Abstract. Using a generalized translation operator, we obtain a generalization of Theorem 5 in [4] for the Bessel transform for functions satisfying the $(\delta, \gamma, 2)$-Bessel Lipschitz condition in $L_{2,\alpha}(\mathbb{R}^+)$. 

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1. Introduction and Preliminaries

Integral transforms and their inverses are widely used solve various in calculus, mechanics, mathematical, physics, and computational mathematics (see, e.g., [7, 9, 10]).

In [4], we proved theorem related the Bessel transform and $(k, \gamma)$-Bessel Lipschitz functions. In this paper, we prove a generalization of this theorem for this transform in the space $L_{2,\alpha}(\mathbb{R}^+)$. For this purpose, we use a generalized translation operator.

Assume that $L_{2,\alpha} = L_{2,\alpha}(\mathbb{R}^+)$, $\alpha > -\frac{1}{2}$, is the Hilbert space of measurable functions $f(x)$ on $\mathbb{R}^+$ with the finite norm

$$
\|f\|_{2,\alpha} = \left(\int_0^\infty |f(x)|^2 x^{2\alpha+1} dx\right)^{1/2}.
$$

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Let
\[ B = \frac{d^2}{dt^2} + \frac{(2\alpha + 1)}{t} \frac{d}{dt}, \]
be the Bessel differential operator.

For \( \alpha > -\frac{1}{2} \), we introduce the Bessel normalized function of the first kind \( j_\alpha \) defined by
\[
j_\alpha(x) = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(n + \alpha + 1)},
\]
where \( \Gamma(x) \) is the gamma-function (see [6]).

**Lemma 1.1** [1] The following inequalities are valid for Bessel function \( j_\alpha \)

1. \( |j_\alpha(x)| \leq 1 \)
2. \( 1 - j_\alpha(x) = O(x^2); \ 0 \leq x \leq 1. \)

**Lemma 1.2** The following inequality is true
\[
|1 - j_\alpha(x)| \geq c,
\]
with \( x \geq 1 \), where \( c > 0 \) is a certain constant.

**Proof** Analog of Lemma 2.9 in [3] ■

The Bessel transform of a function \( f \in L_{2,\alpha} \) is defined (see [5, 6, 8]) by the formula
\[
\hat{f}(\lambda) = \int_0^\infty f(t) j_\alpha(\lambda t) t^{2\alpha+1} dt; \ \lambda \in \mathbb{R}^+.
\]

The inverse Bessel transform is given by the formula
\[
f(t) = (2^{\alpha} \Gamma(\alpha + 1))^{-2} \int_0^\infty \hat{f}(\lambda) j_\alpha(\lambda t) \lambda^{2\alpha+1} d\lambda.
\]

From [5], we have the Parseval’s identity
\[
\|\hat{f}\|_{2,\alpha} = (2^{\alpha} \Gamma(\alpha + 1)) \|f\|_{2,\alpha}.
\]

In \( L_{2,\alpha} \), consider the generalized translation operator \( T_h \) defined by
\[
T_h f(t) = c_\alpha \int_0^\pi f(\sqrt{t^2 + h^2 - 2th \cos \varphi}) \sin^{2\alpha} \varphi d\varphi,
\]
where
\[ c_{\alpha} = \left( \int_{0}^{\pi} \sin^{2\alpha} \varphi d\varphi \right)^{-1} = \frac{\Gamma(\alpha + 1)}{\Gamma(1/2)\Gamma(\alpha + 1/2)} \]

From [2], we have

\[ \widehat{(T_h f)(\lambda)} = j_{\alpha}(\lambda h) \widehat{f}(\lambda) \]  \hspace{1cm} (1)

We note the important property of the Bessel transform: If \( f \in L_{2,\alpha} \) then

\[ \widehat{Bf}(\lambda) = (-\lambda^2) \widehat{f}(\lambda). \]  \hspace{1cm} (2)

The finite differences of the first and higher orders are defined as follows

\[ \Delta_h f(x) = T_h f(x) - f(x) = (T_h - E_{2,\alpha}) f(x), \]

\[ \Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1} f(x)) = (T_h - E_{2,\alpha})^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} T_h^i f(x), \]  \hspace{1cm} (3)

where \( T_h^0 f(x) = f(x) \), \( T_h^i f(x) = T_h(T_h^{i-1} f(x)) \); \( i = 1, 2, ..., k; k = 1, 2, ..., \) and \( E_{2,\alpha} \) is a unit operator in \( L_{2,\alpha} \).

Let \( W_{2,\alpha}^k \) be the Sobolev space constructed by the Bessel operator \( B \), i.e.,

\[ W_{2,\alpha}^k = \{ f \in L_{2,\alpha}, B^j f \in L_{2,\alpha}; j = 1, 2, ..., k \}. \]

In [4], we have the following result

**Theorem 1.3** Let \( f \in L_{2,\alpha} \). Then the following are equivalents

1. \( f \in \text{Lip}(k, \gamma, 2); 0 < k < 1, \gamma \geq 0, \)
2. \( \int_{r}^{\infty} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O(r^{-2k}(\log r)^2\gamma) \) as \( r \to +\infty, \)

where

\[ \text{Lip}(k, \gamma, 2) = \{ f \in L_{2,\alpha}, \|T_h f(t) - f(t)\|_{2,\alpha} = O \left( \frac{h^k}{(\log \frac{1}{h})^\gamma} \right) \text{ as } h \to 0 \}. \]

The main aim of this paper is to establish a generalization of Theorem 1.3

2. **Main Results**

In this section we present the main result of this paper. We first need to define the \((\delta, \gamma, 2)\)-Bessel Lipschitz class.
Definition 2.1 Let $0 < \delta < 1$, $\gamma \geq 0$ and $r = 0, 1, ..., k$. A function $f \in W_{2,\alpha}^k$ is said to be in the $(\delta, \gamma, 2)$-Bessel Lipschitz class, denoted by $\text{Lip}(\delta, \gamma, 2)$; if

$$\|\Delta_h^k B^r f(x)\|_{2,\alpha} = O\left(\frac{h^\delta}{(\log \frac{1}{h})^\gamma}\right) \text{ as } h \to 0.$$ 

Lemma 2.2 Let $f \in W_{2,\alpha}^k$. Then

$$\|\Delta_h^k B^r f(x)\|_{2,\alpha}^2 = \frac{1}{(2^\alpha \Gamma(\alpha + 1))^2} \int_0^\infty \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda,$$

where $r = 0, 1, ..., k$

Proof From formula (2), we have

$$\widehat{B^r f}(\lambda) = (-1)^r \lambda^{2r} \widehat{f}(\lambda)$$ (4)

By formulas (1) and (4), we conclude that

$$\widehat{T_h^i B^r f}(\lambda) = (-1)^r \lambda^{2r} j_\alpha^i(th) \widehat{f}(\lambda), 1 \leq i \leq k.$$ (5)

From formulas (3) and (5) follows that the Bessel transform of $\Delta_h^k B^r f(x)$ is $(-1)^r \lambda^{2r}(j_\alpha(\lambda h) - 1)^k \widehat{f}(\lambda)$.

By Parseval’s identity we have the result. ■

Theorem 2.3 Let $f \in W_{2,\alpha}^k$. Then the followings are equivalents

1) $f \in \text{Lip}(\delta, \gamma, 2)$,

2) $\int_s^\infty \lambda^{4r} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O\left(\frac{s^{-2\delta}}{(\log s)^\gamma}\right) \text{ as } s \to +\infty.$

Proof 1) $\implies$ 2) Assume that $f \in \text{Lip}(\delta, \gamma, 2)$. Then we have from Lemma 2.2

$$\|\Delta_h^k B^r f(x)\|_{2,\alpha}^2 = \frac{1}{(2^\alpha \Gamma(\alpha + 1))^2} \int_0^\infty \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda,$$

If $\lambda \in [\frac{1}{h}, \frac{2}{h})$ then $\lambda h \geq 1$ and Lemma 1.2 implies that

$$1 \leq \frac{1}{c_{4k}} |1 - j_\alpha(\lambda h)|^{2k}.$$ 

Then
\[
\int_{1/h}^{2/h} \lambda^{4r} |\hat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \leq \frac{1}{c \epsilon k} \int_{1/h}^{2/h} \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\hat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda
\]
\[
\leq \frac{1}{c \epsilon k} \int_0^\infty \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\hat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda
\]
\[
= O \left( \frac{h^{2\delta}}{(\log \frac{1}{h})^{2\gamma}} \right).
\]

We have
\[
\int_s^{2s} \lambda^{4r} |\hat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \leq C s^{-2\delta},
\]
where \(C\) is a positive constant.

So that
\[
\int_s^\infty \lambda^{4r} |\hat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = \left[ \int_s^{2s} \lambda^{4r} |\hat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda + \int_{2s}^{4s} \lambda^{4r} |\hat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda + \ldots \right]
\]
\[
\leq C \frac{s^{-2\delta}}{(\log s)^{2\gamma}} + C \frac{(2s)^{-2\delta}}{(\log 2s)^{2\gamma}} + C \frac{(4s)^{-2\delta}}{(\log 4s)^{2\gamma}} + \ldots
\]
\[
\leq C \frac{s^{-2\delta}}{(\log s)^{2\gamma}} \left( 1 + 2^{-2\delta} + (2^{-2\delta})^2 + (2^{-2\delta})^3 + \ldots \right)
\]
\[
\leq CC_\delta \frac{s^{-2\delta}}{(\log s)^{2\gamma}},
\]
where \(C_\delta = (1 - 2^{-2\delta})^{-1}\) since \(2^{-2\delta} < 1\).

This proves that
\[
\int_s^\infty \lambda^{4r} |\hat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O \left( \frac{s^{-2\delta}}{(\log s)^{2\gamma}} \right)\quad \text{as } s \to +\infty
\]

2) \(\Rightarrow\) 1) Suppose now that
\[
\int_s^\infty \lambda^{4r} |\hat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O \left( \frac{s^{-2\delta}}{(\log s)^{2\gamma}} \right)\quad \text{as } s \to +\infty
\]

We write
\[
\int_0^\infty \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\hat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = I_1 + I_2
\]
where
\[ I_1 = \int_0^{1/h} \lambda^{4r} |1 - j_\alpha(\lambda h)| 2^h |\hat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda, \]

and

\[ I_2 = \int_{1/h}^{\infty} \lambda^{4r} |1 - j_\alpha(\lambda h)| 2^h |\hat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda. \]

Estimate the summands \( I_1 \) and \( I_2 \) from above. It follows from the formula \(|j_\alpha(\lambda h)| \leq 1\) that

\[ I_2 = \int_{1/h}^{\infty} \lambda^{4r} |1 - j_\alpha(\lambda h)| 2^h |\hat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \]
\[ \leq 4^k \int_{1/h}^{\infty} \lambda^{4r} |\hat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \]
\[ = O \left( \frac{h^{2\delta}}{(\log \frac{1}{h})^{2\gamma}} \right) \]

To estimate \( I_1 \), we use the inequality (2) of Lemma 1.1

Set

\[ \psi(x) = \int_x^{\infty} \lambda^{4r} |\hat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda. \]

Using integration by parts, we obtain

\[ I_1 \leq -C_1 h^{4k} \int_0^{1/h} t^{4k} \psi'(t) dt \]
\[ \leq C_1 \psi\left( \frac{1}{h} \right) + 4C_1 kh^{4k} \int_0^{1/h} t^{4k-1} \psi(t) dt \]
\[ \leq C_2 h^{4k} \int_0^{1/h} t^{4k-1} \frac{1}{t} - 2^\delta (\log t)^{-2\gamma} dt \]
\[ \leq C_3 h^{-2\delta} (\log \frac{1}{h})^{-2\gamma}, \]

where \( C_1, C_2 \) and \( C_3 \) are positive constants and this ends the proof. \( \blacksquare \)

**Corollary 2.4** Let \( f \in W_{2,\alpha}^k \), and let

\[ f \in \text{Lip}(\delta, \gamma, 2) \]

Then
\[
\int_{s}^{\infty} |\hat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O \left( \frac{s^{-2\delta-4\gamma}}{(\log s)^{2\gamma}} \right) \quad \text{as } s \to +\infty
\]

3. Conclusion
In this work we have succeeded to generalise the theorem 5 in [4] for the Bessel transform in the Sobolev space \( W_{2,\alpha}^k \) constructed by the Bessel operator \( B \). We proved that \( f \in \text{Lip}(\delta, \gamma, 2) \) if and only if \( \int_{s}^{\infty} \lambda^{4r} |\hat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O \left( \frac{s^{-2\delta}}{(\log s)^{2\gamma}} \right) \) as \( s \to +\infty \).

References