 - \alpha_n \in [0, 1] \) for \( 1 \leq i \leq n \). Put \( z_i = c(t_i) \) and \( z = c(t) \) so \( d(z, z_i) = |t - t_i| = \alpha_i l \), for \( 1 \leq i \leq n \). For uniqueness, if \( d(z, z_i) = d(z', z_i) \) for \( 1 \leq i \leq n \), then by (1) and \( i = 1 \), we have \( z = z' \). \( \square \)

**Example 2.2.** Let \( X = [0, 1] \) and put

\[
A = \left\{ (x, 0) : 0 \leq x \leq \frac{2}{3} \right\} \cup \left\{ \left( \frac{2}{3}, y \right) : -\frac{1}{6} \leq y \leq \frac{1}{6} \right\}.
\]

Define \( f : X \to A \subseteq \mathbb{R}^2 \) by

\[
f(x) = \begin{cases} 
(x, 0), & 0 \leq x \leq \frac{2}{3}; \\
\left( \frac{2}{3}, x - \frac{1}{6} \right), & \frac{2}{3} \leq x \leq \frac{5}{6}; \\
\left( \frac{5}{6}, x - \frac{1}{6} \right), & \frac{5}{6} \leq x \leq 1.
\end{cases}
\]

So \( f \) is isometric homeomorphism. For instance let \( \alpha_1 = \frac{2}{3}, \alpha_2 = \alpha_3 = \frac{1}{6} \). Therefore \( z_1 = x = 0, z_2 = \frac{2}{3}, z_3 = y = 1, z = \frac{5}{6} \) and \( l = 1 \). Since \( t = 1 - \alpha_3 = \frac{5}{6} \) and \( t_2 = 1 - \alpha_3 - \alpha_2 = \frac{1}{3} \), we have \( z_2 = c(t_2) = \frac{2}{3}, z = \frac{5}{6} \) and by homeomorphism we have \( z_1 = (0, 0), z_2 = \left( \frac{2}{3}, \frac{1}{6} \right), z_3 = \left( \frac{5}{6}, \frac{1}{6} \right) \) and \( z = \left( \frac{5}{6}, 0 \right) \). And also we have \( d(z, z_i) = \alpha_i l \), for \( 1 \leq i \leq 3 \).
Theorem 2.4. Let $d: [0, 1]^n \to \mathbb{R}$ be a $CAT(0)$ space, let $x, y \in X$ such that $x \neq y$ and $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in [0, 1]^n$ with $\sum_{i=1}^{n} \alpha_i = 1 = \sum_{i=1}^{n} \beta_i$. Then

$$z_{\alpha} = z_{\beta} \iff \alpha = \beta.$$ 

Proof. This is true because,

$$d(z_{\alpha}, z_i) = d(z_{\beta}, z_i) \Rightarrow \alpha_i l = \beta_i l \Rightarrow \alpha_i = \beta_i,$$

for $1 \leq i \leq n$.

Theorem 2.4. Let $(X, d)$ be a $CAT(0)$ space, let $x, y \in X$ such that $x \neq y$ and $d(x, y) = l$. Then

1. $[x, y] = \{ z_{\alpha} | \alpha \in [0, 1]^n, \sum_{i=1}^{n} \alpha_i = 1 \}$.
2. For all $z \in X$ the following holds:

$$\exists z_1, \ldots, z_n \in [x, y] \text{ such that } \sum_{i=1}^{n} d(z, z_i) = d(x, y) \iff z \in [x, y].$$
3. The mapping $f: [0, 1]^n \to [x, y], f(\alpha) = z_{\alpha}$ is continuous and bijective.

Proof. (1) The case of $n = 2$ is proved in [2, Lemma 2.1]. Now let $z \in [x, y]$. By induction, suppose there exists $\beta \in [0, 1]^{n-1}$, such that $\sum_{i=1}^{n-1} \beta_i = 1$ and $z = z_{\beta}$. Put $\alpha_i = \beta_i$ for $1 \leq i \leq n-2$ and $\alpha_{n-1} = \alpha_n = \frac{\beta_{n-1} + \beta_{n-2}}{2}$ therefore $\sum_{i=1}^{n} \alpha_i = 1$ and there exists $z' = c \frac{\beta_{n-1} l}{2}$ that $d(z', x) = (\sum_{i=1}^{n-2} \beta_i + \frac{\beta_{n-1}}{2}) l$ and $d(z, z') = \frac{\beta_{n-1} l}{2}$. Now $z' = (\sum_{i=1}^{n-2} \beta_i + \frac{\beta_{n-1}}{2}) z_{\beta} + \frac{\beta_{n-1}}{2} y$ thus $z' \in [x, y]$ and $d(z, z') = \alpha_{n-1} l$. To prove (2) let for every $z \in X$ there exist $z_1, \ldots, z_n \in [x, y]$ such that $\sum_{i=1}^{n} d(z, z_i) = d(x, y)$. Put $\alpha_i = \frac{d(z, z_i)}{l}$ where $z_i \in [x, y]$ and $1 \leq i \leq n$, so there exists $z_{\alpha}$ such that $z_{\alpha} = z$. 

Notation: By the point $z_{\alpha}$, we mean the unique point

$$z_{\alpha} = \alpha_1 z_1 \oplus \alpha_2 z_2 \oplus \cdots \oplus \alpha_n z_n$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in [0, 1]^n$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and $z_i \in X$ for $1 \leq i \leq n$. Also $z_{\alpha}$ can be written as

$$z_{\alpha} = (1 - \alpha_n) z' \oplus \alpha_n z_n,$$

where $z' = \frac{\alpha_1}{1 - \alpha_n} z_1 \oplus \cdots \oplus \frac{\alpha_{n-1}}{1 - \alpha_n} z_{n-1}$ where $\alpha_n \neq 1$.

Remark 2.3. Let $(X, d)$ be a $CAT(0)$ space, let $x, y \in X$ such that $x \neq y$ and $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in [0, 1]^n$ with $\sum_{i=1}^{n} \alpha_i = 1 = \sum_{i=1}^{n} \beta_i$. Then

$$z_{\alpha} = z_{\beta} \iff \alpha = \beta.$$
Conversely, if $z \in [x, y]$ then $z = z_\alpha$ for some $\alpha$ and $z_1, \ldots, z_n$ such that $d(z, z_i) = \alpha_i d(x, y)$ so $\sum_{i=1}^n d(z, z_i) = d(x, y)$.

To prove (3) applying (1) and Remark 2.3, we get that $f$ is well defined and bijective. The continuity of $f$ is obvious by induction, because $f$ can be written as $f(\alpha) = g(\beta) \oplus h(\alpha_n)$ where $g(\beta) = \beta_1 z_1 \oplus \cdots \oplus \beta_{n-1} z_{n-1}$, $\beta_i := \frac{\alpha_i}{1-\alpha_n}$ for $1 \leq i \leq n - 1$ and $h(\alpha_n) = \alpha_n z_n$.

**Lemma 2.5.** Let $(X, d)$ be a CAT(0) space. Then

1. $d(z_\alpha, z) \leq \sum_{i=1}^n \alpha_i d(z_i, z) \leq \max\{d(z_i, z) : 1 \leq i \leq n\}$,
2. $d(z_\alpha, z)^2 \leq \sum_{i=1}^n \alpha_i d(z_i, z)^2 \leq \max\{d(z_i, z)^2 : 1 \leq i \leq n\}$,
3. $d(z_\alpha, z'_{\beta}) \leq \sum_{i,j=1}^{n} \alpha_i \beta_j d(z_i, z_j') \leq \max\{d(z_i, z_j') : 1 \leq i, j \leq n\}$,

for $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in [0, 1]^n$ with $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i = 1$ and $z, z_i, z_i' \in X$ for $1 \leq i \leq n$ which $z_\alpha = \alpha_1 z_1 \oplus \alpha_2 z_2 \oplus \cdots \oplus \alpha_n z_n, z_{\beta}' = \beta_1 z_1' \oplus \beta_2 z_2' \oplus \cdots \oplus \beta_n z_n'$.

**Proof.** By Lemma 1.2 it is true for $n = 2$. So by induction let

$$z_\alpha = \alpha_1 z_1 \oplus \alpha_2 z_2 \oplus \cdots \oplus \alpha_n z_n$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in [0, 1]^n$ such that $\sum_{i=1}^n \alpha_i = 1$ and $z_i \in X$ for $1 \leq i \leq n$.

Put $\gamma := \left(\frac{\alpha_1}{1-\alpha_n}, \ldots, \frac{\alpha_{n-1}}{1-\alpha_n}\right)$ that $\sum_{k=1}^{n-1} \frac{\alpha_k}{1-\alpha_n} = 1$ by Theorem 2.1 there exists $v_\gamma \in [x, z_{n-1}]$ such that $v_\gamma = \frac{\alpha_1}{1-\alpha_n} z_1 \oplus \cdots \oplus \frac{\alpha_{n-1}}{1-\alpha_n} z_{n-1}$ and we have $z_\alpha = (1-\alpha_\gamma) v_\gamma + \alpha_n z_n$.

$$d(z_\alpha, z) = d((1-\alpha_n) v_\gamma + \alpha_n z_n, z) \leq (1-\alpha_n)d(v_\gamma, z) + \alpha_n d(z_n, z)$$

$$= (1-\alpha_n)d\left(\frac{\alpha_1}{1-\alpha_n} z_1 \oplus \cdots \oplus \frac{\alpha_{n-1}}{1-\alpha_n} z_{n-1}, z\right) + \alpha_n d(z_n, z) \leq \sum_{i=1}^n \alpha_i d(z_i, z) \leq \max\{d(z_i, z) : 1 \leq i \leq n\}.$$

This proves (1).

(2) can easily proved according to Lemma 1.3 and again by induction on $n \geq 2$.

**Lemma 2.6.** Let $(X, d)$ be a hyperbolic space. Then

$$d(z_\alpha, z'_{\beta}) \leq \sum_{i=1}^n \alpha_i d(z_i, z_i') \leq \max\{d(z_i, z_i') : 1 \leq i \leq n\},$$

for $\alpha = (\alpha_1, \ldots, \alpha_n) \in [0, 1]^n$ with $\sum_{i=1}^n \alpha_i = 1$ and $z_i, z_i' \in X$ for $1 \leq i \leq n$ which $z_\alpha = \alpha_1 z_1 \oplus \alpha_2 z_2 \oplus \cdots \oplus \alpha_n z_n, z_{\beta}' = \alpha_1 z_1' \oplus \alpha_2 z_2' \oplus \cdots \oplus \alpha_n z_n'$. 
Proof. By the property of (W4) it is true for $n = 2$. The remaining is similar to the proof of the lemma 2.5. □

3 Fixed points and approximate fixed points for $T_{\alpha}$ maps

In 2008 T. Suzuki [5], defined condition (C) for mappings on a subset of a Banach space, as following: "Let $T$ be a mapping on a subset $C$ of a Banach space $E$. Then $T$ is said to satisfy condition (C) if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

This condition is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. In that paper, he has presented fixed point theorems and convergence theorems for mappings satisfying condition (C). Also Examples 1 and 2 in the same paper stated that there exists a map $T$ which satisfies condition (C), but $T$ is not nonexpansive, and there exists a map $T$ which is quasi-nonexpansive, but it does not satisfy condition (C).

Recently B. Nanjaras, B. Panyanaka and W. Phuengrattana in [6], A. Razani and H. Salahifard in [7] and other mathematicians has proved some theorems according to single-valued mappings or multi-valued mappings which are satisfying Suzuki’s condition (C) in a $CAT(0)$ space.

Some basic properties on condition (C) by [6, Propositions 3.2, 3.3], [7, Theorems 2.3, 2.7 and Corollary 2.8] and [8, Theorem 1.3] are:

$P_1$ ([6, Lemma 2.5]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a $CAT(0)$ space $X$ and let $\{\alpha_n\} \subseteq [0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_n \alpha_n < 1$. Suppose that $x_{n+1} = \alpha_n y_n \oplus (1 - \alpha_n)x_n$ and $d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$. Then $\lim_{n \to \infty} d(y_n, x_n) = 0$.

$P_2$ ([6, Proposition 3.2]) Let $K$ be a nonempty subset of a $CAT(0)$ space $X$. If $T : K \to K$ be a nonexpansive mapping, then $T$ satisfies condition (C).

$P_3$ ([6, Proposition 3.3]) Let $K$ be a nonempty subset of a $CAT(0)$ space $X$. If $T : K \to K$ satisfies condition (C) and has a fixed point, then $T$ is a quasi-nonexpansive mapping.

$P_4$ ([7, Theorem 2.3]) Let $K$ be a bounded closed convex subset of a complete $CAT(0)$ space $X$. If $T : K \to K$ satisfies the condition (C) and $F(T) \neq \emptyset$, then $F(T)$ is $\Delta$-closed and convex set.
**P5** ([7, Theorem 2.7]) Let $K$ be a bounded closed convex subset of a complete $CAT(0)$ space $X$. If $T : K \to K$ satisfies condition (C), then $F(T)$ is nonempty.

**P6** ([7, Corollary 2.8]) Let $K$ be a bounded closed convex subset of a complete $CAT(0)$ space $X$. If $T : K \to K$ satisfies condition (C), then $F(T)$ is nonempty, $\Delta$-closed and convex.

**P7** ([8, Theorem 1.3]) Let $(X,d)$ be a convex subset of a $CAT(0)$ space and $f : X \to X$ a quasi-nonexpansive map whose fixed point set is nonempty. Then $F(f)$ is closed, convex and hence contractible.

And now, we start our results by following definitions.

**Definition 3.1.** ([5]) Let $T$ be a mapping on a subset $K$ of a $CAT(0)$ space $(X,d)$. Then $T$ is said to satisfy condition (C) if

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow d(Tx,Ty) \leq d(x,y),$$

for all $x, y \in K$.

The following we will use this notation $T_\alpha = \alpha_1 T_1 \oplus \cdots \oplus \alpha_n T_n$ where $T_1, \cdots, T_n : X \to [x,y]$ for $1 \leq i \leq n$ and $\alpha = (\alpha_1, \cdots, \alpha_n) \in [0,1]^n$ a multiindex satisfying $\sum_{i=1}^n \alpha_i = 1$.

**Definition 3.2.** ([9-10]) Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in [0,1]^n$ be a multiindex satisfying $\sum_{i=1}^n \alpha_i = 1$. The maps $T_1, \cdots, T_n$ on $X$ are said to be $\alpha$-nonexpansive if

$$\sum_{i=1}^n \alpha_id(T_i x, T_i y) \leq d(x,y), \quad (3.1)$$

for all $x, y \in X$.

**Theorem 3.3.** Let $K$ be a bounded closed convex subset of a complete $CAT(0)$ space $(X,d)$. If $T_\alpha : K \to K$ is defined by $T_\alpha = \alpha_1 T_1 \oplus \cdots \oplus \alpha_n T_n$ which $T_1, \cdots, T_n$ are selfmapps on $K$, which commute each other and satisfy condition (C), then $T_\alpha$ has a fixed point.

**Proof.** By P5, $F(T_i) \neq \emptyset$ for $1 \leq i \leq n$. We say $\bigcap_{i=1}^n F(T_i) \neq \emptyset$. By induction we assume that $L := \bigcap_{i=1}^{n-1} F(T_i) \neq \emptyset$. Let $x \in L$ so we have

$$T_n x = T_n (T_1 x) = T_1 (T_n x),$$

thus $T_n x \in F(T_1)$ for $1 \leq i \leq n - 1$. Therefore $T_n x \in L$ hence $T_n(L) \subseteq L$. By P6, $F(T_i)$ nonempty and convex and since $T_i$ satisfy the condition (C) by P3, $T_i$ is a quasinonexpansive map and by P7, $F(T_i)$ closed and convex, for $(1 \leq i \leq n)$, therefore $L$ and $F(T_n)$ are nonempty, bounded closed convex subsets of a complete
Thus $T : L \to L$ satisfies the condition of the $P4$, hence $T_n x$ has a fixed point in $L$, that is,

$$L \cap F(T_n) = \bigcap_{i=1}^{n} F(T_i) \neq \emptyset.$$ 

If we let $x \in \bigcap_{i=1}^{n} F(T_i)$, then

$$d(x, T_{\alpha} x) \leq \sum_{i=1}^{n} \alpha_i d(x, T_i x) = 0,$$

namely $x \in F(T_{\alpha})$. \hfill $\Box$

**Theorem 3.4.** Let $K$ be a bounded closed convex subset of a complete $CAT(0)$ space $(X, d)$. If $T_{\alpha} : K \to K$ defined by $T_{\alpha} = \alpha_1 T_1 \oplus \cdots \oplus \alpha_n T_n$ which $T_1, \cdots, T_n$ are selfmaps on $K$, which $T_1$ satisfies the condition $(C)$ and $d(x, T_n x) \leq d(x, T_1 x)$ for every $x \in K$, then $\inf_{x \in K} d(x, T_{\alpha} x) = 0$.

**Proof.** Let $x_1 \in K$, define sequence $\{x_n\} \subseteq K$ by $x_{n+1} := tT_1 x_n \oplus (1-t)x_n$ for $n \in \mathbb{N}$, where $t \in \left[\frac{1}{2}, 1\right)$. Then by the assumption $\frac{1}{2}d(x_n, T_1 x_n) \leq td(x_n, T_1 x_n) = d(x_n, x_{n+1})$ for $n \in \mathbb{N}$ hence $d(T_1 x_{n+1}, T_1 x_n) \leq d(x_{n+1}, x_n)$. So by $P1$ we have $\inf_{x \in K} d(x, T_1 x) = 0$. So

$$d(x, T_{\alpha} x) \leq d(x, T_1 x) + d(T_1 x, T_{\alpha} x),$$

$$= d(x, T_1 x) + \alpha_1 d(T_1 x, T_{n} x),$$

$$\leq d(x, T_1 x) + d(T_1 x, x) + d(x, T_n x),$$

$$\leq 3d(x, T_1 x),$$

therefore there exists $\{x_n\} \subseteq K$ such that $d(x_n, T_1 x_n) \to 0$ as $n \to \infty$ thus $d(x_n, T_{\alpha} x_n) \to 0$. \hfill $\Box$

**Corollary 3.5.** ([7, Lemma 2.5]) Let $K$ be a bounded closed convex subset of a complete $CAT(0)$ space $(X, d)$. If $T : K \to K$ satisfies the condition $(C)$, then there exists an approximate fixed point sequence for $T$, i.e., $\inf_{x \in K} d(x, T x) = 0$.

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**References**


