Elliptic Function solutions of (2+1)-Dimensional Breaking Soliton Equation by Sinh-Cosh Method and Sinh-Gordon Expansion Method

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Abstract

In this paper, based on sinh-cosh method and sinh-Gordon expansion method, families of solutions of (2+1)-dimensional breaking soliton equation are obtained. These solutions include Jacobi elliptic function solution, soliton solution, trigonometric function solution.

Key words: sinh-cosh method, soliton, Jacobi elliptic function, sinh-Gordon expansion method.

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1 Introduction

There exist many methods for obtaining solutions of the (2+1)-Dimensional breaking soliton equation, such as the Generalized Jacobi elliptic function method [2], \((G'/G)\) Expansion method [3] and so on. In this paper, by using the sinh-cosh method [1] and sinh-Gordon expansion method [4,5], we construct elliptic function solutions in the (2+1)-dimensional breaking soliton equation.

\[ u_t - bu_{xxx} + 4b(uv)_x = 0, \]  
\[ v_x - u_y = 0, \]

(1.1) (1.2)

Where \(b\) is an arbitrary constant, the system (1)-(2) was used to describe the (2+1)-dimensional interaction of Riemann was propagated along the y-axis with long wave propagated along the x-axis and it seems to have been investigated extensively where overlapping solutions have been derived.

2 Methods

Consider a given (2+1)-dimensional breaking soliton equation with independent variable \(x = (t, x_1, x_2, ...)\) and dependent variables \(u(x)\). The following formal solution of the given (2+1)-dimensional breaking soliton equation will be sought by the following ansatz

\[ u(x) = A_0 + \sum_{i=1}^{n} \cosh^{-1}(w) [A_i \sinh(w) + B_i \cosh(w)], \]

(2.1)

Where \(n\) is an integer which is determined by balancing the highest order derivative term with the highest order nonlinear term in the given(1)-(2) [5], and \(A_0 = A_0(x), ..., A_n = A_n(x), B_1 = B_1(x), ..., B_n = B_n(x), w = w(\mu), \mu = \alpha x + p + q\) are all differentiable function.
satisfies \( \omega \)

\[
\left( \frac{dw}{dx} \right)^2 = \sinh^2(w(\mu)) + c,
\]

(2.2)

Or in another form

\[
\frac{d^2w}{dx^2} = \sinh(w) \cosh(w),
\]

(2.3)

Where \( c = 1 - m^2 \) and \( m \) is the modulus of Jacobi elliptic function. Equation (2.2) has the following solution:

\[
\sinh(w) = cs(\mu, m) = \frac{cn(\mu, m)}{sn(\mu, m)},
\]

(2.4)

\[
\cosh(w) = ns(\mu, m) = \frac{1}{sn(\mu, m)},
\]

(2.5)

Where \( sn(\mu, m), cn(\mu, m) \) are Jacobian elliptic sine function and the Jacobian elliptic cosine function respectively. We can also seek (2+1)-dimensional breaking soliton equation solution in the up form where \( w = a(\xi), \xi = k(x + \alpha y - \beta t) \) where \( \xi \) a real parameter and \( k, \alpha, \beta \) are constant.

3 the application of methods

3.1 the application of sinh-cosh method

In order to solve (1) and (2) by using our method, we first reduce (1) and (2) to a differential equations. We make transformations

\[
u(x, y, t) = u(\mu), v(x, yt) = v(\mu), \]

(3.1)

\[
\mu = \alpha x + p + q,
\]

(3.2)
Where $\alpha$ is a nonzero constant and $p$ is the function of, $q$ is a function of.

The substitutions of (8) and (9) into (1) and (2) yields

\[
\begin{align*}
q'(t)u' - b\alpha^2 p' \beta(y)u'' + 4b\alpha u'v + 4b\alpha uv' &= 0, \\
\alpha v' - p'(y)u' &= 0,
\end{align*}
\]

(3.3) \hspace{1cm} (3.4)

And integrating yields, (10) and (11)

\[
\begin{align*}
q'(t) - b\alpha^2 p' \beta(y)u'' + 4b\alpha uv &= 0, \\
\alpha v - p'(y)u &= 0,
\end{align*}
\]

(3.5) \hspace{1cm} (3.6)

The substitutions of $v = \frac{p'(y)}{\alpha} u$ into (12) yields

\[
q'(t)u - b\alpha^2 p' \beta(y)u'' + 4bp'u^2 = 0.
\]

(3.7)

Balancing $u^2$ with $u''$ then gives $n = 2.$

According to method we assume that (14) has the solution

\[
u(x) = A_0 + A_1 \sinh(w) + B_1 \cosh(w) + A_2 \sinh(w) \cosh(w) + B_2 \cosh^2(w),
\]

(3.8)

Substituting (15) into (14) along with (4) and (5), yields a differential equation about setting the coefficients of $\sinh^i(w) \cosh^j(w)(\sinh^2(w) + \frac{c}{k}), i = 1, 2, \ldots; j = 0, 1; k = 0, 1.$

\[
sinh^i(w) \cosh^j(w)(\sinh^2(w) + \frac{c}{k}), i = 1, 2, \ldots; j = 0, 1; k = 0, 1 \text{ to zero, we}
\]
get the overdetermined equations:

\[
q'(t)A_0 + q'(t)B_2 + 2b\alpha^2 p'(y)B_2 c + 4bp'(y)A_0^2 + 4bp'(y)B_1^2 + 4bp'(y)B_2^2 \\
+ 8bp'(y)A_0 B_2 = 0,
\]
\[
q'(t)A_1 - b\alpha^2 p'(y)A_1 - b\alpha^2 p'(y)A_1 c + 8bp'(y)A_0 A_1 + 8bp'(y)A_1 B_2 \\
+ 8bp'(y)B_1 A_2 = 0,
\]
\[
q'(t)A_1 - b\alpha^2 p'(y)B_1 c + 8bp'(y)A_0 B_1 \\
+ 8bp'(y)B_1 B_2 = 0,
\]
\[
q'(t)A_2 - 4b\alpha^2 p'(y)A_2 c - 8bp'(y)B_2 c + 8bp'(y)A_1 B_1 + 8bp'(y)A_2 B_2 \\
+ 8bp'(y)A_0 A_2 = 0,
\]
\[
q'(t)B_2 - 4b\alpha^2 p'(y)A_2 c - 8bp'(y)B_2 c + 4bp'(y)A_1^2 + 4bp'(y)A_2^2 \\
+ 8bp'(y)B_2^2 \\
+ 8bp'(y)A_0 B_2 + 4bp'(y)B_1^2 = 0,
\]
\[
8bp'(y)B_1 A_2 - 2b\alpha^2 p'(y)A_1 + 8bp'(y)A_1 B_2 = 0,
\]
\[
8bp'(y)A_1 A_2 + 8bp'(y)B_1 B_2 - 2b\alpha^2 p'(y)B_1 = 0,
\]
\[
- 6b\alpha^2 p'(y)A_2 + 8bp'(y)A_2 B_2 = 0,
\]
\[
- 6b\alpha^2 p'(y)B_2 + 4bp'(y)A_2^2 + 4bp'(y)B_2^2 = 0.
\]

Solving equations with Maple, we derive the solutions of the partial differential equations.

\[
A_0 = \frac{1}{2} \alpha^2 \sqrt{\frac{1}{16} - c + c^2 - \frac{5}{8} \alpha^2 + \frac{1}{2} \alpha^2 c}, A_1 = 0, B_1 = 0,
\]
\[
A_2 = -\frac{3}{4} \alpha^2, B_2 = \frac{3}{4} \alpha^2, p = \frac{1}{\alpha^2 y}, q = \left[ -4b \sqrt{\frac{1}{16} - c + c^2} \right] t
\]
We have obtained solutions of (12) and (13) if $v = \frac{1}{\alpha'}(y)$, these solutions are

$$
\begin{align*}
  u_{11} &= \frac{1}{2} \alpha^2 \sqrt{\frac{1}{16} - c + c^2} - \frac{5}{8} \alpha^2 + \frac{1}{2} \alpha^2 c \\
  &\quad - \frac{3}{4} \alpha^2 \text{csch} \left( \alpha x + \frac{1}{\alpha^2} y + \left[ -4b \sqrt{\frac{1}{16} - c + c^2} \right] t, m \right) \\
  &\quad + ns \left( \alpha x + \frac{1}{\alpha^2} y + \left[ -4b \sqrt{\frac{1}{16} - c + c^2} \right] t, m \right) \\
  &\quad + \frac{3}{4} \alpha^2 n^2 \left( \alpha x + \frac{1}{\alpha^2} y + \left[ -4b \sqrt{\frac{1}{16} - c + c^2} \right] t, m \right) \\
  v_{11} &= \frac{1}{2} \alpha \sqrt{\frac{1}{16} - c + c^2} - \frac{5}{8 \alpha} + \frac{1}{2} \alpha c \\
  &\quad - \frac{3}{4} \alpha \text{csch} \left( \alpha x + \frac{1}{\alpha^2} y + \left[ -4b \sqrt{\frac{1}{16} - c + c^2} \right] t, m \right) \\
  &\quad + ns \left( \alpha x + \frac{1}{\alpha^2} y + \left[ -4b \sqrt{\frac{1}{16} - c + c^2} \right] t, m \right) \\
  &\quad + \frac{3}{4} \alpha n^2 \left( \alpha x + \frac{1}{\alpha^2} y + \left[ -4b \sqrt{\frac{1}{16} - c + c^2} \right] t, m \right) \\
\end{align*}
$$

(3.10)

When $m \to 1, \text{cs}(\mu, m) \to \text{csch}(\mu)$ and $\text{ns}(\mu, m) \to \text{coth}(\mu), c \to 0$ so obtain the following soliton solutions of (1) and (2). (figure 1)

$$
\begin{align*}
  u_{12} &= -\frac{1}{2} \alpha^2 - \frac{3}{4} \alpha^2 \text{csch} \left( \alpha x + \frac{1}{\alpha^2} y - bt \right) \text{coth} \left( \alpha x + \frac{1}{\alpha^2} y - bt \right) \\
  &\quad + \frac{3}{4} \alpha^2 \text{coth}^2 \left( \alpha x + \frac{1}{\alpha^2} y - bt \right) \\
  v_{12} &= -\frac{1}{2} \alpha - \frac{3}{4} \alpha \text{csch} \left( \alpha x + \frac{1}{\alpha^2} y - bt \right) \text{coth} \left( \alpha x + \frac{1}{\alpha^2} y - bt \right) \\
  &\quad + \frac{3}{4} \alpha^2 \text{coth}^2 \left( \alpha x + \frac{1}{\alpha^2} y - bt \right) \\
\end{align*}
$$

(3.11)

When $m \to 0, \text{cs}(\mu, m) \to \text{coth}(\mu)$ and $\text{ns}(\mu, m) \to \text{csc}(\mu), c \to 1$ so obtain the following trigonometric function solutions of (1) and (2). (figure
\[
\begin{align*}
\begin{cases}
  u_{13} & = -\frac{3}{4} \alpha^2 \coth \left( \alpha x + \frac{1}{\alpha^2} y - bt \right) \csc \left( \alpha x + \frac{1}{\alpha^2} y - bt \right) \\
  & + \frac{3}{4} \alpha^2 \csc^2 \left( \alpha x + \frac{1}{\alpha^2} y - bt \right), \\
  v_{13} & = -\frac{3}{4} \alpha \coth \left( \alpha x + \frac{1}{\alpha^2} y - bt \right) \csc \left( \alpha x + \frac{1}{\alpha^2} y - bt \right) \\
  & + \frac{3}{4} \alpha \csc^2 \left( \alpha x + \frac{1}{\alpha^2} y - bt \right)
\end{cases}
\end{align*}
\tag{3.12}
\]

3.2 the application of sinh-Gordon expansion method

In order to solve (1) and (2) by using our method, we first reduce (1) and (2) to differential equations. We make transformations

\[
u(x, y, t) = u(\xi), \quad v(x, y, t) = v(\xi) \tag{3.13}
\]

\[
\xi = k(x + \alpha y - \beta t) \tag{3.14}
\]

Where \(\xi\) is a real parameter and \(k, \alpha, \beta\) are constants. The substitutions of (20) and (21) into (1) and (2) yield

\[
-k\beta u' - bk^3 \alpha u''' + 4bkuv' + 4bkuu' = 0, \tag{3.15}
\]

\[
kv' - k\alpha u' = 0, \tag{3.16}
\]

And integrating yields, (22) and (23)

\[
-k\beta u - bk^3 \alpha u''' + 4bkuv = 0, \tag{3.17}
\]

\[
kv - k\alpha u = 0, \tag{3.18}
\]

The substitutions of \(v = \alpha u\) into (24) yield

\[
-k\beta u - bk^3 \alpha u''' + 4b\alpha u^2 = 0. \tag{3.19}
\]
Balancing $u''$ with $u'$ the gives $n = 2$.
According to method we assume that (26) has the solution

$$u(\xi) = A_0 + A_1 \sinh(w) + B_1 \cosh(w) + A_2 \sinh(w) \cosh(w) + B_2 \cosh^2(w),$$

(3.20)

Substituting (27) and (26) along with (4) and (5), yields a hyperbolic polynomial about

$$w^s \sinh^i(w) \cosh^j(w) \ (i = 0, 1; s = 0, 1; j = 0, 1, 2, ...).$$

(3.21)

Setting the coefficients of (28) to zero, we get the following of equations:

\begin{align*}
-k\beta A_0 - 2bk^3\alpha B_2 + 2bk^3\alpha B_2c + 4bk\alpha A_0^2 - 4bk\alpha A_1^2 &= 0, \\
bk^3\alpha A_1 - bk^3\alpha A_1c + 8bk\alpha A_0 A_1 - k\beta A_1 &= 0, \\
k\beta B_1 + 2bk^3\alpha B_1 - bk^3\alpha B_1c + 8bk\alpha A_0 B_1 - 8bk\alpha A_1 A_2 &= 0, \\
k\beta A_1 + 5bk^3\alpha A_2 - 4bk^3\alpha A_2c + 8bk\alpha A_1 B_1 - 8bk\alpha A_0 A_2 &= 0, \\
k\beta B_2 + 8bk^3\alpha B_2 - 4bk^3\alpha B_2c + 4bk\alpha A_1^2 - 4bk\alpha A_2^2 + 8bk\alpha A_0 B_2 + 4bk\alpha B_1^2 &= 0, \\
-2bk^3\alpha A_1 + 8bk\alpha A_1 B_2 + 8bk\alpha B_1 A_2 &= 0, \\
-2bk^3\alpha B_1 + 8bk\alpha A_1 A_2 + 8bk\alpha B_1 B_2 &= 0, \\
-6bk^3\alpha A_2 + 8bk\alpha A_2 B_2 &= 0, \\
-6bk^3\alpha B_2 + 4bk\alpha A_1^2 + 4bk\alpha B_2^2 &= 0.
\end{align*}

Solving equations with Maple, we derive the following solutions:

\begin{align*}
A_0 &= \frac{\beta}{8bk} - \frac{5}{8}k^2 + \frac{1}{2}k^2c, A_1 = 0, B_1 = 0, \\
A_2 &= -\frac{3}{4}\alpha k^2, B_2 = \frac{3}{4}k^2, \beta = -4bka^2 k^2  \sqrt{16 + c^2 - c}
\end{align*}

(3.22)
We have obtained solutions of (24) and (25) if $v = \alpha u$, these solutions are

$$
\begin{align*}
\begin{cases}
u_{21} = \left( \frac{\beta}{8\alpha} - \frac{5}{8} k^2 + \frac{1}{2} k^2 \right) - \frac{3}{4} k^2 c_s (k(x + \alpha y - \beta t), m) \\
ns (k(x + \alpha y - \beta t), m) + \frac{3}{4} k^2 ns^2 (k(x + \alpha y - \beta t), m),
\end{cases} \\
v_{21} = \left( \frac{\beta}{8\alpha} - \frac{5}{8} k^2 + \frac{1}{2} k^2 \right) - \frac{3}{4} k^2 c_s (k(x + \alpha y - \beta t), m) \\
ns (k(x + \alpha y - \beta t), m) + \frac{3}{4} k^2 ns^2 (k(x + \alpha y - \beta t), m)
\end{align*}
$$

(3.23)

When $m \to 1$, $cs(\xi, m) \to \csc(\xi)$ and $ns(\xi, m) \to \coth(\xi)$, $c \to 0$. so we obtain the following soliton solutions of (1) and (2). (figure 3)

$$
\begin{align*}
\begin{cases}
u_{22} = -\frac{3}{4} k^2 c_s c_s (k(x + \alpha y + b\alpha k^2 t)) \coth (k(x + \alpha y + b\alpha k^2 t)) \\
+ \frac{3}{4} k^2 c_s (k(x + \alpha y + b\alpha k^2 t)),
\end{cases} \\
v_{22} = -\frac{3}{4} k^2 c_s c_s (k(x + \alpha y + b\alpha k^2 t)) \coth (k(x + \alpha y + b\alpha k^2 t)) \\
+ \frac{3}{4} k^2 c_s (k(x + \alpha y + b\alpha k^2 t))
\end{align*}
$$

(3.24)

when $m \to 0$, $cs(\xi, m) \to \coth(\xi)$ and $ns(\xi, m) \to \csc(\xi)$, $c \to 1$ so we obtain the following trigonometric function solutions of (1) and (2)

$$
\begin{align*}
\begin{cases}
u_{23} = -\frac{1}{4} k^2 c_s c_s (k(x + \alpha y + b\alpha k^2 t)) \csc (k(x + \alpha y + b\alpha k^2 t)) \\
+ \frac{3}{4} k^2 c_s c_s (k(x + \alpha y + b\alpha k^2 t)),
\end{cases} \\
v_{23} = -\frac{1}{4} k^2 c_s c_s (k(x + \alpha y + b\alpha k^2 t)) \csc (k(x + \alpha y + b\alpha k^2 t)) \\
+ \frac{3}{4} k^2 c_s c_s (k(x + \alpha y + b\alpha k^2 t))
\end{align*}
$$

(3.25)

Some of the properties of these solutions of (1) and (2) are shown by means of figures as follows: figure 1 and figure 2 and figure 3 show the properties of $u_{12}, v_{12}$ and $u_{13}, v_{13}$ and $u_{22}, v_{22}$, respectively, where we select
parameters as follows:

\[ k = \frac{1}{2}, \alpha = \frac{1}{2}, b = 4 \]

Fig. 1. The soliton solutions \(u_{12}, v_{12}\) of the (2+1)-dimensional breaking soliton equation are shown at \(x = 0\).

Fig. 2. Trigonometric function solutions \(u_{13}, v_{13}\) of the (2+1)-dimensional breaking soliton equation are shown at \(x = 0\).
Fig. 3. the soliton solutions $u_{22}, v_{22}$ of the (2+1)-dimensional breaking soliton equation are shown at $x = 0$.

In summary, we have the sinh-Gordon expansion method and sinh-cosh method to the (2+1)-dimensional breaking soliton equation. As a result, Jacobi elliptic function solutions are obtained. When $m \to 1$, we get the soliton solutions; while when $m \to 0$, we get the trigonometric function solutions.

References


