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A Strong Computational Method for Solving of System of Infinite Boundary Integro-Differential Equations

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Abstract. The introduced method in this study consists of reducing a system of infinite boundary integro-differential equations (IBI-DE) into a system of algebraic equations, by expanding the unknown functions, as a series in terms of Laguerre polynomials with unknown coefficients. Properties of these polynomials and operational matrix of integration are first presented. Finally, two examples illustrate the simplicity and the effectiveness of the proposed method have been presented.

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1. Introduction

In recent years, many different orthogonal functions and polynomials have been used to approximate the solution of various functional equations. The main goal of using orthogonal basis is that the equation under study reduces to a system of linear or nonlinear algebraic equations. This can be done by truncating series of functions with orthogonal basis for the solution of equations and using the operational matrices. In this letter, Laguerre polynomials basis, on the infinite

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interval $[0, \infty)$, have been considered for solving systems of (IBI-DE). Mathematical modeling for many problems in different disciplines, such as engineering, chemistry, physics and biology leads to integral equations, or system of integral equations. It's the reason of great interest for solving these equations. We consider the following system of infinite boundary integro-differential equations as:

$$U'(x) = F(x) + \lambda \int_0^\infty e^{-t} K(x,t) U(t) dt, \tag{1}$$

along with initial condition $U(0) = A$, where $\lambda \in R$, and

$$\begin{aligned} U(x) &= [u_1(x), u_2(x), ..., u_m(x)]^T, \\ F(x) &= [f_1(x), f_2(x), ..., f_m(x)]^T, \\ K(x,t) &= [k_{ij}], \quad i,j = 1,2,...,m, \\ A &= [a_1, a_2, ..., a_m]^T. \end{aligned} \tag{2}$$

In system (1), the known kernel $K(x,t)$ might has singularity in the region $D = \{(x,t) : 0 \leqslant x,t < \infty\}$ and $F(x)$ is continuous function and A is fixed constant vector, and $U(x)$ is the unknown vector function of the solution that will be determined. Many researchers have developed the approximate method to solve infinite boundary integral equation using Galerkin and Collocation methods with Laguerre and Hermite polynomials as a bases function or CAS wavelet method constructed on the unit interval and spline Collocation as basis [2–4, 6, 8, 9]. Moreover there are several numerical methods for solving system (1) when the limit of integration is finite. For example Tau method [7], He's homotopy perturbation method (HPM)[7], rationalized Haar functions method [1]. However, method of solution for system (1) is too rear in the literature. Our aim in this paper is to obtain the analytical-numerical solutions by using the Laguerre polynomials for the system of (IBI-DE). The layout of this paper is organized as follows: In section 2, we introduce some necessary definitions and give some relevant properties of the Laguerre polynomials and approximate the function $f(x)$ and also the kernel function $k(x,t)$ by these polynomials and related operational matrices. Section 3 is devoted to present a computational method for solving system (1) utilizing Laguerre polynomials and approximate the unknown function $u(x)$. Section 4, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering illustrative examples. Finally, we conclude the article in Section 5.

2. Preliminary Notes

Let $\Lambda = \{x : 0 \leqslant x < \infty\} = [0, \infty)$ and $w(x) = e^{-x}$ be a weight function on Λ in the usual sense. We define the following:

$$L^2_w(\Lambda) = \{v : v \ \ is\ measurable\ on \ \ \Lambda \ \ and \ \ \|v\|_w < \infty\}, \tag{3}$$

equipped with the following inner product and norm:

$$(u,v)_w = \int_\Lambda u(x)v(x)w(x)dx, \quad \|v\|_w = (v,v)^{\frac{1}{2}}_w. \tag{4}$$

Next, suppose $L_n(x)$ be the Laguerre polynomials of degree n, defined by the following:

$$L_n(x) = \frac{1}{n!}e^x\partial_x^n(x^n e^{-x}), \qquad n = 0, 1, ... \tag{5}$$

They satisfy the equations

$$\partial_x(xe^{-x}\partial_x L_n(x)) + ne^{-x}L_n(x) = 0, \qquad x \in \Lambda, \tag{6}$$

and

$$L_n(x) = \partial_x L_n(x) - \partial_x L_{n+1}(x), \qquad n \geqslant 0, \tag{7}$$

where $L_0(x) = 1$ and $L_1(x) = 1 - x$. The set $\{L_n(x) : n = 0, 1, ...\}$ in Hilbert space $L_w^2(\Lambda)$ is a complete orthogonal set, namely,

$$\int_0^\infty L_i(x)L_j(x)w(x)dx = \delta_{ij}, \qquad \forall i, j \geqslant 0, \tag{8}$$

where δ_{ij} is the Kronecher function.

2.1 *Function Approximation*

A function $f(x) \in L_w^2(\Lambda)$ may be expressed in terms of Laguerre polynomials as:

$$f(x) = \sum_{i=0}^{\infty} f_i L_i(x), \tag{9}$$

where the Laguerre coefficients f_i are given by

$$f_i = \int_0^\infty f(x)L_i(x)w(x)dx, \quad i = 0, 1, ... \tag{10}$$

In practice, only the first $(n + 1)$ terms of Laguerre polynomials are considered. Then we have

$$f(x) \simeq \sum_{i=0}^{n} f_i L_i(x) = F^T L_x, \tag{11}$$

where the Laguerre coefficient vector F and the Laguerre vector L_x are given by as follows:

$$F = [f_0,\ f_1,\ f_2, \ldots,\ f_n]^T, \quad and \quad L_x = [L_0(x),\ L_1(x),\ L_2(x), \ldots,\ L_n(x)]^T. \tag{12}$$

We can also approximate the function of two variables, $k(x, t) \in L_w^2(\Lambda^2)$ as follows:

$$k(x, t) \simeq \sum_{i=0}^{n}\sum_{j=0}^{n} L_i(x)k_{ij}L_j(t) = L_x^T K L_t. \tag{13}$$

Here the entries of matrix $K = [k_{ij}]_{(n+1)\times(n+1)}$ will be obtained by

$$k_{ij} = (L_i(x), (k(x,t), L_j(t))), \qquad for \quad i,j = 0,1,...,n. \tag{14}$$

The integration of the product of two Laguerre vector functions with respect to the weight function $w(x)$, is obtained as:

$$I = \int_0^{\infty} e^{-x} L_x L_x^T dx, \tag{15}$$

where I is an identity matrix.

2.2 *Operational Matrix of Integration*

The main objective of this subsection is to derive the integration of the Laguerre vector L_x defined in Eq. (12).

Theorem 1. Let L_x be the Laguerre vector then

$$\int_0^x L_t dt \simeq P L_x, \tag{16}$$

where P is the $(n+1)\times(n+1)$ operational matrix for integration as follows:

$$P = \begin{bmatrix} \Omega(0,0) & \Omega(0,1) & \Omega(0,2) & \cdots & \Omega(0,n) \\ \Omega(1,0) & \Omega(1,1) & \Omega(1,2) & \cdots & \Omega(1,n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega(i,0) & \Omega(i,1) & \Omega(i,2) & \cdots & \Omega(i,n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega(n,0) & \Omega(n,1) & \Omega(n,2) & \cdots & \Omega(n,n) \end{bmatrix}, \tag{17}$$

where

$$\Omega(i,j) = \sum_{k=0}^{i}\sum_{r=0}^{j} \frac{(-1)^{k+r} i! j! \Gamma(k+r+2)}{(i-k)!(j-r)!(k+1)!k!(r!)^2}. \tag{18}$$

Proof. The analytic form of the Laguerre polynomials $L_i(x)$ of degree i is given as follows:

$$L_i(x) = \sum_{k=0}^{i} (-1)^k \frac{i!}{(i-k)!(k!)^2} x^k, \tag{19}$$

where $L_i(0) = 1$. Using Eq.(19), and since the integration is a linear operation, we get the following:

$$\begin{aligned} \int_0^x L_i(t)dt &= \sum_{k=0}^{i} (-1)^k \frac{i!}{(i-k)!(k!)^2} \int_0^x t^k dt \\ &= \sum_{k=0}^{i} (-1)^k \frac{i!}{(i-k)!(k+1)!(k!)} x^{k+1}, \quad i = 0,1,...,n. \end{aligned} \tag{20}$$

Now, by approximating x^{k+1} by the $n+1$ terms of the Laguerre series, we have

$$x^{k+1} = \sum_{j=0}^{n} b_j L_j(x), \tag{21}$$

where b_j is given from Eq. (10) with $f(x) = x^{k+1}$, that is,

$$b_j = \sum_{r=0}^{j} \frac{(-1)^r j! \Gamma(k+r+2)}{(j-r)!(r!)^2}, \quad j = 0, 1, ..., n. \tag{22}$$

In virtue of Eqs. (20) and (21), we get:

$$\int_0^x L_i(t)dt = \sum_{j=0}^{n} \Omega(i,j) L_j(x), \quad i = 0, 1, ..., n, \tag{23}$$

where

$$\Omega(i,j) = \sum_{k=0}^{i} \sum_{r=0}^{j} \frac{(-1)^{k+r} i! j! \Gamma(k+r+2)}{(i-k)!(j-r)!(k+1)!k!(r!)^2}, \quad j = 0, 1, ..., n. \tag{24}$$

Accordingly, Eq. (23) can be written in a vector form as follows:

$$\int_0^x L_i(t)dt \simeq [\Omega(i,0), \Omega(i,1), \Omega(i,2), ..., \Omega(i,n)] L_x, \quad i = 0, 1, ..., n. \tag{25}$$

Eq. (25) leads to the desired result.

3. Method of Solution

In this section, we solve the special type of system of infinite boundary integro-differential equations of the second kind (1). To this end, we consider the *ith* equation of (1) as follows:

$$u_i'(x) = f_i(x) + \lambda \int_0^\infty e^{-t} \sum_{j=1}^{m} k_{ij}(x,t) u_j(t) dt, \quad u_i(0) = a_i, \quad i = 1, 2, ..., m, \tag{26}$$

where $f_i \in L_w^2(\Lambda)$, $k_{ij} \in L_w^2(\Lambda^2)$, and $u_i'(x)$ represents the first order derivative of $u_i(x)$ with respect to x, a_i are constants that give the initial conditions and u_i is an unknown function. In order to approximate the solution of equation (26), we approximate functions $f_i(x)$, $u_i(x)$ and $k_{ij}(x,t)$ with respect to Laguerre polynomials (basis) by the way mentioned in Section 2 as:

$$f_i(x) \simeq F_i^T L_x, \quad u_i'(x) \simeq C_i'^T L_x, \quad u_i(0) \simeq C_{i0}^T L_x, \quad k_{ij}(x,t) \simeq L_x^T K_{ij} L_t, \tag{27}$$

where F_i, C_i' for $i = 1, ..., m$ are known $(n+1)\times 1$ vectors and K_{ij} for $i, j = 1, 2, ..., m$ are known $(n+1) \times (n+1)$ matrices. Then for $i = 1, ..., m$, we have:

$$\begin{aligned} u_i(x) &= \int_0^x u_i'(t)dt + u_i(0) \simeq \int_0^x C_i'^T L_t dt + C_{i0}^T L_x \\ &\simeq C_i'^T P L_x + C_{i0}^T L_x = (C_i'^T P + C_{i0}^T) L_x, \end{aligned} \tag{28}$$

where P is the $(n+1)\times(n+1)$ operational matrix of integration given in Eq. (16). After substituting the approximation equations (27) and (28) into (26), we get the following:

$$\begin{aligned} L_x^T C_i' &= L_x^T F_i + \lambda \int_0^\infty e^{-t} \sum_{j=1}^m L_x^T K_{ij} L_t L_t^T (p^T C_j' + C_{j0}) dt \\ &= L_x^T F_i + \lambda L_x^T \sum_{j=1}^m K_{ij} \{ \int_0^\infty e^{-t} L_t L_t^T dt \} (p^T C_j' + C_{j0}) \\ &= L_x^T F_i + \lambda L_x^T \sum_{j=1}^m K_{ij} (p^T C_j' + C_{j0}). \end{aligned} \tag{29}$$

Then we have following system of linear equations:

$$C_i' = F_i + \lambda \sum_{j=1}^m K_{ij} (p^T C_j' + C_{j0}), \quad i = 1, ..., m. \tag{30}$$

By solving above linear system, we can achieve the vector C_i' for $i = 1, ..., m$, then we will have

$$C_i^T = C_i'^T P + C_{i0}^T \Longrightarrow u_i(x) \simeq C_i^T L_x, \quad i = 1, ..., m. \tag{31}$$

That are the approximate solution for our system of (IBI-DE) (1). Also one can check the accuracy of the method. Since the truncated Laguerre series are approximate the solutions of the systems (1), so the error function $e(u_i(x))$ is constructed as follows

$$e(u_i(x)) = |u_i(x) - C_i^T L_x|. \tag{32}$$

If we set $x = x_j$ where $x_j \in [0, 1]$, the error values can be obtained.

4. Illustrative Examples

To demonstrate the effectiveness of the proposed method in the present paper, two test examples are carried out in this section. For each example we find the approximate solutions using different degree of Laguerre polynomials. The results obtained by the present methods reveal that the present method is very effective and convenient for system (1) on the half line. The computations associated with the examples were performed in a personal computer using Matlab.

Example 4.1. For the first example, consider the following system of infinite boundary integro-differential equations(constructed):

$$u_1'(x) = f_1(x) + \int_0^\infty e^{-t}(2x+t^2)(u_1(t)+u_2(t))dt,$$
$$u_2'(x) = f_2(x) + \int_0^\infty e^{-t}(t-x^2)(u_1(t)-u_2(t))dt, \quad (33)$$

where $f_1(x) = 3x^2 - 24x - 158$ and $f_2(x) = 6x^2 + 2x - 22$. Subject to initial conditions $u_1(0) = 1$ and $u_2(0) = 1$. The exact solutions of this problem are $u_1(x) = x^3 + 2x + 1$ and $u_2(x) = x^2 + 1$. If we apply the presented method in this paper and solve equation (33) with $n = 3$. For this system we get:

$$u_1(x) = (9)L_0(x) + (-20)L_1(x) + (18)L_2(x) + (-6)L_3(x) = x^3 + 2x + 1,$$
$$u_2(x) = (3)L_0(x) + (-4)L_1(x) + (2)L_2(x) + (0)L_3(x) = x^2 + 1, \quad (34)$$

which is the exact solution. Also, if we choose $n \geqslant 4$, we get the same approximate solution as obtained in equation (34). Numerical results will not be presented since the exact solution is obtained.

Example 4.2. For the second example, consider the following system of infinite boundary integro-differential equations(constructed):

$$u_1'(x) = f_1(x) + \int_0^\infty e^{-t-x}(sin(t-x)u_1(t) + tu_2(t))dt,$$
$$u_2'(x) = f_2(x) + \int_0^\infty e^{-t}(xtu_1(t) - e^{-x}u_2(t))dt, \quad (35)$$

with $f_1(x) = 1 - \frac{1}{4}(1 + 2cosx)e^{-x}$ and $f_2(x) = -2x - \frac{1}{2}e^{-x}$ and with the exact solutions $u_1(x) = x$, $u_2(x) = e^{-x}$ and boundary conditions $u_1(0) = 0$ and $u_2(0) = 1$. The Laguerre series approach is applied for solving Eq. (35) and numerical results are provided in table 1 that shows the absolute errors for $n = 8$, and $n = 12$ using the present method in equally divided interval $[0, 1]$ for $u_2(x)$.

Absolute errors for $u_2(x)$

i	x_i	$n = 8$	$n = 12$
0	0.0	$2.0000e-003$	$1.2207e-004$
1	0.1	$4.0892e-004$	$1.6620e-007$
2	0.2	$4.6337e-004$	$4.8478e-005$
3	0.3	$8.5134e-004$	$5.3272e-005$
4	0.4	$9.0567e-004$	$3.5120e-005$
5	0.5	$7.4563e-004$	$8.2898e-006$
6	0.6	$4.6392e-004$	$1.8048e-005$
7	0.7	$1.3087e-004$	$3.8553e-005$
8	0.8	$2.0185e-004$	$5.0686e-005$
9	0.9	$4.9795e-004$	$5.3884e-005$
10	1.0	$7.3375e-004$	$4.8904e-005$

Note that absolute errors for $u_1(x)$ is zero.

Corollary: If the exact solution to system (1) be a polynomial, then the proposed method will obtain in the real solution.

5. Conclusion

In this article, we develop an efficient and powerful method for solving system of infinite boundary integro-differential equations of the second kind along with initial conditions on a semi-infinite domain by using of Laguerre polynomials. By some useful properties of these polynomials such as, operational matrix, orthogonal basis and coefficient matrix together with Galerkin method, a system of infinite boundary integro-differential equations can be transformed to a linear system of algebraic equations. The numerical results given in the previous section show that the proposed algorithm with a small number of Laguerre polynomials is giving a satisfactory result.

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