Redefined (anti) fuzzy $BM$-algebras

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Abstract

In this paper by using the notion of anti fuzzy points and its besideness to and non-quasi-coincidence with a fuzzy set the concepts of an anti fuzzy subalgebras in $BM$-algebras are generalized and their inter-relations and related properties are investigated.

Keywords: non-quasi coincident, $\left(\alpha, \beta\right)^*$-fuzzy subalgebra, $BM$-algebras.

1 Introduction

Y. Imai and K. Iseki introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras [6, 7]. It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. In [4, 5] Q. P. Hu and X. Li introduced a wide class of abstract algebras: $BCH$-algebras. They have shown that the class of $BCI$-algebras is a proper subclass of the class of $BCH$-algebras. J. Neggers and H. S. Kim [13] introduced the notion of $d$-algebras which is another generalization of $BCK$-algebras, and also they introduced the notion of $B$-algebras [14, 15]. Moreover, Y. B. Jun, E. H. Roh and H. S. Kim [11] introduced a new notion, called a $BH$-algebra, which is a generalization of $BCH/BCI/BCK$-algebras. Walendziak obtained the another equivalent axioms for $B$-algebra [18]. H. S. Kim, Y. H. Kim and J. Neggers [9] introduced the notion a (pre-) Coxeter algebra and showed that a Coxeter algebra is equivalent to an abelian group all of whose elements have order 2, i.e., a Boolean group. C. B. Kim and H. S. Kim [8] introduced the notion of a $BM$-algebra which is a specialization of $B$-algebras.

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The concept of a fuzzy set was introduced in [19] by L. A. Zadeh. Since then it has become a vigorous area of research in engineering, medical science, social science, physics, statistics, graph theory, etc. In this paper, we introduce the concept of an anti fuzzy subalgebra of $BM$-algebras by using the notion of anti fuzzy points and its besideness to and non-quasi-coincidence with a fuzzy set, and investigate their inter-relations and related properties.

2 Preliminaries

Definition 2.1. [8] A $BM$-algebra is a non-empty set $X$ with a consonant $0$ and a binary operation $*$ satisfying the following axioms:

(I) $x * 0 = x$,
(II) $(z * x) * (z * y) = y * x$,
for all $x, y, z \in X$.

In $X$ we can define a binary relation by $x \leq y$ if and only if $x * y = 0$.

Proposition 2.2. [8] Let $X$ be a $BM$-algebra. Then for any $x, y$ and $z$ in $X$, the following hold:

(a) $x * x = 0$,
(b) $0 * (0 * x) = x$,
(c) $0 * (x * y) = y * x$,
(d) $(x * z) * (y * z) = x * y$,
(e) $x * y = 0$ if and only if $y * x = 0$,
(f) $(x * y) * z = (x * z) * y$.

Definition 2.3. A non-empty subset $S$ of a $BM$-algebra $X$ is called a subalgebra of $X$ if $x * y \in S$ for any $x, y \in S$.

A mapping $f : X \rightarrow Y$ of $BM$-algebras is called a $BM$-homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

We now review some fuzzy logic concept (see [19]).

Let $X$ be a set. A fuzzy set $A$ in $X$ is characterized by a membership function $\mu_A : X \rightarrow [0, 1]$. Let $f$ be a mapping from the set $X$ to the set $Y$ and let $B$ be a fuzzy set in $Y$ with membership function $\mu_B$.

The inverse image of $B$, denoted $f^{-1}(B)$, is the fuzzy set in $X$ with membership function $\mu_{f^{-1}(B)}$ defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all $x \in X$.

Conversely, let $A$ be a fuzzy set in $X$ with membership function $\mu_A$. Then the image of $A$, denoted by $f(A)$, is the fuzzy set in $Y$ such that:

$$
\mu_{f(A)}(y) = \begin{cases} 
\sup_{z \in f^{-1}(y)} \mu_A(z) & \text{if } f^{-1}(y) = \{x : f(x) = y\} \neq \emptyset, \\
0 & \text{otherwise}
\end{cases}
$$
A fuzzy set $\mathcal{A}$ in $X$ of the form

$$\mathcal{A}(y) := \begin{cases} t \in [0,1) & \text{if } y = x, \\ 1 & \text{if } y \neq x \end{cases}$$

is called an anti fuzzy point with support $x$ and value $t$ and is denoted by $x_t$. A fuzzy set $\mathcal{A}$ in $X$ is said to be non-unit if there exists $x \in X$ such that $\mathcal{A}(x) < 1$.

A fuzzy set $\mathcal{A}$ in a BM-algebra $X$ is called an anti-fuzzy subalgebra of $X$ if it satisfies

$$\forall x, y \in X \ (\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y)\}).$$

(2.1)

3 Redefined (anti) fuzzy subalgebras

From now $(X, *, 0)$ or simply $X$ is a BM-algebra.

Definition 3.1. An anti-fuzzy point $x_t$ is said to beside to (resp. be non-quasi coincident with) a fuzzy set $\mathcal{A}$, denoted by $x_t \prec \mathcal{A}$ (resp. $x_t \prec \mathcal{A}$), if $\mathcal{A}(x) < t$ (resp. $\mathcal{A}(x) + t < 1$). We say that $\prec$ (resp. $\prec$) is a beside to relation (resp. non-quasi coincident with relation) between anti-fuzzy points and fuzzy sets.

If $x_t \prec \mathcal{A}$ or $x_t \prec \mathcal{A}$ (resp. $x_t \prec \mathcal{A}$ and $x_t \prec \mathcal{A}$), we say that $x_t \prec \mathcal{A}$ (resp. $x_t \prec \mathcal{A}$).

Proposition 3.2. Let $\mathcal{A}$ be a fuzzy set in a BM-algebra $X$. Then $\mathcal{A}$ satisfies the condition (2.1) if and only if it satisfies the following condition.

$$(\forall x, y \in X) \ (\forall t_1, t_2 \in [0,1)) \ (x_{t_1}, y_{t_2} \prec \mathcal{A} \Rightarrow (x * y)_{\max\{t_1, t_2\}} \prec \mathcal{A}).$$

(3.1)

Proof. Assume that $\mathcal{A}$ satisfies the condition (2.1). Let $x, y \in X$ and $t_1, t_2 \in [0,1)$ satisfy $x_{t_1}, y_{t_2} \prec \mathcal{A}$. Then $\mathcal{A}(x) \leq t_1$ and $\mathcal{A}(y) \leq t_2$. Using (2.1) induces that

$$\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y)\} \leq \max\{t_1, t_2\}.$$}

Hence $(x * y)_{\max\{t_1, t_2\}} \prec \mathcal{A}$.

Conversely, suppose that the condition (3.1) is valid. Since $x_{\mathcal{A}(x)} \prec \mathcal{A}$ and $y_{\mathcal{A}(y)} \prec \mathcal{A}$ for all $x, y \in X$, it follows from (3.1) that

$$(x * y)_{\max\{\mathcal{A}(x), \mathcal{A}(y)\}} \prec \mathcal{A}$$

so that $\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y)\}$. This completes the proof.

Note that if $\mathcal{A}$ is a fuzzy set in $X$ such that $\mathcal{A}(x) \geq 0.5$ for all $x \in X$, then the set \{x_t | x_t \prec \mathcal{A}\} is empty. In what follows let $\alpha$ and $\beta$ denote any one of $\prec$, $\prec$, $\prec \lor \mathcal{A}$, or $\prec \land \mathcal{A}$ unless otherwise specified. To say that $x_t \alpha \mathcal{A}$ means that $x_t \alpha \mathcal{A}$ does not hold.
Definition 3.3. A fuzzy set $\mathcal{A}$ in a $BM$-algebra $X$ is called an $(\alpha, \beta)^*$-fuzzy subalgebra of $X$, where $\alpha \neq \emptyset \land \mathcal{Y}$, if it satisfies the following implication:

$$(\forall x, y \in X) (\forall t_1, t_2 \in [0, 1]) (x \cdot t_1 \alpha \mathcal{A}, y \cdot t_2 \alpha \mathcal{A} \Rightarrow (x \cdot y)_{\max\{t_1, t_2\}} \beta \mathcal{A}). \quad (3.2)$$

Example 3.4. [3] Let $X = \{0, 1, 2\}$ be a set with the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
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<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(X, *, 0)$ is a $BM$-algebra. Let $\mathcal{A}$ be a fuzzy set in $X$ defined by $\mathcal{A}(0) = 0.4$, $\mathcal{A}(1) = 0.3$, and $\mathcal{A}(2) = 0.7$. It is routine to verify that $\mathcal{A}$ is a $(\emptyset, \emptyset \lor \mathcal{Y})^*$-fuzzy subalgebra of $X$.

Theorem 3.5. In a $BM$-algebra, every $(\emptyset \lor \mathcal{Y}, \emptyset \lor \mathcal{Y})^*$-fuzzy subalgebra is a $(\emptyset, \emptyset \lor \mathcal{Y})^*$-fuzzy subalgebra.

Proof. Let $\mathcal{A}$ be a $(\emptyset \lor \mathcal{Y}, \emptyset \lor \mathcal{Y})^*$-fuzzy subalgebra of a $BM$-algebra $X$. Let $x, y \in X$ and $t_1, t_2 \in [0, 1]$ satisfy $x \cdot t_1 \emptyset \mathcal{A}$ and $y \cdot t_2 \emptyset \mathcal{A}$. Then $x \cdot t_1 \emptyset \mathcal{Y} \mathcal{A}$ and $y \cdot t_2 \emptyset \mathcal{Y} \mathcal{A}$, which imply that $(x \cdot y)_{\max\{t_1, t_2\}} \emptyset \mathcal{Y} \mathcal{A}$. Hence $\mathcal{A}$ is a $(\emptyset, \emptyset \lor \mathcal{Y})^*$-fuzzy subalgebra of $X$.

The converse of Theorem 3.5 is not true in general. For example, the $(\emptyset, \emptyset \lor \mathcal{Y})^*$-fuzzy subalgebra $\mathcal{A}$ of $X$ in Example 3.4 is not a $(\emptyset \lor \mathcal{Y}, \emptyset \lor \mathcal{Y})^*$-fuzzy subalgebra of $X$ since $1_{0.5} \emptyset \lor \mathcal{Y} \mathcal{A}$ and $0_{0.4} \emptyset \lor \mathcal{Y} \mathcal{A}$, but $(0 \cdot 1)_{\max\{0.5, 0.4\}} = 2_{0.5} \emptyset \lor \mathcal{Y} \mathcal{A}$.

Obviously any $(\emptyset, \emptyset \lor \mathcal{Y})^*$-fuzzy subalgebra is a $(\emptyset, \emptyset \lor \mathcal{Y})^*$-fuzzy subalgebra, but the converse is not true. For example, the $(\emptyset, \emptyset \lor \mathcal{Y})^*$-fuzzy subalgebra $\mathcal{A}$ of $X$ in Example 3.4 is not a $(\emptyset, \emptyset \lor \mathcal{Y})^*$-fuzzy subalgebra, but $(\emptyset \cdot 1)_{\max\{0.34, 0.38\}} = 0_{0.38} \emptyset \mathcal{A}$.

Also $\mathcal{A}$ is a $(\emptyset, \emptyset \lor \mathcal{Y})^*$-fuzzy subalgebra of $X$ may not be a $(\mathcal{Y}, \emptyset \lor \mathcal{Y})^*$-fuzzy subalgebra. For example, the $(\emptyset, \emptyset \lor \mathcal{Y})^*$-fuzzy subalgebra $\mathcal{A}$ of $X$ in Example 3.4 is not a $(\mathcal{Y}, \emptyset \lor \mathcal{Y})^*$-fuzzy subalgebra of $X$ since $1_{0.38} \emptyset \mathcal{A}$ and $1_{0.34} \emptyset \mathcal{A}$, but $(1 \cdot 2)_{\max\{0.6, 0.1\}} = 2_{0.6} \emptyset \lor \mathcal{Y} \mathcal{A}$.

Theorem 3.6. Let $\mathcal{A}$ be a fuzzy set in a $BM$-algebra $X$. Then the left diagram shows the relationship between $(\alpha, \beta)^*$-fuzzy subalgebras of $X$, where $\alpha, \beta$ are one of $\emptyset$ and $\mathcal{Y}$. Also we have the right diagram.
Proposition 3.7. Let $A$ be a fuzzy set in a $BM$-algebra $X$ which is non-unit. If $A$ is an $(\alpha, \beta)^*$-fuzzy subalgebra of $X$, then $A(0) < 1$.

Proof. Assume that $A(0) = 1$. Since $A$ is non-unit, there exists $x \in X$ such that $A(x) = t < 1$. If $\alpha = \varepsilon$ or $\alpha = \varepsilon \lor T$, then $x_0 \alpha A$, but $(x * x)_{\text{max}(t, t)} = 0 \beta A$. This is a contradiction. If $\alpha = T$, then $x_0 \alpha A$ because $A(x) + 0 = t + 0 = t < 1$. But $(x * x)_{\text{max}(0, 0)} = 0 \beta A$, which is a contradiction. Hence $A(0) < 1$.

Proposition 3.8. Let $A$ be a fuzzy set in a $BM$-algebra $X$. If $A$ is a $(\varepsilon, \varepsilon)^*$-fuzzy subalgebra of $X$, then $A(0) \leq A(x)$, for all $x \in X$.

Proof. Since $x * x = 0$, for all $x \in X$. Then we get that $A(0) = A(x * x) \leq \text{max}(A(x), A(x)) = A(x)$.

For a fuzzy set $A$ in a $BM$-algebra $X$, we denote

$$X^* := \{x \in X \mid A(x) < 1\}.$$ 

Theorem 3.9. Let $A$ be a fuzzy set in a $BM$-algebra $X$ which is non-unit. If $A$ is an $(\alpha, \beta)^*$-fuzzy subalgebra of $X$ where $(\alpha, \beta)$ is one of the following:

- $(\varepsilon, \varepsilon)$,
- $(\varepsilon, T)$,
- $(T, \varepsilon)$,
- $(T, T)$,

then the set $X^*$ is a subalgebra of $X$.

Proof. (i) Assume that $A$ is a $(\varepsilon, \varepsilon)^*$-fuzzy subalgebra of $X$. Let $x, y \in X^*$. Then $A(x) < 1$ and $A(y) < 1$. Assume that $A(x * y) = 1$. Note that $x_0 A(x) \varepsilon A$ and $y_0 A(y) \varepsilon A$. But, since $A(x * y) = 1 > \text{max}(A(x), A(y))$, we get $(x * y)_{\text{max}(x, y)} T \subseteq A$. This is a contradiction, and so $A(x * y) < 1$ which shows that $x * y \in X^*$. Hence $X^*$ is a subalgebra of $X$.

(ii) Assume that $A$ is a $(\varepsilon, T)^*$-fuzzy subalgebra of $X$. Let $x, y \in X^*$. Then $A(x) < 1$ and $A(y) < 1$. If $A(x * y) = 1$, then

$$A(x * y) + \text{max}(A(x), A(y)) \geq 1.$$

Hence $(x * y)_{\text{max}(A(x), A(y))} T A$, which is a contradiction since $x_0 A(x) \varepsilon A$ and $y_0 A(y) \varepsilon A$. Thus $A(x * y) < 1$, and so $x * y \in X^*$. Therefore $X^*$ is a subalgebra of $X$.

(iii) Assume that $A$ is a $(T, \varepsilon)^*$-fuzzy subalgebra of $X$. Let $x, y \in X^*$. Then $A(x) < 1$ and $A(y) < 1$. Thus $x_0 T A$ and $y_0 T A$. If $A(x * y) = 1$, then $A(x * y) = 1 > 0 = \text{max}(0, 0)$. Therefore $(x * y)_{\text{max}(0, 0)} T A$, which is a contradiction. Hence $A(x * y) < 1$, and so $x * y \in X^*$.

(iv) Assume that $A$ is a $(T, T)^*$-fuzzy subalgebra of $X$. Let $x, y \in X^*$. Then $A(x) < 1$ and $A(y) < 1$. If $A(x * y) = 1$, then $A(x * y) + \text{max}(0, 0) = 1$ and so $(x * y)_{\text{max}(0, 0)} T A$. This is impossible, and hence $A(x * y) < 1$, i.e., $x * y \in X^*$. This completes the proof.
Corollary 3.10. Let $\mathcal{A}$ be a fuzzy set in a $BM$-algebra $X$ which is non-unit. If $\mathcal{A}$ is an $(\alpha, \beta)^*$-fuzzy subalgebra of $X$ where $(\alpha, \beta)$ is one of the following:

- $(\ll, \ll \land \Upsilon)$,  
- $(\Upsilon, \ll \land \Upsilon)$,  
- $(\ll \lor \Upsilon, \ll \lor \Upsilon)$,  
- $(\ll \lor \Upsilon, \ll \land \Upsilon)$,

then the set $X^*$ is a subalgebra of $X$.

Proof. By Theorem 3.6, it is enough to prove for the cases:

1. $(\ll, \ll \lor \Upsilon)$ and $(\Upsilon, \ll \lor \Upsilon)$.

(i) Let $x, y \in X^*$. Then $\mathcal{A}(x) < 1$ and $\mathcal{A}(y) < 1$, and so $\mathcal{A}(x) = t_1$ and $\mathcal{A}(y) = t_2$ for some $t_1, t_2 \in [0, 1)$. It follows that $x_{t_1} \ll \mathcal{A}$ and $y_{t_2} \ll \mathcal{A}$ so that $(x \ast y)_{\max\{t_1, t_2\}} \ll \Upsilon \mathcal{A}$, i.e., $(x \ast y)_{\max\{t_1, t_2\}} \ll \mathcal{A}$ or $(x \ast y)_{\max\{t_1, t_2\}} \Upsilon \mathcal{A}$. If $(x \ast y)_{\max\{t_1, t_2\}} \ll \mathcal{A}$, then $\mathcal{A}(x \ast y) \leq \max\{t_1, t_2\} < 1$ and thus $x \ast y \in X^*$. If $(x \ast y)_{\max\{t_1, t_2\}} \Upsilon \mathcal{A}$, then $\mathcal{A}(x \ast y) \leq \mathcal{A}(x \ast y) + \max\{t_1, t_2\} < 1$. Hence $x \ast y \in X^*$. For the case (ii), let $x, y \in X^*$. Then $\mathcal{A}(x) < 1$ and $\mathcal{A}(y) < 1$, which imply that $x_0 \Upsilon \mathcal{A}$ and $y_0 \Upsilon \mathcal{A}$. Since $\mathcal{A}$ is a $(\Upsilon, \ll \lor \Upsilon)^*$-fuzzy subalgebra, $(x \ast y)_0 = (x \ast y)_{\max\{0, 0\}} \ll \Upsilon \mathcal{A}$, i.e., $(x \ast y)_0 \ll \mathcal{A}$ or $(x \ast y)_0 \Upsilon \mathcal{A}$. If $(x \ast y)_0 \ll \mathcal{A}$, then $\mathcal{A}(x \ast y) = 0 < 1$. If $(x \ast y)_0 \Upsilon \mathcal{A}$, then $\mathcal{A}(x \ast y) = \mathcal{A}(x \ast y) + 0 < 1$. Therefore $x \ast y \in X^*$. This completes the proof.

Theorem 3.11. Let $\mathcal{A}$ be a fuzzy set in a $BM$-algebra $X$ which is non-unit. Then every $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of $X$ is a constant on $X^*$.

Proof. Let $\mathcal{A}$ be a $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of $X$ which is non-unit. Assume that $\mathcal{A}$ is not constant on $X^*$. Then there exists $y \in X^*$ such that $t_y = \mathcal{A}(y) \neq \mathcal{A}(0) = t_0$. Then either $t_y > t_0$ or $t_y < t_0$. If $t_y < t_0$, then $\mathcal{A}(y) + (1 - t_0) = t_0 + 1 - t_0 < 1$ and so $y_{1-t_0} \Upsilon \mathcal{A}$. Since

$$\mathcal{A}(y \ast y) + (1 - t_0) = \mathcal{A}(0) + 1 - t_0 = t_0 + 1 - t_0 = 1,$$

we have $(y \ast y)_{\max\{1-t_0,1-t_0\}} \Upsilon \mathcal{A}$. This is a contradiction. Now assume that $t_y > t_0$. Choose $t_1, t_2 \in (0, 1)$ such that $t_1 < 1 - t_y < t_2 < 1 - t_0$. Then $\mathcal{A}(0) + t_2 = t_0 + t_2 < 1$ and $\mathcal{A}(y) + t_1 = t_y + t_1 < 1$. Thus $y_{t_1} \Upsilon \mathcal{A}$ and $y_{t_2} \Upsilon \mathcal{A}$. Since

$$\mathcal{A}(y \ast 0) + \max\{t_1, t_2\} = \mathcal{A}(y) + t_2 = t_y + t_2 > 1,$$

we get $(y \ast 0)_{\max\{t_1, t_2\}} \Upsilon \mathcal{A}$, which is a contradiction. Therefore $\mathcal{A}$ is a constant on $X^*$.

Theorem 3.12. Let $\mathcal{A}$ be a fuzzy set in a $BM$-algebra $X$. Then $\mathcal{A}$ is a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of $X$ if and only if there exists a subalgebra $S$ of $X$ such that

$$\mathcal{A}(x) := \begin{cases} t \in [0, 1) & \text{if } x \in S, \\ 1 & \text{otherwise} \end{cases} \quad (3.3)$$
Let $A$ be a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of $X$. Then by Proposition 3.7 and Theorems 3.11 and 3.9 we get that $A(x) < 1$, for all $x \in X$ and $x^*$ is a subalgebra of $X$, and

$$A(x) := \begin{cases} A(0) & \text{if } x \in X^*, \\ 1 & \text{otherwise} \end{cases}$$

Conversely, let $S$ be a subalgebra of $X$ which satisfy (3.3). Assume that $x_s \Upsilon A$ and $y_r \Upsilon A$ for some $s, r \in [0, 1)$. Then $A(x) + s < 1$ and $A(y) + r < 1$, and so $A(x) \neq 1$ and $A(y) \neq 1$. Thus $x, y \in S$ and so $x * y \in S$. It follows that $A(x * y) + \max\{s, r\} = t + \max\{s, r\} < 1$ so that $(x * y)_{\max\{s, r\}} \Upsilon A$. Therefore $A$ is a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of $X$.

**Theorem 3.13.** Let $S$ be a subalgebra of a $BM$-algebra $X$ and let $A$ be a fuzzy set in $X$ such that

(i) $(\forall x \in X \setminus S) (A(x) = 1),$

(ii) $(\forall x \in S) (A(x) \leq 0.5).$

Then $A$ is a $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of $X$.

**Proof.** Let $x, y \in X$ and $t_1, t_2 \in [0, 1)$ be such that $x_t \Upsilon A$ and $y_t \Upsilon A$, that is, $A(x) + t_1 < 1$ and $A(y) + t_2 < 1$. If $x * y \notin S$, then $x \in X \setminus S$ or $y \in X \setminus S$, i.e., $A(x) = 1$ or $A(y) = 1$. It follows that $t_1 < 0$ or $t_2 < 0$. This is a contradiction, and so $x * y \in S$. Hence $A(x * y) \leq 0.5$. If $\max\{t_1, t_2\} < 0.5$, then $A(x * y) + \max\{t_1, t_2\} < 1$ and thus $(x * y)_{\max\{t_1, t_2\}} \Upsilon A$. If $\max\{t_1, t_2\} \geq 0.5$, then $A(x * y) \leq 0.5 \leq \max\{t_1, t_2\}$ and so $(x * y)_{\max\{t_1, t_2\}} \Upsilon A$. Therefore $(x * y)_{\max\{t_1, t_2\}} \Upsilon A$. This completes the proof.

**Theorem 3.14.** Let $A$ be a $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of a $BM$-algebra $X$ such that $A$ is not constant on $X^*$. Then there exists $x \in X$ such that $A(x) \leq 0.5$. Moreover $A(x) \leq 0.5$ for all $x \in X^*$.

**Proof.** Assume that $A(x) > 0.5$ for all $x \in X$. Since $A$ is not constant on $X^*$, there exists $x \in X^*$ such that $t_x = A(x) \neq A(0) = t_0$. Then either $t_0 > t_x$ or $t_0 < t_x$. For the first case, choose $\delta < 0.5$ such that $t_x + \delta < 1 < t_0 + \delta$. It follows that $x_\delta \Upsilon A$,

$$A(x * x) = A(0) = t_0 > \delta = \max\{\delta, \delta\},$$

so that $(x * x)_{\max\{\delta, \delta\}} \Upsilon A$. This is a contradiction. For the second case, we can choose $\delta < 0.5$ such that $t_x + \delta > 1 > t_0 + \delta$. Then $0_\delta \Upsilon A$ and $x_1 \Upsilon A$, but $(x * 0)_{\max\{1, \delta\}} = x_1 \Upsilon A$ since $A(x) > 0.5 > \delta$ and $A(x) + \delta = t_x + \delta > 1$. This leads to a contradiction. Therefore $A(x) \leq 0.5$ for some $x \in X$. We now show that $A(0) \leq 0.5$. Assume that $A(0) = t_0 > 0.5$. Since there exists $x \in X$ such that $A(x) = t_x \leq 0.5$, we have $t_0 > t_x$. Choose $t_1 < t_0$ such that $t_x + t_1 < 1 < t_0 + t_1$. Then $A(x) + t_1 = t_x + t_1 < 1$, and so $x_1 \Upsilon A$. Now we get

$$A(x * x) + \max\{t_1, t_1\} = A(0) + t_1 = t_0 + t_1 > 1,$$
Hence \((x * x)_{\max(t_1, t_2)} \leq \vee t \wedge A\), a contradiction. Therefore \(A(0) \leq 0.5\). Finally suppose that \(t_x = A(x) > 0.5\) for some \(x \in X^*\). Let \(t\) be such that \(0 < t < 0.5\) and \(t_x > 0.5 + t\). Therefore \(A(x) + 0 < 1\) and \(A(0) + (0.5 - t) < 1\) which imply that \(x_0 \wedge A\) and \(0 \wedge (0.5 - t) \wedge A\).

But \((x * 0)_{\max(0.5 - t)} = x_{0.5 - t}\) and so \(A(x) > 0.5 - t\) and \(A(x) + 0.5 - t > 1\), thus \((x * 0)_{0.5 - t} \leq \vee t \wedge A\), which is a contradiction. Hence \(A(x) \leq 0.5\).

We give a characterization of a \((\leq, \leq \vee T)^*\)-fuzzy subalgebra.

**Theorem 3.15.** Let \(A\) be a fuzzy set in a \(BM\)-algebra \(X\). Then \(A\) is a \((\leq, \leq \vee T)^*\)-fuzzy subalgebra of \(X\) if and only if it satisfies the following inequality.

\[
(\forall x, y \in X) \ (A(x * y) \leq \max\{A(x), A(y), 0.5\}). \tag{3.4}
\]

**Proof.** Assume that \(A\) is a \((\leq, \leq \vee T)^*\)-fuzzy subalgebra of \(X\). Let \(x, y \in X\) be such that \(\max\{A(x), A(y)\} > 0.5\). Then \(A(x * y) \leq \max\{A(x), A(y)\}\). If it is not true, then \(A(x * y) < t < \max\{A(x), A(y)\}\) for some \(t \in (0.5, 1)\). It follows that \(x_t < A\) and \(y_t < A\), but \((x * y)_{\max(t, t)} = (x * y)_{t < \vee t \wedge A}\) which is a contradiction. Hence \(A(x * y) \leq \max\{A(x), A(y)\}\) whenever \(\max\{A(x), A(y)\} > 0.5\). If \(\max\{A(x), A(y)\} \leq 0.5\), then \(x_0.5 < A\) and \(y_0.5 < A\) which imply that \((x * y)_{0.5} = (x * y)_{\max(0.5, 0.5)} < \vee t \wedge A\). Therefore \(A(x * y) \leq 0.5\) because if \(A(x * y) > 0.5\), then \(A(x * y) + 0.5 > 0.5 + 0.5 = 1\), a contradiction. Hence \(A(x * y) \leq \max\{A(x), A(y), 0.5\}\) for all \(x, y \in X\).

Conversely, assume that \(A\) satisfies (3.4). Let \(x, y \in X\) and \(t_1, t_2 \in [0, 1]\) be such that \(x_{t_1} < A\) and \(y_{t_2} < A\). Then \(A(x) \leq t_1\) and \(A(y) \leq t_2\). Suppose that \(A(x * y) > \max\{t_1, t_2\}\). If \(\max\{A(x), A(y)\} > 0.5\) then \(A(x * y) \leq \max\{A(x), A(y), 0.5\} = \max\{A(x), A(y)\} \leq \max\{t_1, t_2\}\).

This is a contradiction, and so \(\max\{A(x), A(y)\} \leq 0.5\). It follows that

\[
A(x * y) + \max\{t_1, t_2\} < 2A(x * y) \leq 2 \max\{A(x), A(y), 0.5\} \leq 1
\]

so that \((x * y)_{\max\{t_1, t_2\}} A\). Hence \((x * y)_{\max\{t_1, t_2\}} \leq \vee t \wedge A\), and consequently \(A\) is a \((\leq, \leq \vee T)^*\)-fuzzy subalgebra of \(X\).

**Theorem 3.16.** For any subset \(S\) of a \(BM\)-algebra \(X\), let \(\chi_S\) denote the characteristic function of \(S\). Then the function \(\chi_S^e : X \to [0, 1]\) defined by \(\chi_S^e(x) = 1 - \chi_S(x)\) for all \(x \in X\) is a \((\leq, \leq \vee T)^*\)-fuzzy subalgebra of \(X\) if and only if \(S\) is a subalgebra of \(X\).

**Proof.** Assume that \(\chi_S^e\) is a \((\leq, \leq \vee T)^*\)-fuzzy subalgebra of \(X\) and let \(x, y \in S\). Then \(\chi_S(x) = 1 - \chi_S^e(x) = 0\) and \(\chi_S^e(y) = 1 - \chi_S(y) = 0\). Hence \(x_0 < \chi_S^e\) and \(y_0 < \chi_S^e\), which imply that \((x * y)_{0} = (x * y)_{\max(0, 0)} < \vee t \wedge A\). Thus \(\chi_S(x * y) \leq 0\) or \(\chi_S^e(x * y) + 0 < 1\). If \(\chi_S(x * y) \leq 0\), then \(1 - \chi_S(x * y) = 0\), i.e., \(\chi_S(x * y) = 1\). Hence \(x * y \in S\). If \(\chi_S^e(x * y) + 0 < 1\), then \(\chi_S(x * y) > 0\). Thus \(\chi_S(x * y) = 1\), and so \(x * y \in S\). Therefore \(S\) is a subalgebra of \(X\).

Conversely, suppose that \(S\) is a subalgebra of \(X\). Let \(x, y \in X\). If \(x, y \in S\), then \(x * y \in S\), and thus

\[
\chi_S(x * y) = \max\{\chi_S^e(x), \chi_S^e(y)\} \leq \max\{\chi_S^e(x), \chi_S^e(y), 0.5\}.
\]
If any one of $x$ and $y$ does not belong to $S$, then $\chi_S^x(x) = 1$ or $\chi_S^y(y) = 1$. Hence $\chi_S^x(x \ast y) \leq \max\{\chi_S^x(x), \chi_S^y(y)\} \leq \max\{\chi_S^x(x), \chi_S^y(y), 0.5\}$. Using Theorem 3.15, we know that $\chi_S^x$ is a $(\prec, \prec \lor Y)^*$-fuzzy subalgebra of $X$.

**Theorem 3.17.** A fuzzy set $\mathcal{A}$ in a $BM$-algebra $X$ is a $(\prec, \prec \lor Y)^*$-fuzzy subalgebra of $X$ if and only if the set

$$L(\mathcal{A}; t) := \{x \in X \mid \mathcal{A}(x) \leq t\}, t \in [0.5, 1)$$

is a subalgebra of $X$.

**Proof.** Assume that $\mathcal{A}$ is a $(\prec, \prec \lor Y)^*$-fuzzy subalgebra of $X$ and let $x, y \in L(\mathcal{A}; t)$. Then $\mathcal{A}(x) \leq t$ and $\mathcal{A}(y) \leq t$, and so $x \prec \mathcal{A}$ and $y \prec \mathcal{A}$. It follows from Theorem 3.15 that

$$\mathcal{A}(x \ast y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\} \leq \max\{t, 0.5\} = t$$

so that $x \ast y \in L(\mathcal{A}; t)$. Hence $L(\mathcal{A}; t)$ is a subalgebra of $X$.

Conversely, let $\mathcal{A}$ be a fuzzy set in $X$ such that the set $L(\mathcal{A}; t) := \{x \in X \mid \mathcal{A}(x) \leq t\}$ is a subalgebra of $X$ for all $t \in [0.5, 1)$. If there exist $x, y \in X$ such that $\mathcal{A}(x \ast y) > \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}$, then we can take $t \in (0, 1)$ such that

$$\max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\} < t < \mathcal{A}(x \ast y).$$

Thus $x, y \in L(\mathcal{A}; t)$ and $t > 0.5$, and so $x \ast y \in L(\mathcal{A}; t)$, i.e., $\mathcal{A}(x \ast y) \leq t$. This is a contradiction. Therefore $\mathcal{A}(x \ast y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}$ for all $x, y \in X$. Using Theorem 3.15, we conclude that $\mathcal{A}$ is a $(\prec, \prec \lor Y)^*$-fuzzy subalgebra of $X$.

**Proposition 3.18.** Let $\mathcal{A}$ be a fuzzy set in a $BM$-algebra $X$. Then $\mathcal{A}$ is a $(\prec, \prec^\lor Y)^*$-fuzzy subalgebra of $X$ if and only if for all $t \in [0, 1)$, the nonempty level set $L(\mathcal{A}; t)$ is a subalgebra of $X$.

**Proof.** The proof follows from Proposition 3.2.

**Theorem 3.19.** Let $\mathcal{A}$ be a fuzzy set in a $BM$-algebra $X$. Then $\mathcal{A}$ is a non-unit $(Y, Y)^*$-fuzzy subalgebra of $X$ if and only if $L(\mathcal{A}; \mathcal{A}(0)) = X^*$ and for all $t \in [0, 1)$, the nonempty level set $L(\mathcal{A}; t)$ is a subalgebra of $X$.

**Proof.** Let $\mathcal{A}$ be a non-unit $(Y, Y)^*$-fuzzy subalgebra of $X$. Then by Theorem 3.12 we have

$$\mathcal{A}(x) = \begin{cases} \mathcal{A}(0) & \text{if } x \in X^* \\ 1 & \text{otherwise} \end{cases}$$

So it is easy to check that $L(\mathcal{A}; \mathcal{A}(0)) = X^*$. Let $x, y \in L(\mathcal{A}; t)$, for $t \in [0, 1]$. Then $\mathcal{A}(x) \leq t$ and $\mathcal{A}(y) \leq t$. If $t = 1$, then it is clear that $x \ast y \in L(\mathcal{A}; 1)$. Now let $t \in [0, 1)$. Then $x \in X^*$ and so $x \ast y \in X^*$. Hence $\mathcal{A}(x \ast y) = \mathcal{A}(0) \leq t$. Therefore $L(\mathcal{A}; t)$ is a subalgebra of $X$.

Conversely, since $L(\mathcal{A}; \mathcal{A}(0)) = X^*$ and $0 \in L(\mathcal{A}; \mathcal{A}(0))$, $X^*$ is a subalgebra of $X$.
and $A$ is non-unit. Now let $x \in X^*$. Then $A(x) \geq A(0)$ and $A(x) > 0$. Since $L(A; A(x)) \neq \emptyset$, so $L(A; A(x))$ is a subalgebra of $X$. Then $0 \in L(A; A(x))$ implies that $A(0) \geq A(x)$. Hence $A(x) = A(0)$, for all $x \in X^*$. Therefore

$$A(x) = \begin{cases} 
A(0) & \text{if } x \in X^* \\
1 & \text{otherwise}
\end{cases}$$

Hence by Theorem 3.12 $A$ is a $(\Upsilon, \Upsilon^*)$-fuzzy subalgebra of $X$.

**Theorem 3.20.** Every $(\Upsilon, \Upsilon^*)$-fuzzy subalgebra is a $(\leq, \leq^*)$-fuzzy subalgebra.

**Proof.** The proof follows from Theorem 3.19 and Proposition 3.18.

**Theorem 3.21.** Let $A$ be a non-unit $(\Upsilon, \leq \lor \Upsilon)$*-fuzzy subalgebra of $X$. Then the nonempty level set $L(A; t)$ is a subalgebra of $X$, for all $t \in [0, 1]$.

**Proof.** If $A$ is a constant on $X^*$, then by Theorem 3.12, $A$ is a $(\Upsilon, \Upsilon^*)$-fuzzy subalgebra. Thus by Theorem 3.19 we have the nonempty level set $L(A; t)$ is a subalgebra of $X$, for $t \in [0, 1]$. If $A$ is not a constant on $X^*$, then by Theorem 3.12, we have

$$A(x) = \begin{cases} 
\alpha & \text{if } x \in X^* \\
1 & \text{otherwise}
\end{cases}$$

where $\alpha \leq 0.5$. Now we show that the nonempty level set $L(A; t)$ is a subalgebra of $X$ for $t \in [0.5, 1]$. If $t = 1$, then it is clear that $L(A; t)$ is a subalgebra of $X$. Now let $t \in [0.5, 1)$ and $x, y \in L(A; t)$. Then $A(x), A(y) \leq t < 1$ imply that $x, y \in X^*$. Thus $x \land y \in X^*$ and so $A(x \land y) \leq 0.5 \leq t$. Therefore $x \land y \in L(A; t)$.

**Theorem 3.22.** Let $A$ be a non-unit fuzzy set of $BM$ algebra $X$, $L(A; 0.5) = X^*$ and the nonempty level set $L(A; t)$ is a subalgebra of $X$, for all $t \in [0, 1]$. Then $A$ is a $(\Upsilon, \leq \lor \Upsilon)$*-fuzzy subalgebra of $X$.

**Proof.** Since $A \neq 1$ we get that $X^* \neq \emptyset$. Thus by hypothesis we have $L(A; 0.5) \neq \emptyset$ and so $X^*$ is a subalgebra of $X$. Also $A(x) \leq 0.5$, for all $x \in X^*$ and $A(x) = 1$, if $x \notin X^*$. Therefore by Theorem 3.21, $A$ is a $(\Upsilon, \leq \lor \Upsilon)$*-fuzzy subalgebra of $X$.

**Theorem 3.23.** Let $A$ be an $(\Upsilon, \leq \lor \Upsilon)$*-fuzzy subalgebra of $BM$ algebra $X$. Then for all $t \in [0.5, 1]$, the nonempty level set $L(A; t)$ is a subalgebra of $X$. Conversely, if the nonempty level set $A$ is a subalgebra of $X$, for all $t \in [0, 1]$, then $A$ is an $(\Upsilon, \leq \lor \Upsilon)$*-fuzzy subalgebra of $X$.

**Proof.** Let $A$ be an $(\Upsilon, \leq \lor \Upsilon)$*-fuzzy subalgebra of $X$. If $t = 1$, then $L(A; t)$ is a subalgebra of $X$. Now let $L(A; t) \neq \emptyset$, $0.5 \leq t < 1$ and $x, y \in L(A; t)$. Then $A(x), A(y) \leq t$. Thus by hypothesis we have $A(x \land y) \leq \max(A(x), A(y)), 0.5) \leq \max(t, 0.5) \leq t$. Therefore $L(A; t)$ is a subalgebra of $X$.

Conversely, let $x, y \in X$. Then we have

$$A(x), A(y) \leq \max(A(x), A(y), 0.5) = t_0$$

Hence $x, y \in L(A; t_0)$, for $t_0 \in [0, 1]$ and so $x \land y \in L(A; t_0)$. Therefore $A(x \land y) \leq t_0 = \max(A(x), A(y), 0.5)$, then $A$ is a $(\Upsilon, \leq \lor \Upsilon)$*-fuzzy subalgebra of $X$. 


For any fuzzy set \( \mathcal{A} \) in \( X \) and \( t \in [0, 1) \), we denote
\[
\mathcal{A}_t := \{ x \in X \mid x \in \mathcal{A} \}
\]
and \( |\mathcal{A}|_t := \{ x \in X \mid x \leq \mathcal{A} \} \).

Obviously \( |\mathcal{A}|_t = L(\mathcal{A}; t) \cup \mathcal{A}_t \).

**Theorem 3.24.** A fuzzy set \( \mathcal{A} \) in a \( BM \)-algebra \( X \) is a \((\leq, \leq \vee \Upsilon)^\ast\)-fuzzy subalgebra of \( X \) if and only if \( |\mathcal{A}|_t \) is a subalgebra of \( X \) for all \( t \in [0, 1) \).

**Proof.** Let \( \mathcal{A} \) be a \((\leq, \leq \vee \Upsilon)^\ast\)-fuzzy subalgebra of \( X \) and let \( x, y \in |\mathcal{A}|_t \) for \( t \in [0, 1) \). Then \( x \leq \vee \Upsilon \mathcal{A} \) and \( y \leq \vee \Upsilon \mathcal{A} \), that is, \( \mathcal{A}(x) \leq t \) or \( \mathcal{A}(x) + t > 1 \), and \( \mathcal{A}(y) \leq t \) or \( \mathcal{A}(y) + t > 1 \). Since \( \mathcal{A}(x \ast y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\} \) by Theorem 3.15, we have \( \mathcal{A}(x \ast y) \leq \max\{t, 0.5\} \). If it is not true, then \( x \leq \vee \mathcal{A} \) or \( y \leq \vee \mathcal{A} \), a contradiction. If \( t \geq 0.5 \), then \( \mathcal{A}(x \ast y) \leq \max\{t, 0.5\} = t \) and so \( x \ast y \in L(\mathcal{A}; t) \subseteq |\mathcal{A}|_t \).

Conversely, let \( \mathcal{A} \) be a fuzzy set in \( X \) and \( t \in [0, 1) \) be such that \( |\mathcal{A}|_t \) is a subalgebra of \( X \). Let \( \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\} < t < \mathcal{A}(x \ast y) \) for some \( t \in (0.5, 1) \). Then \( x, y \in L(\mathcal{A}; t) \subseteq |\mathcal{A}|_t \), which implies that \( x \ast y \in |\mathcal{A}|_t \). Hence \( \mathcal{A}(x \ast y) \leq \max\{t, 0.5\} \). Therefore \( \mathcal{A}(x \ast y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\} \) for all \( x, y \in X \). Using Theorem 3.15, we know that \( \mathcal{A} \) is a \((\leq, \leq \vee \Upsilon)^\ast\)-fuzzy subalgebra of \( X \).

**Theorem 3.25.** Let \( \{\mathcal{A}_i \mid i \in \Lambda\} \) be a family of \((\leq, \leq \vee \Upsilon)^\ast\)-fuzzy subalgebras of a \( BM \)-algebra \( X \). Then \( \mathcal{A} := \bigcap_{i \in \Lambda} \mathcal{A}_i \) is a \((\leq, \leq \vee \Upsilon)^\ast\)-fuzzy subalgebra of \( X \).

**Proof.** By Theorem 3.15 we have \( \mathcal{A}_i(x \ast y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\} \), and so
\[
\mathcal{A}(x \ast y) = \inf_{i \in \Lambda} \mathcal{A}_i(x \ast y) \\
\leq \inf_{i \in \Lambda} \max\{\mathcal{A}_i(x), \mathcal{A}_i(y), 0.5\} \\
= \max\{\inf_{i \in \Lambda} \mathcal{A}_i(x), \inf_{i \in \Lambda} \mathcal{A}_i(y), 0.5\} \\
= \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}.
\]

By Theorem 3.15 we know that \( \mathcal{A} \) is a \((\leq, \leq \vee \Upsilon)^\ast\)-fuzzy subalgebra of \( X \).

**Theorem 3.26.** Let \( \{\mathcal{A}_i \mid i \in \Lambda\} \) be a family of \((\alpha, \beta)^\ast\)-fuzzy subalgebras of \( X \). Then \( \mathcal{A} := \bigcap_{i \in \Lambda} \mathcal{A}_i \) is an \((\alpha, \beta)^\ast\)-fuzzy subalgebra of \( X \), where \((\alpha, \beta)\) is one of the following forms
\[
\begin{align*}
(i) & \quad (\leq, \Upsilon), \\
(ii) & \quad (\leq, \leq \Upsilon), \\
(iii) & \quad (\Upsilon, \leq), \\
(iv) & \quad (\Upsilon, \leq \Upsilon), \\
(v) & \quad (\leq \Upsilon, \leq \Upsilon), \\
(vi) & \quad (\leq \Upsilon, \leq \Upsilon), \\
(vii) & \quad (\leq \Upsilon, \leq \Upsilon), \\
(viii) & \quad (\Upsilon, \leq \Upsilon), \\
(ix) & \quad (\Upsilon, \Upsilon).
\end{align*}
\]
Proof. We prove theorem for an \((Y, Y)^*\)-fuzzy subalgebra. The proof of the other cases is similar.

If there exists \(i \in \Lambda\) such that \(A_i = 0\), then \(A = 0\). So \(A\) is a \((Y, Y)^*\)-fuzzy subalgebra.

Let \(A_i \neq 0\) for all \(i \in \Lambda\). Then by Theorem 3.12 we have

\[
A_i(x) = \begin{cases} 
A_i(0) & \text{if } x \in X_i^* \\
1 & \text{otherwise}
\end{cases}
\]

for all \(i \in \Lambda\). So it is clear that

\[
A(x) = \begin{cases} 
A(0) & \text{if } x \in \bigcap_{i \in \Lambda} X_i^* \\
1 & \text{otherwise}
\end{cases}
\]

Since \(\bigcap_{i \in \Lambda} X_i^*\) is a subalgebra of \(X\), then by Theorem 3.12 \(A\) is a \((Y, Y)^*\)-fuzzy subalgebra of \(X\).

Theorem 3.27. Let \(\{A_i \mid i \in \Lambda\}\) be a family of \((\prec, \preceq)^*\)-fuzzy subalgebras of a \(BM\)-algebra \(X\). Then \(A := \bigcup_{i \in \Lambda} A_i\) is a \((\prec, \preceq)^*\)-fuzzy subalgebra of \(X\).

Proof. Let \(x_i \prec A\) and \(y_r \preceq A\), where \(t, r \in [0, 1]\). Then \(A(x) \leq t\) and \(A(y) \leq r\).

Thus for all \(i \in \Lambda\), we have \(A_i(x) \leq t\) and \(A_i(y) \leq r\) and so \(A_i(x \ast y) \leq \max(t, r)\).

Therefore \(A(x \ast y) \leq \max(t, r)\). Hence \((x \ast y)_{\max(t, r)} \preceq A\).

The following is our question: Is the union of two \((\prec, \preceq \lor Y)^*\)-fuzzy subalgebras of a \(BM\)-algebra \(X\) a \((\prec, \preceq \lor Y)^*\)-fuzzy subalgebra of \(X\)?

Lemma 3.28. Let \(f : X \to Y\) be a \(BM\)-homomorphism and \(G\) be a fuzzy set of \(Y\) with membership function \(A_G\). Then \(x_i \alpha A_{f^{-1}(G)} \Leftrightarrow f(x_i) \alpha A_G\), for all \(\alpha \in \{Y, \prec, \prec \lor Y, \preceq \lor Y\}\).

Proof. Let \(\alpha = \prec\). Then

\[
x_i \alpha A_{f^{-1}(G)} \Leftrightarrow A_{f^{-1}(G)}(x) \leq t \Leftrightarrow A_G(f(x)) \leq t \Leftrightarrow (f(x))_\alpha A_G
\]

The proof of the other cases is similar to above argument.

Theorem 3.29. Let \(f : X \to Y\) be a \(BM\)-homomorphism and \(G\) be a fuzzy set of \(Y\) with membership function \(A_G\).

(i) If \(G\) is an \((\alpha, \beta)^*\)-fuzzy subalgebra of \(Y\), then \(f^{-1}(G)\) is an \((\alpha, \beta)^*\)-fuzzy subalgebra of \(X\).

(ii) Let \(f\) be epimorphism. If \(f^{-1}(G)\) is an \((\alpha, \beta)^*\)-fuzzy subalgebra of \(X\), then \(G\) is an \((\alpha, \beta)^*\)-fuzzy subalgebra of \(Y\).

Proof. (i) Let \(x_i \alpha A_{f^{-1}(G)}\) and \(y_r \alpha A_{f^{-1}(G)}\), for \(t, r \in [0, 1]\). Then by Lemma 3.28, we get that \((f(x))_\alpha A_G\) and \((f(y))_r A_G\). Hence by hypothesis \((f(x) \ast f(y))_{\max(t, r)} \beta A_G\). Then \((f(x \ast y))_{\max(t, r)} \beta A_G\) and so \((x \ast y)_{\max(t, r)} \beta A_{f^{-1}(G)}\).
(ii) Let \( x, y \in Y \). Then by hypothesis there exist \( x', y' \in X \) such that \( f(x') = x \) and \( f(y') = y \). Assume that \( x_i \alpha_{G} \) and \( y_i \alpha_{G} \), then \( f(x')_i \alpha_{G} \) and \( f(y')_i \alpha_{G} \). Thus \( x_i \alpha_{f^{-1}(G)} \) and \( y_i \alpha_{f^{-1}(G)} \) and therefore \( (x * y')_{\max(t,r)} \beta_{f^{-1}(G)} \). So

\[
(f(x' * y'))_{\max(t,r)} \beta_{A_G} \Rightarrow (f(x') * f(y'))_{\max(t,r)} \beta_{A_G} \Rightarrow (x * y)_{\max(t,r)} \beta_{A_G}.
\]

**Theorem 3.30.** Let \( f : X \to Y \) be a BM-homomorphism and \( H \) be a \((\leq, \leq \vee \Upsilon)^*\)-fuzzy subalgebra of \( X \) with membership function \( A_H \). If \( A_H \) is \( f \)-invariant, then \( f(H) \) is a \((\leq, \leq \vee \Upsilon)^*\)-fuzzy subalgebra of \( Y \).

**Proof.** Let \( y_1 \) and \( y_2 \in Y \). If \( f^{-1}(y_1) \) or \( f^{-1}(y_2) = \emptyset \), then \( A_{f(H)}(y_1 * y_2) \leq \max(A_{f(H)}(y_1), A_{f(H)}(y_2), 0.5) \). Now let \( f^{-1}(y_1) \) and \( f^{-1}(y_2) \neq \emptyset \). Then there exist \( x_1, x_2 \in X \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \). Thus by hypothesis we have

\[
A_{f(H)}(y_1 * y_2) = \sup_{t \in f^{-1}(y_1 * y_2)} A_H(t) = \sup_{t \in f^{-1}(x_1 * x_2)} A_H(t) = A_H(x_1 * x_2) \leq \max(A_H(x_1), A_H(x_2), 0.5) = \max(\sup_{t \in f^{-1}(y_1)} A_H(t), \sup_{t \in f^{-1}(y_2)} A_H(t), 0.5) = \max(A_{f(H)}(y_1), A_{f(H)}(y_2), 0.5).
\]

So by Theorem 3.15, \( f(H) \) is a \((\leq, \leq \vee \Upsilon)^*\)-fuzzy subalgebra of \( Y \).

**Lemma 3.31.** Let \( f : X \to Y \) be a BM-homomorphism.

(i) If \( S \) is a subalgebra of \( X \), then \( f(S) \) is a subalgebra of \( Y \),

(ii) If \( S' \) is a subalgebra of \( Y \), then \( f^{-1}(S') \) is a subalgebra of \( X \).

**Proof.** The proof is easy.

**Theorem 3.32.** Let \( f : X \to Y \) be a BM-homomorphism. If \( H \) is a non-unit \((\Upsilon, \Upsilon)^*\)-fuzzy subalgebra of \( X \) with membership function \( A_H \), then \( f(H) \) is a non-unit \((\Upsilon, \Upsilon)^*\)-fuzzy subalgebra of \( Y \).

**Proof.** Let \( H \) be a non-unit \((\Upsilon, \Upsilon)^*\)-fuzzy subalgebra of \( X \). Then by Theorem 3.12, we have

\[
A_H(x) = \begin{cases} 
A_H(0) & \text{if } x \in X^* \\
1 & \text{otherwise}
\end{cases}
\]

Now we show that

\[
A_{f(H)}(y) = \begin{cases} 
A_H(0) & \text{if } y \in f(X^*) \\
1 & \text{otherwise}
\end{cases}
\]
Let $y \in Y$. If $y \in f(X^*)$, then there exists $x \in X^*$ such that $f(x) = y$. Thus $A_{f(H)}(y) = \sup_{t \in f^{-1}(y)} A_H(t) = A_H(0)$. If $y \notin f(X^*)$, then it is clear that $A_{f(H)}(y) = 1$.

Since $X^*$ is a subalgebra of $X$, $f(X^*)$ is a subalgebra of $Y$. Therefore by Theorem 3.12, $f(H)$ is a non-unit $(\Upsilon, \Upsilon^*)$-fuzzy subalgebra of $Y$.

**Theorem 3.33.** Let $f : X \to Y$ be a $BM$-homomorphism. If $H$ is an $(\alpha, \beta)$-fuzzy subalgebra of $X$ with membership function $A_H$, then $f(H)$ is an $(\alpha, \beta)$-fuzzy subalgebra of $Y$, where $(\alpha, \beta)$ is one of the following forms:

- $(i)$ $(\leq, \Upsilon^*)$,
- $(ii)$ $(\leq, \leq \wedge \Upsilon^*)$,
- $(iii)$ $(\Upsilon, \leq)$,
- $(iv)$ $(\Upsilon, \leq \wedge \Upsilon^*)$,
- $(v)$ $(\leq \vee \Upsilon, \Upsilon)$,
- $(vi)$ $(\leq \vee \Upsilon, \leq \wedge \Upsilon^*)$,
- $(vii)$ $(\leq \vee \Upsilon, \leq)$,
- $(viii)$ $(\Upsilon, \leq \vee \Upsilon^*)$.

**Theorem 3.34.** Let $f : X \to Y$ be a $BM$-homomorphism and $H$ be an $(\leq, \leq)$-fuzzy subalgebra of $X$ with membership function $A_H$. If $A_H$ is an $f$-invariant, then $f(H)$ is an $(\leq, \leq)$-fuzzy subalgebra of $Y$.

**Proof.** Let $z_t \leq A_{f(H)}$ and $y_r \leq A_{f(H)}$, where $t, r \in [0, 1)$. Then $A_{f(H)}(z) \leq t$ and $A_{f(H)}(y) \leq r$. Thus $f^{-1}(z), f^{-1}(y) \neq \emptyset$ imply that there exist $x_1, x_2 \in X$ such that $f(x_1) = z$ and $f(x_2) = y$. Since $A_H$ is $f$-invariant, then $A_{f(H)}(z) \leq t$ and $A_{f(H)}(y) \leq r$ imply that $A_H(x_1) \leq t$ and $A_H(x_2) \leq r$. So by hypothesis we have

\[
A_{f(H)}(z \ast y) = \sup_{t \in f^{-1}(z \ast y)} A_H(t) = \sup_{t \in f^{-1}(f(x_1 \ast x_2))} A_H(t) = A_H(x_1 \ast x_2) \leq \max(t, r)
\]

Therefore $(z \ast y)_{\max(t, r)} \in A_{f(H)}$, and hence $f(H)$ is a $(\leq, \leq)$-fuzzy subalgebra of $Y$.

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**References**


Redefined (anti) fuzzy BM-algebras