Some Results for CAT(0) Spaces

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Abstract

We shall generalize the concept of $z = (1 - t)x \oplus ty$ to $n$ times which contains to verify some their properties and inequalities in CAT(0) spaces. In the sequel with introducing of $\alpha$-nonexpansive mappings, we obtain some fixed points and approximate fixed points theorems.

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1 Introduction

Let $(X, d)$ be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$) is a map $c$ from a closed interval $[0, l] \subseteq R$ to $X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t_0)) = |t - t_0|$ for all $t, t_0 \in [0, l]$. In particular, $c$ is an isometry and $d(x, y) = l$. The image $\alpha$ of $c$ is called a geodesic (or metric) segment joining $x$ and $y$. When it is unique, this geodesic is denoted by $[x, y]$. The space $(X, d)$ is said to be a geodesic space if every two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if $Y$ includes every

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geodesic segment joining any two of its points.

A geodesic triangle \( \triangle(x_1, x_2, x_3) \) in a geodesic metric space \((X, d)\) consists of three points in \(X\) (the vertices of \(\triangle\)) and a geodesic segment between each pair of vertices (the edges of \(\triangle\)). A comparison triangle for a geodesic triangle \( \triangle(x_1, x_2, x_3) \) in \((X, d)\) is a triangle \( \triangle(x_1, x_2, x_3) := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3) \) in the Euclidean plane \(\mathbb{E}^2\) such that \(d_{\mathbb{E}^2}(\bar{x}_i, \bar{y}_j) = d(x_i, y_j)\) for \(i, j \in \{1, 2, 3\}\).

A geodesic metric space is said to be a CAT(\(0\)) space if all geodesic triangles of appropriate size satisfy the following comparison axiom:

"Let \(\triangle\) be a geodesic triangle in \(X\) and let \(\overline{\triangle}\) be a comparison triangle for \(\triangle\). Then \(\overline{\triangle}\) is said to satisfy the CAT(\(0\)) inequality if for all \(x, y, \overline{\triangle}\) and all comparison points \(\bar{x}, \bar{y} \in \overline{\triangle}\),

\[
d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y}).
\]

**Definition 1.1.** ([1]) A hyperbolic space is a triple \((X, d, W)\) where \((X, d)\) is a metric space and \(W : X \times X \times [0, 1] \to X\) is such that

\(W1\) \(d(z, W(x, y, t)) \leq (1 - t)d(z, x) + td(z, y)\)

\(W2\) \(d(W(x, y, t), W(x, y, s)) = |t - s|d(x, y)\)

\(W3\) \(W(x, y, t) = W(y, x, 1 - t)\)

\(W4\) \(d(W(x, z, t), W(y, w, t)) \leq (1 - t)d(x, y) + td(z, w)\)

for all \(x, y, z, w \in X\) and \(t, s \in [0, 1]\).

If \(x, y \in X\) and \(t \in [0, 1]\) then we use the notation \((1 - t)x \oplus ty\) for \(W(x, y, t)\). We shall denote by \([x, y]\) the set \(\{(1 - t)x \oplus ty : t \in [0, 1]\}\). A nonempty subset \(C \subseteq X\) is convex if \([x, y] \subseteq C\) for all \(x, y \in C\).

We remark that any normed space \((X, \|\cdot\|)\) is a hyperbolic space, with

\[
(1 - t)x \oplus ty := (1 - t)x + ty.
\]

Here we recall a couple of lemmas which will be used next.

**Lemma 1.2.** ([2, Lemma 2.4]) Let \((X, d)\) be a CAT(\(0\)) space. Then

\[
d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z) \leq \max\{d(x, z), d(y, z)\},
\]

for \(x, y, z \in X\) and \(t \in [0, 1]\).

**Lemma 1.3.** ([2, Lemma 2.5]) Let \((X, d)\) be a CAT(\(0\)) space. Then

\[
d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2,
\]

for all \(x, y, z \in X\) and \(t \in [0, 1]\).

In particular by Lemma 1.3 we have

\[
d(z, \frac{1}{2}x + \frac{1}{2}y)^2 \leq \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2,
\]

for all \(x, y, z \in X\), which is called (CN) inequality of Bruhat-Tits, as it was shown in [3]. In fact (cf. [4], p. 163), a geodesic space is a CAT(\(0\)) space if and only if it satisfies the (CN) inequality.
2 Main results

Throughout this section we let \( n \in \mathbb{N} \), \( z_1 = x \) and \( z_n = y \) until Definition 3.2.

**Lemma 2.1.** Let \((X, d)\) be a \(\text{CAT}(0)\) space. Then

1. Let \( x, y \in X \), \( x \neq y \) and \( z_i, z'_i \in [x, y] \) such that \( d(x, z_i) = d(x, z'_i) \) for all \( 1 \leq i \leq n \). Then \( z_i = z'_i \) for \( 1 \leq i \leq n \).

2. Let \( x, y \in X \), then for each \( \alpha = (\alpha_1, \cdots, \alpha_n) \in [0, 1]^n \) with \( \sum_{i=1}^n \alpha_i = 1 \) there exist points \( z_1, \cdots, z_n \in [x, y] \) and unique point \( z \in [x, y] \) such that \( d(z, z_i) = \alpha_i d(x, y) \) for \( 1 \leq i \leq n \).

**Proof.** Since \( z_i, z'_i \in [x, y] \), there exist \( t_i, t'_i \in [0, l] \) such that \( c(t_i) = z_i \) and \( c(t'_i) = z'_i \). Thus \( d(x, z_i) = d(c(0), c(t_i)) = t_i \) and similarly \( d(x, z'_i) = t'_i \). Since \( d(x, z_i) = d(x, z'_i) \), we have \( t_i = t'_i \), and consequently \( z_i = z'_i \) for \( 1 \leq i \leq n \), which proves (1).

To prove (2), by [2, Lemma 2.1(iv)], this is true for \( n = 2 \), because for \( \alpha = (\alpha_1, \alpha_2) \) with \( \alpha_1 + \alpha_2 = 1 \) there exists unique point \( z \in [x, y] \) such that \( d(x, z) = \alpha_1 l, d(z, y) = \alpha_2 l \) that for convention we had shown with \( z = \alpha_1 x + \alpha_2 y \).

Now by induction let it holds for \( n - 1 \) and choose \( \alpha = (\alpha_1, \cdots, \alpha_n) \in [0, 1]^n \) such that \( \sum_{i=1}^{n-1} \alpha_i = 1 \). Put \( \beta_i := \frac{\alpha_i}{1-\alpha_n} \) for \( 1 \leq i \leq n - 1 \). Thus \( \sum_{i=1}^{n-1} \beta_i = 1 \) and by hypothesis of induction there exists unique point \( z' \in [z_{n-1}, z_n] \) such that \( d(z', z_i) = \beta_i l \) for \( 1 \leq i \leq n - 1 \), now there exists unique point \( z \in [z', z_n] \) such that \( d(z, z_n) = \alpha_n l, d(z, z') = (1-\alpha_n) l \).

To prove (2) directly, let \( t_i = 1 - \alpha_n - \alpha_i, t = 1 - \alpha_n \in [0, 1] \) for \( 1 \leq i \leq n \). Put \( z_i = c(t_i) \) and \( z = c(t) \) so \( d(z, z_i) = |t - t_i| l = \alpha_n l \) for \( 1 \leq i \leq n \). For uniqueness, if \( d(z, z_i) = d(z', z_i) \) for \( 1 \leq i \leq n \), then by (1) and \( i = 1 \), we have \( z = z' \). \( \square \)

**Example 2.2.** Let \( X = [0, 1] \) and put

\[
A = \left\{ (x, 0) : 0 \leq x \leq \frac{2}{3} \right\} \cup \left\{ \left( \frac{2}{3}, y \right) : -\frac{1}{6} \leq y \leq \frac{1}{6} \right\}.
\]

Define \( f : X \to A \subseteq \mathbb{R}^2 \) by

\[
f(x) = \begin{cases} 
(x, 0), & 0 \leq x \leq \frac{2}{3}; \\
\left( \frac{2}{3}, x - \frac{2}{3} \right), & \frac{2}{3} \leq x \leq \frac{5}{6}; \\
\left( \frac{2}{3}, x - \frac{5}{6} \right), & \frac{5}{6} \leq x \leq 1.
\end{cases}
\]

So \( f \) is isometric homeomorphism. For instance let \( \alpha_1 = \frac{2}{3}, \alpha_2 = \alpha_3 = \frac{1}{6} \). Therefore \( z_1 = x = 0, z_2 = \frac{2}{3}, z_3 = y = 1, z = \frac{5}{6} \) and \( l = 1 \). Since \( t = 1 - \alpha_3 = \frac{5}{6} \) and \( t_2 = 1 - \alpha_3 - \alpha_2 = \frac{2}{3} \), so \( z_2 = c(t_2) = \frac{2}{3}, z = \frac{5}{6} \) and by homeomorphism we have \( z_1 = (0, 0), z_2 = \left( \frac{2}{3}, \frac{1}{6} \right), z_3 = \left( \frac{5}{6}, \frac{1}{6} \right) \) and \( z = \left( \frac{5}{6}, 0 \right) \). And also we have \( d(z, z_i) = \alpha_i l \), for \( 1 \leq i \leq 3 \).
Notation: By the point $z_\alpha$, we mean the unique point

$$z_\alpha = \alpha_1 z_1 \oplus \alpha_2 z_2 \oplus \cdots \oplus \alpha_n z_n$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in [0, 1]^n$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and $z_i \in X$ for $1 \leq i \leq n$.

Also $z_\alpha$ can be written as

$$z_\alpha = (1 - \alpha_n) z' \oplus \alpha_n z_n,$$

where $z' = \frac{\alpha_1}{1 - \alpha_n} z_1 \oplus \cdots \oplus \frac{\alpha_{n-1}}{1 - \alpha_n} z_{n-1}$ where $\alpha_n \neq 1$.

Remark 2.3. Let $(X, d)$ be a $\text{CAT}(0)$ space, let $x, y \in X$ such that $x \neq y$ and $\alpha = (\alpha_1, \cdots, \alpha_n), \beta = (\beta_1, \cdots, \beta_n) \in [0, 1]^n$ with $\sum_{i=1}^{n} \alpha_i = 1 = \sum_{i=1}^{n} \beta_i$. Then

$$z_\alpha = z_\beta \iff \alpha = \beta.$$

Proof. This is true because,

$$d(z_\alpha, z_i) = d(z_\beta, z_i) \Rightarrow \alpha_i l = \beta_i l \Rightarrow \alpha_i = \beta_i,$$

for $1 \leq i \leq n$.

Theorem 2.4. Let $(X, d)$ be a $\text{CAT}(0)$ space, let $x, y \in X$ such that $x \neq y$ and $d(x, y) = l$. Then

1. $[x, y] = \{z_\alpha | \alpha \in [0, 1]^n, \sum_{i=1}^{n} \alpha_i = 1\}$.

2. For all $z \in X$ the following holds:

$$(\exists z_1, \cdots, z_n \in [x, y] \text{ such that } \sum_{i=1}^{n} d(z, z_i) = d(x, y)) \iff z \in [x, y].$$

3. The mapping $f : [0, 1]^n \to [x, y], f(\alpha) = z_\alpha$ is continuous and bijective.

Proof. (1) The case of $n = 2$ is proved in [2, Lemma 2.1]. Now let $z \in [x, y]$. By induction, suppose there exists $\beta \in [0, 1]^{n-1}$, such that $\sum_{i=1}^{n-1} \beta_i = 1$ and $z = z_\beta$. Put $\alpha_i = \beta_i$ for $1 \leq i \leq n - 2$ and $\alpha_{n-1} = \alpha_n = \frac{\beta_{n-1}}{2}$. Therefore $\sum_{i=1}^{n} \alpha_i = 1$ and there exists $z' = c(\frac{\beta_{n-1}}{2} l)$ that $d(z', x) = (\sum_{i=1}^{n-2} \beta_i + \frac{\beta_{n-1}}{2}) l$ and $d(z, z') = \frac{\beta_{n-1}}{2} l$. Now $z' = (\sum_{i=1}^{n-2} \beta_i + \frac{\beta_{n-1}}{2}) z_\beta \oplus \frac{\beta_{n-1}}{2} y$ thus $z' \in [x, y]$ and $d(z, z') = \alpha_n l$.

To prove (2) let for every $z \in X$ there exist $z_1, \cdots, z_n \in [x, y]$ such that $\sum_{i=1}^{n} d(z, z_i) = d(x, y)$. Put $\alpha_i = \frac{d(z, z_i)}{l}$ where $z_i \in [x, y]$ and $1 \leq i \leq n$, so there exists $z_\alpha$ such that $z_\alpha = z$. 


Conversely, if \( z \in [x, y] \) then \( z = z_\alpha \) for some \( \alpha \) and \( z_1, \ldots, z_n \) such that \( d(z, z_i) = \alpha_i d(x, y) \) so \( \sum_{i=1}^n d(z, z_i) = d(x, y) \).

To prove (3) applying (1) and Remark 2.3, we get that \( f \) is well defined and bijective. The continuity of \( f \) is obvious by induction, because \( f \) can be written as \( f(\alpha) = g(\beta) \oplus h(\alpha_n) \) where \( g(\beta) = z_\beta = \beta_1 z_1 \oplus \cdots \oplus \beta_{n-1} z_{n-1}, \beta_i := \frac{\alpha_i}{1-\alpha_n} \) for \( 1 \leq i \leq n - 1 \) and \( h(\alpha_n) = \alpha_n z_n \). \( \square \)

**Lemma 2.5.** Let \((X, d)\) be a CAT(0) space. Then

1. \( d(z_\alpha, z) \leq \sum_{i=1}^n \alpha_i d(z_i, z) \leq \max\{d(z_i, z) : 1 \leq i \leq n\} \),
2. \( d(z_\alpha, z)^2 \leq \sum_{i=1}^n \alpha_i d(z_i, z)^2 \leq \max\{d(z_i, z)^2 : 1 \leq i \leq n\} \),
3. \( d(z_\alpha, z'_\beta) \leq \sum_{i,j=1}^n \alpha_i \beta_j d(z_i, z'_j) \leq \max\{d(z_i, z'_j) : 1 \leq i, j \leq n\} \),

for \( \alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in [0,1]^n \) with \( \sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i = 1 \) and \( z, z_i, z'_j \in X \) for \( 1 \leq i, j \leq n \) which \( z_\alpha = \alpha_1 z_1 \oplus \alpha_2 z_2 \oplus \cdots \oplus \alpha_n z_n, z'_\beta = \beta_1 z'_1 \oplus \beta_2 z'_2 \oplus \cdots \oplus \beta_n z'_n \).

**Proof.** By Lemma 1.2 it is true for \( n = 2 \). So by induction let

\[
z_\alpha = \alpha_1 z_1 \oplus \alpha_2 z_2 \oplus \cdots \oplus \alpha_n z_n
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_n) \in (0,1)^n \) such that \( \sum_{i=1}^n \alpha_i = 1 \) and \( z_i \in X \) for \( 1 \leq i \leq n \).

Put \( \gamma := \left( \frac{\alpha_1}{1-\alpha_n}, \ldots, \frac{\alpha_{n-1}}{1-\alpha_n} \right) \) that \( \sum_{k=1}^{n-1} \frac{\alpha_k}{1-\alpha_n} = 1 \) by Theorem 2.1 there exists \( v_\gamma \in [x, z_{n-1}] \) such that \( v_\gamma = \frac{\alpha_1}{1-\alpha_n} z_1 \oplus \cdots \oplus \frac{\alpha_{n-1}}{1-\alpha_n} z_{n-1} \) and we have \( z_\alpha = (1-\alpha_n) v_\gamma + \alpha_n z_n \) so

\[
d(z_\alpha, z) = d((1-\alpha_n) v_\gamma \oplus \alpha_n z_n, z)
\]

\[
\leq (1-\alpha_n) d(v_\gamma, z) + \alpha_n d(z_n, z)
\]

\[
= (1-\alpha_n) d\left( \frac{\alpha_1}{1-\alpha_n} z_1 \oplus \cdots \oplus \frac{\alpha_{n-1}}{1-\alpha_n} z_{n-1}, z \right) + \alpha_n d(z_n, z)
\]

\[
\leq \sum_{i=1}^n \alpha_i d(z_i, z)
\]

\[
\leq \max\{d(z_i, z) : 1 \leq i \leq n\}. \square
\]

This proves (1).

(2) can easily proved according to Lemma 1.3 and again by induction on \( n \geq 2 \). \( \square \)

**Lemma 2.6.** Let \((X, d)\) be a hyperbolic space. Then

\[
d(z_\alpha, z'_\beta) \leq \sum_{i=1}^n \alpha_i d(z_i, z'_i) \leq \max\{d(z_i, z'_i) : 1 \leq i \leq n\},
\]

for \( \alpha = (\alpha_1, \ldots, \alpha_n) \in [0,1]^n \) with \( \sum_{i=1}^n \alpha_i = 1 \) and \( z_i, z'_i \in X \) for \( 1 \leq i \leq n \) which \( z_\alpha = \alpha_1 z_1 \oplus \alpha_2 z_2 \oplus \cdots \oplus \alpha_n z_n, z'_\alpha = \alpha_1 z'_1 \oplus \alpha_2 z'_2 \oplus \cdots \oplus \alpha_n z'_n \).
Proof. By the property of \((W4)\) it is true for \(n = 2\). The remaining is similar to the proof of the lemma 2.5. □

3 Fixed points and approximate fixed points for \(T_\alpha\) maps

In 2008 T. Suzuki [5], defined condition (C) for mappings on a subset of a Banach space, as following: "Let \(T\) be a mapping on a subset \(C\) of a Banach space \(E\). Then \(T\) is said to satisfy condition (C) if
\[
\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|
\]
for all \(x, y \in C\)."

This condition is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. In that paper, he has presented fixed point theorems and convergence theorems for mappings satisfying condition (C). Also Examples 1 and 2 in the same paper stated that there exists a map \(T\) which satisfies condition (C), but \(T\) is not nonexpansive, and there exists a map \(T\) which is quasi-nonexpansive, but it does not satisfy condition (C).

Recently B. Nanjaras, B. Panyanaka and W. Phuengrattana in [6], A. Razani and H. Salahifard in [7] and other mathematicians has proved some theorems according to single-valued mappings or multi-valued mappings which are satisfying Suzuki’s condition (C) in a \(CAT(0)\) space.

Some basic properties on condition (C) by [6, Propositions 3.2, 3.3], [7, Theorems 2.3, 2.7 and Corollary 2.8] and [8, Theorem 1.3] are:

\(P1\) ([6, Lemma 2.5]) Let \(\{x_n\}\) and \(\{y_n\}\) be bounded sequences in a \(CAT(0)\) space \(X\) and let \(\{\alpha_n\} \subseteq [0, 1)\) such that \(\sum_{n=1}^{\infty} \alpha_n = \infty\) and \(\limsup_{n} \alpha_n < 1\). Suppose that \(x_{n+1} = \alpha_n y_n \oplus (1 - \alpha_n) x_n\) and \(d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n)\) for all \(n \in \mathbb{N}\). Then \(\lim_{n \to \infty} d(y_n, x_n) = 0\).

\(P2\) ([6, Proposition 3.2]) Let \(K\) be a nonempty subset of a \(CAT(0)\) space \(X\). If \(T : K \to K\) be a nonexpansive mapping, then \(T\) satisfies condition (C).

\(P3\) ([6, Proposition 3.3]) Let \(K\) be a nonempty subset of a \(CAT(0)\) space \(X\). If \(T : K \to K\) satisfies condition (C) and has a fixed point, then \(T\) is a quasi-nonexpansive mapping.

\(P4\) ([7, Theorem 2.3]) Let \(K\) be a bounded closed convex subset of a complete \(CAT(0)\) space \(X\). If \(T : K \to K\) satisfies the condition (C) and \(F(T) \neq \emptyset\), then \(F(T)\) is \(\Delta\)-closed and convex set.
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P5 ([7, Theorem 2.7]) Let $K$ be a bounded closed convex subset of a complete CAT(0) space $X$. If $T : K \to K$ satisfies condition (C), then $F(T)$ is nonempty.

P6 ([7, Corollary 2.8]) Let $K$ be a bounded closed convex subset of a complete CAT(0) space $X$. If $T : K \to K$ satisfies condition (C), then $F(T)$ is nonempty, $\Delta$-closed and convex.

P7 ([8, Theorem 1.3]) Let $(X,d)$ be a convex subset of a CAT(0) space and $f : X \to X$ a quasi-nonexpansive map whose fixed point set is nonempty. Then $F(f)$ is closed, convex and hence contractible.

And now, we start our results by following definitions.

Definition 3.1. ([5]) Let $T$ be a mapping on a subset $K$ of a CAT(0) space $(X,d)$. Then $T$ is said to satisfy condition (C) if

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow d(Tx,Ty) \leq d(x,y),$$

for all $x,y \in K$.

The following we will use this notation $T_\alpha = \alpha_1 T_1 + \cdots + \alpha_n T_n$ where $T_1, \cdots, T_n : X \to [x,y]$ for $1 \leq i \leq n$ and $\alpha = (\alpha_1, \cdots, \alpha_n) \in [0,1]^n$ a multiindex satisfying $\sum_{i=1}^n \alpha_i = 1$.

Definition 3.2. ([9-10]) Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in [0,1]^n$ be a multiindex satisfying $\sum_{i=1}^n \alpha_i = 1$. The maps $T_1, \cdots, T_n$ on $X$ are said to be $\alpha$-nonexpansive if

$$\sum_{i=1}^n \alpha_i d(T_i x, T_i y) \leq d(x,y),$$

for all $x,y \in X$.

Theorem 3.3. Let $K$ be a bounded closed convex subset of a complete CAT(0) space $(X,d)$. If $T_\alpha : K \to K$ is defined by $T_\alpha = \alpha_1 T_1 + \cdots + \alpha_n T_n$ which $T_1, \cdots, T_n$ are selfmaps on $K$, which commute each other and satisfy condition (C), then $T_\alpha$ has a fixed point.

Proof. By $P5$, $F(T_i) \neq \emptyset$ for $1 \leq i \leq n$. We say $\bigcap_{i=1}^n F(T_i) \neq \emptyset$. By induction we assume that $L := \bigcap_{i=1}^{n-1} F(T_i) \neq \emptyset$. Let $x \in L$ so we have

$$T_n x = T_n(T_1 x) = T_1(T_n x),$$

thus $T_n x \in F(T_i)$ for $1 \leq i \leq n - 1$. Therefore $T_n x \in L$ hence $T_n(L) \subseteq L$. By $P6$, $F(T_i)$ nonempty and convex and since $T_i$ satisfy the condition (C) by $P3$, $T_i$ is a quasi-nonexpansive map and by $P7$, $F(T_i)$ closed and convex, for $(1 \leq i \leq n)$, therefore $L$ and $F(T_n)$ are nonempty, bounded closed convex subsets of a complete
Thus $T : L \to L$ satisfies the condition of the $P4$, hence $T_n x$ has a fixed point in $L$, that is,

$$L \cap F(T_n) = \bigcap_{i=1}^{n} F(T_i) \neq \emptyset.$$ 

If we let $x \in \bigcap_{i=1}^{n} F(T_i)$, then

$$d(x, T_\alpha x) \leq \sum_{i=1}^{n} \alpha_i d(x, T_i x) = 0,$$

namely $x \in F(T_\alpha). \square$

**Theorem 3.4.** Let $K$ be a bounded closed convex subset of a complete $CAT(0)$ space $(X, d)$. If $T_\alpha : K \to K$ defined by $T_\alpha = \alpha_1 T_1 + \cdots + \alpha_n T_n$ which $T_1, \ldots, T_n$ are selfmaps on $K$, which $T_1$ satisfies the condition (C) and $d(T_n x, x) \leq d(T_1 x, x)$ for every $x \in K$, then $\inf_{x \in K} d(x, T_\alpha x) = 0$.

**Proof.** Let $x_1 \in K$, define sequence $\{x_n\} \subseteq K$ by $x_{n+1} := t T_1 x_n + (1-t) x_n$ for $n \in \mathbb{N}$, where $t \in \left[ \frac{1}{2}, 1 \right)$. Then by the assumption $\frac{1}{2} d(x_n, T_1 x_n) \leq t d(x_n, T_1 x_n) = d(x_n, x_{n+1})$ for $n \in \mathbb{N}$ hence $d(T_1 x_{n+1}, T_1 x_n) \leq d(x_{n+1}, x_n)$. So by $P1$ we have $\inf_{x \in K} d(x, T_1 x) = 0$. So

$$d(x, T_\alpha x) \leq d(x, T_1 x) + d(T_1 x, T_\alpha x),$$

$$= d(x, T_1 x) + \alpha_1 d(T_1 x, T_\alpha x),$$

$$\leq d(x, T_1 x) + d(T_1 x, x) + d(x, T_\alpha x),$$

$$\leq 3d(x, T_1 x),$$

therefore there exists $\{x_n\} \subseteq K$ such that $d(x_n, T_1 x_n) \to 0$ as $n \to \infty$ thus $d(x_n, T_\alpha x_n) \to 0$. □

**Corollary 3.5.** ([7, Lemma 2.5]) Let $K$ be a bounded closed convex subset of a complete $CAT(0)$ space $(X, d)$. If $T : K \to K$ satisfies the condition (C), then there exists an approximate fixed point sequence for $T$, i.e., $\inf_{x \in K} d(x, T x) = 0$.

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**References**


