Recognition by prime graph of the almost simple group PGL(2, 25)

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Abstract. Throughout this paper, every groups are finite. The prime graph of a group $G$ is denoted by $\Gamma(G)$. Also $G$ is called recognizable by prime graph if for every finite group $H$ with $\Gamma(H) = \Gamma(G)$, we conclude that $G \cong H$. Until now, it is proved that if $k$ is an odd number and $p$ is an odd prime number, then $\text{PGL}(2,p^k)$ is recognizable by prime graph. So if $k$ is even, the recognition by prime graph of $\text{PGL}(2,p^k)$, where $p$ is an odd prime number, is an open problem. In this paper, we generalize this result and we prove that the almost simple group $\text{PGL}(2,25)$ is recognizable by prime graph.

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1. Introduction

Let $\mathbb{N}$ denotes the set of natural numbers. If $n \in \mathbb{N}$, then we denote by $\pi(n)$, the set of all prime divisors of $n$. Let $G$ be a finite group. The set $\pi(|G|)$ is denoted by $\pi(G)$. Also the set of element orders of $G$ is denoted by $\pi_e(G)$. We denote by $\mu(S)$, the maximal numbers of $\pi_e(G)$ under the divisibility relation. The prime graph of $G$ is a graph whose vertex set is $\pi(G)$ and two distinct primes $p$ and $q$ are joined by an edge (and we write $p \sim q$), whenever $G$ contains an element of order $pq$. The prime graph of $G$ is denoted by $\Gamma(G)$. A finite group $G$ is called recognizable by prime graph if for every finite group $H$ such that $\Gamma(H) = \Gamma(G)$, then we have $G \cong H$.

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In [10], it is proved that if \( p \) is a prime number which is not a Mersenne or Fermat prime and \( p \neq 11, 19 \) and \( \Gamma(G) = \Gamma(\text{PGL}(2, p)) \), then \( G \) has a unique nonabelian composition factor which is isomorphic to \( \text{PSL}(2, p) \) and if \( p = 13 \), then \( G \) has a unique nonabelian composition factor which is isomorphic to \( \text{PSL}(2, 13) \) or \( \text{PSL}(2, 27) \). In [3], it is proved that if \( q = p^\alpha \), where \( p \) is a prime and \( \alpha > 1 \), then \( \text{PGL}(2, q) \) is uniquely determined by its element orders. Also in [1], it is proved that if \( q = p^\alpha \), where \( p \) is an odd prime and \( \alpha \) is an odd natural number, then \( \text{PGL}(2, q) \) is uniquely determined by its prime graph. In this paper as the main result we consider the recognition by prime graph of almost simple group \( \text{PGL}(2, 25) \).

2. Preliminary Results

**Lemma 2.1** ([8]) Let \( G \) be a finite group and \( |\pi(G)| \geq 3 \). If there exist prime numbers \( r, s, t \in \pi(G) \), such that \( \{tr, ts, rs\} \cap \pi_e(G) = \emptyset \), then \( G \) is non-solvable.

**Lemma 2.2** (see [20]) Let \( G \) be a Frobenius group with kernel \( F \) and complement \( C \). Then every subgroup of \( C \) of order \( pq \), with \( p, q \) (not necessarily distinct) primes, is cyclic. In particular, every Sylow subgroup of \( C \) of odd order is cyclic and a Sylow 2-subgroup of \( C \) is either cyclic or generalized quaternion group. If \( C \) is a non-solvable group, then \( C \) has a subgroup of index at most 2 isomorphic to \( SL(2, 5) \times M \), where \( M \) has cyclic Sylow \( p \)-subgroups and \( |M|, 30| = 1 \).

Using [14, Theorem A], we have the following result:

**Lemma 2.3** Let \( G \) be a finite group with \( t(G) \geq 2 \). Then one of the following holds:

(a) \( G \) is a Frobenius or 2-Frobenius group;

(b) there exists a nonabelian simple group \( S \) such that \( S \leq \overline{G} := G/N \leq \text{Aut}(S) \) for some nilpotent normal subgroup \( N \) of \( G \).

**Lemma 2.4** ([21]) Let \( G = L^e_n(q) \), \( q = p^m \), be a simple group which acts absolutely irreducibly on a vector space \( W \) over a field of characteristic \( p \). Denote \( H = W \rtimes G \). If \( n = 2 \) and \( q \) is odd then \( 2p \in \pi_e(H) \).

3. Main Results

**Theorem 3.1** The almost simple group \( \text{PGL}(2, 25) \) is recognizable by prime graph.

**Proof.** Throughout this proof, we suppose that \( G \) is a finite group such that \( \Gamma(G) = \Gamma(\text{PGL}(2, 25)) \).

First of all, we remark that by [19, Lemma 7], we have \( \mu(\text{PGL}(2, 25)) = \{5, 24, 26\} \). Therefore, the prime graph of \( \text{PGL}(2, 25) \) has two connected components which are \( \{5\} \) and \( \pi(5^4 - 1) \). Also we conclude that the subsets \( \{2, 5\} \) and \( \{3, 5, 13\} \) are two independent subsets of \( \Gamma(G) \). In the sequel, we prove that \( G \) is neither a Frobenius nor a 2-Frobenius group.

Let \( G = K : C \) be a Frobenius group with kernel \( K \) and complement \( C \). By Lemma 2.2, we know that \( K \) is nilpotent and \( \pi(C) \) is a connected component of the prime graph of \( G \). Hence we conclude that \( \pi(K) = \{5\} \) and \( \pi(C) = \{2, 3, 13\} \), since 5 is an isolated vertex in \( \Gamma(G) \).

If \( C \) is non-solvable, then by Lemma 2.2, \( C \) consists a subgroup isomorphic to \( SL(2, 5) \). This implies that \( 5 \in \pi(\text{SL}(2, 5)) \subseteq \pi(C) \), which is a contradiction since \( \pi(C) = \{2, 3, 13\} \). Therefore, \( C \) is solvable and so it contains a \( \{3, 13\} \)-Hall subgroup,
say \( H \). Since \( K \) is a normal subgroup of \( G \), \( KH \) is a subgroup of \( G \). Also we have \( \pi(KH) = \{3, 5, 13\} \). Thus \( KH \) is a subgroup of odd order and so it is a solvable subgroup of \( G \). On the other hand, in the prime graph of \( G \), the subset \{3, 5, 13\} is independent. Hence \( KH \) is a solvable subgroup of \( G \) such that its prime graph contains no edge, which contradicts to Lemma 2.1. Therefore, we get that \( G \) is a Frobenius group.

Let \( G \) be a 2-Frobenius group with the normal series \( 1 < H < K < G \), where \( K \) is a Frobenius group with kernel \( H \) and \( G/H \) is a Frobenius group with kernel \( K/H \). We know that \( G \) is a solvable group. This implies that \( G \) contains a \{3, 5, 13\}-Hall subgroup, say \( T \). Again similar to the previous discussion, we get a contradiction.

By the above argument, the finite group \( G \) is neither Frobenius nor 2-Frobenius. So by Lemma 2.3, we conclude that there exists a nonabelian simple group \( S \) such that:

\[
S \leq \overline{G} := \frac{G}{K} \leq \text{Aut}(S)
\]

in which \( K \) is the Fitting subgroup of \( G \). We know that \( \pi(S) \subseteq \pi(G) \). Since \( \pi(G) = \{2, 3, 5, 13\} \), so by [13, Table 8], we get that \( S \) is isomorphic to one of the simple group \( A_5, A_6, \text{PSU}_3(2), \text{PSU}_4(2), \text{PSL}_3(3), S_4(5), \) or \( \text{PSL}_2(25) \). Now we consider each possibility for the simple group \( S \), step by step.

**Step 1.** Let \( S \) be isomorphic to the alternating group \( A_5 \) or \( A_6 \). Since \( \pi(S) \cup \pi(\text{Out}(S)) = \{2, 3, 5\} \), we conclude that \( 13 \in \pi(K) \). We know that the alternating groups \( A_5 \) and \( A_6 \) consist a Frobenius subgroup \( 2^2 : 3 \). Hence since \( 13 \in \pi(K) \), by [17, Lemma 3.1], we deduce that \( 13 \sim 3 \), which is a contradiction.

**Step 2.** Let \( S \) be isomorphic to the simple group \( \text{PSU}_4(2) \). By [5], the finite group \( S \) contains a Frobenius group \( 2^2 : 3 \), so similar to the above argument we get a contradiction.

**Step 3.** Let \( S \) be isomorphic to the simple group \( \text{PSU}_3(4) \). Again by [4], in the prime graph of the simple group \( S \), \( 5 \) is not an isolated vertex.

**Step 4.** Let \( S \) be isomorphic to the simple group \( \text{PSL}_3(3) \). Since \( \pi(\text{PSL}_3(3)) = \{2, 3, 13\} \), we get that \( 5 \in \pi(K) \). On the other hand Sylow 3-subgroups of \( \text{PSL}_3(3) \) are not cyclic. Hence \( 5 \notin \pi(K) \), since \( 5 \) and \( 3 \) are nonadjacent in \( \Gamma(G) \).

**Step 5.** Let \( S \) be isomorphic to the simple group \( S_4(5) \). Again by [4], in the prime graph of the simple group \( S \), \( 5 \) is not an isolated vertex.

**Step 4.** Let \( S \) be isomorphic to \( \text{PSL}_2(25) \). Hence \( \text{PSL}_2(25) \leq \overline{G} \leq \text{Aut}(\text{PSL}_2(25)) \).

Let \( \pi(K) \) contains a prime \( r \) such that \( r \neq 5 \). Since \( K \) is a vector space over a field with \( r \) elements. Hence the prime graph of the semidirect product \( K \rtimes \text{PSL}_2(25) \) is a subgraph of \( \Gamma(G) \). Let \( B \) be a Sylow 5-subgroup of \( \text{PSL}_2(25) \). We know that \( B \) is not cyclic. On the other hand \( K \rtimes B \) is a Frobenius group such that \( \pi(K \times B) = \{r, 5\} \). Hence \( B \) should be cyclic which is a contradiction.

Let \( \pi(K) = \{5\} \). In this case, by Lemma 2.4, we get that there is an edge between \( 2 \) and \( 5 \) in the prime graph of \( G \) which is a contradiction. Therefore, by the above discussion, we deduce that \( K = 1 \). Also this implies that \( \text{PSL}_2(25) \leq G \leq \text{Aut}(\text{PSL}_2(25)) \).

We know that \( \text{Aut}(\text{PSL}_2(25)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). Since in the prime graph of \( \text{PSL}_2(25) \) there is not any edge between \( 13 \) and \( 2 \), we get that \( G \neq \text{PSL}_2(25) \). Also if \( G = \text{PSL}_2(25) : \langle \theta \rangle \) where \( \theta \) is a field automorphism, then we get that \( 2 \) and \( 5 \) are adjacent in \( G \), which is a contradiction. If \( G = \text{PSL}_2(25) : \langle \gamma \rangle \), where \( \gamma \) is a diagonal-field automorphism, then we get that \( G \) does not contain any element with order \( 2 \cdot 13 \) (see [3, Lemm 12]), which is contradiction, since in \( \Gamma(G) \), \( 2 \sim 13 \). This argument shows that \( G \cong \text{PGL}_2(25) \), which completes the proof. 

\[\blacksquare\]
References

[17] A. Mahmoudifar, On finite groups with the same prime graph as the projective general linear group PGL(2, 81), (to appear).