Computational aspect to the nearest southeast submatrix that makes multiple a prescribed eigenvalue

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Abstract. Given four complex matrices A, B, C and D where $A \in \mathbb{C}^{n \times n}$ and $D \in \mathbb{C}^{m \times m}$ and let the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a normal matrix and assume that $\lambda$ is a given complex number that is not eigenvalue of matrix $A$. We present a method to calculate the distance norm (with respect to 2-norm) from $D$ to the set of matrices $X \in \mathbb{C}^{m \times m}$ such that, $\lambda$ be a multiple eigenvalue of matrix $\begin{pmatrix} A & B \\ C & X \end{pmatrix}$. We also find the nearest matrix $X$ to the matrix $D$.

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1. Introduction

In paper[4], A.N. Malyshev obtained the following formula for the 2-norm distance $\text{rsep}(A)$ from a complex $n \times n$ matrix to a closest matrix with a multiple eigenvalue:

$$\text{rsep}(A) = \min_{\lambda \in \mathbb{C}} \max_{\gamma \geq 0} \sigma_{2n-1}(G(\gamma)),$$

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where,

\[
G(\gamma) = \begin{pmatrix}
\lambda I - A & \gamma I \\
0 & \lambda I - A
\end{pmatrix},
\]

and \( \sigma_{2n-1}(G(\gamma)) \) is the penultimate singular value of the matrix \( G \), assuming that the singular values are numbered in decreasing order. Ikramov and Nazari in [2] introduced a correction for Malyshev’s formula for a normal matrix. In recent paper [5] Nazari and Rajabi used the same correction to [2] for the paper of Lippert [3] for normal matrices.

In the recent paper [1], Gracia and Velasco obtained the following formula for the 2-norm distance \( D \) from a complex \( m \times m \) matrix to a closest matrix with a multiple eigenvalue:

\[
\min_{X \in \mathbb{C}^{m \times m}, \|M(\alpha, X)\| \geq 2} \|X - D\| = \sup_{t \in \mathbb{R}} \sigma_{2m-1}(S_2(t)),
\]

where

\[
M(\alpha, X) = \begin{pmatrix}
A & B \\
C & X
\end{pmatrix}, \quad \text{and} \quad S_2(t) = \begin{pmatrix} M & tN \\
0 & M \end{pmatrix},
\]

that

\[
M = (D - \lambda I_m) - C(A - \lambda I_n)^{-1} B,
\]

\[
N = I_m + C(A - \lambda I_n)^{-2} B,
\]

where \( \lambda \) is not eigenvalue of matrix \( A \) and \( \sigma_{2m-1}(S_2(t)) \) is the penultimate singular value of the matrix \( S_2(t) \), where \( \alpha = (A, B, C) \in L_{n,m} = \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times m} \times \mathbb{C}^{m \times n} \). In the two Theorems that follows, we briefly describe the article of Gracia and Velasco.

**Theorem 1.1** Let \( t^* > 0 \) be a local optimizer of function \( s_2(t) = \sigma_{2m-1}(S_2(t)) \). Suppose \( \sigma^* = s_2(t^*) > 0 \), then there exists a pair of normalized singular vectors associated with the singular value \( t^* \) of \( s_2(t^*) \), namely a left vector

\[
u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1, u_2 \in \mathbb{C}^m
\]

and a corresponding right vector

\[
u = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad v_1, v_2 \in \mathbb{C}^m
\]

such that

\[
\text{Re}(u_1^* N v_2) = 0.
\]

where the matrix \( N \) is defined by (3). Moreover, the matrices

\[
U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},
\]
satisfy the relation
\[ V^*V = U^*U \in \mathbb{C}^{2\times2}. \] (6)

**Theorem 1.2** Let \( \lambda \) is not eigenvalue of matrix \( A \) and \( t^* \) in Theorem (1.1) is a positive number. The matrix \( D + \Delta \), where
\[ \Delta = -\sigma^* UV^\dagger, \] (7)

is the closest (with respect to the 2-norm) matrix to matrix \( X \), such that the matrix \( \begin{pmatrix} A & B \\ C & X \end{pmatrix} \) having multiple eigenvalue \( \lambda \) and
\[ \| \Delta \|_2 = \sigma^*, \] (8)

where denote by \( V^\dagger \) the Moore-Penrose inverse matrix of \( V \).

Let \( G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \). By a similar method that introduced in [2], we discuss some issues related to the computer implementation of this method. It turns out that the case of a general matrix \( G \) is substantially different from that of a normal matrix \( G \).

2. Normal matrix

Let \( G \) be a normal matrix. Let
\[ A = \begin{pmatrix} 12 & 7 & 7 \\ 7 & 16 & 10 \\ 7 & 10 & 12 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 5 \\ 3 & 3 \\ 11 & 11 \end{pmatrix}, \]
\[ C = \begin{pmatrix} 5 & 3 & 11 \\ 5 & 3 & 11 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}, \]

then it is easy to see that the matrix \( G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) is normal matrix. Assume that \( \lambda = 0 \). By MATLAB software with long format for computation, we found the values \( t^* = 7.60093750000001, \sigma^* = 7.60093621516323 \). The singular value \( \sigma_{2m-2} \) equals \( 7.60093750000000 \). These two values are approximately the same, namely
\[ \sigma_{2m-1}(S_2(t^*)) \simeq \sigma_{2m-2}(S_2(t^*)). \]

Thus, in the optimal matrix \( S_2(t^*) \), the value \( \sigma^* \) is iterated. Let \( u^{(2m-1)}, v^{(2m-1)} \) and \( u^{(2m-2)}, v^{(2m-2)} \) be the pairs of singular vectors of \( S_2(t^*) \) associated with \( \sigma_{2m-1} \) and \( \sigma_{2m-2} \), respectively, that MATLAB gives us. An attempt to use any of these pairs for implementing the construction described in Theorem (1.2) leads to catastrophic results. Namely, for the matrix
\[ \Delta^{2m-1} = -\sigma^* U^{(2m-1)} V^{(2m-1)} \dagger, \]
we obtain
\[ \| \Delta^{2m-1} \| = 2.3455800417827 \times 10^{16}, \]
while \( \Delta^{2m-2} = \sigma^* U^{(2m-2)} V^{(2m-2)\dagger} \) has the norm
\[ \| \Delta^{2m-2} \| = 2.34558396903085 \times 10^{16}. \]

It is easy to find the reason why equality (8) is violated in both cases. The value of
\[ u_1^{2m-1} v_2^{2m-1}, \text{ for two vectors } u^{(2m-1)} = \begin{pmatrix} u_1^{2m-1} \\ u_2^{2m-1} \end{pmatrix} \text{ and } v^{(2m-1)} = \begin{pmatrix} v_1^{2m-1} \\ v_2^{2m-1} \end{pmatrix}, \]
is
\[-0.78923059741806\]
and for the pair vectors \( u^{(2m-2)} = \begin{pmatrix} u_1^{2m-2} \\ u_2^{2m-2} \end{pmatrix} \) and \( v^{(2m-1)} = \begin{pmatrix} v_1^{2m-1} \\ v_2^{2m-1} \end{pmatrix}, \) is
\[ 1.00000000000000. \]

In any case above, equality (4), even approximately does not hold. It follows that equality (6) is violated. i.e.
\[ U^* U = \begin{pmatrix} 0.10538463458652 & 0.30704838931280 \\ 0.30704838931280 & 0.89461536541348 \end{pmatrix}, \]
\[ V^* V = \begin{pmatrix} 0.89461536541348 & 0.30704838931280 \\ 0.30704838931280 & 0.10538463458652 \end{pmatrix}. \]

If we calculate the eigenvalues of matrix \( G \), we see that
\[
10^7 \times \left\{ \begin{array}{l}
\text{The eigenvalues of matrix } G = \\
0.00000371178081,
0.00000065233548,
-0.00000139525368,
0.00000069856511 + 4.06720386782124i,
0.00000069856511 - 4.06720386782124i
\end{array} \right\}
\]

Since \( \lambda = 0 \), by Theorem (1.2) we must have a multiple eigenvalue zero in matrix \( G \), and all of eigenvalues that calculated above far from zero.

The situation can be rectified as follows. Consider the number
\[ \sigma^* = \sigma_{2m-1}(S_2(t)) \]
as a double singular value of \( S_2(t) \) and the vectors \( u^{(2m-1)} \) and \( u^{(2m-2)} \) as an orthonormal basis in the left singular subspace associated with \( \sigma^* \). In this subspace, we look for a normalized vector
\[ u = \alpha u^{(2m-1)} + \beta u^{(2m-2)}, \quad |\alpha|^2 + |\beta|^2 = 1, \quad (9) \]
and combined with the associated right singular vector

\[ v = \alpha v^{(2m-1)} + \beta v^{(2m-2)} \]  

(10)

in order to satisfy relation (4). From (4) we have

\[ \text{Re}(u_1^* N v_2) = 0. \]  

(11)

Substituting (9) and (10) into (11), we achieve the relation

\[ (\alpha \beta) \text{Re} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0, \]  

(12)

in which

\[ W = \begin{pmatrix} u_1^{(2m-1)H} * N * v_2^{(2m-1)} & u_1^{(2m-1)H} * N * v_2^{(2m-2)} \\ u_1^{(2m-2)H} * N * v_2^{(2m-1)} & u_1^{(2m-2)H} * N * v_2^{(2m-2)} \end{pmatrix} \]  

(13)

and

\[ W_f = \text{Re}(W) = \begin{pmatrix} \text{Re}(W_{11}) \frac{(W_{21}+W_{12})}{2} \\ \frac{(W_{12}+W_{21})}{2} \text{Re}(W_{22}) \end{pmatrix} \]  

(14)

The existence of a nontrivial solution for Eq. (12) is ensured by the fact that the Hermitian matrix (12) is indefinite. In fact, let us call \( g(t) = \sigma_{2m-2}(S_2(t)) \). Let \( \mu_1 \geq \mu_2 \) be the eigenvalues of the matrix \( \text{Re}W \). Then the right derivatives of the functions \( S_2 \) and \( g \) at \( t^* \) are equal to \( \mu_2 \) and \( \mu_1 \)

\[ S_2'(t^+) = \mu_2, \quad g'(t^+) = \mu_1 \]

respectively. Since \( S_2 \) is decreasing and \( g \) is increasing at right of \( t^* \), we deduce that

\[ \mu_2 < 0 \quad \text{and} \quad \mu_1 > 0. \]

The numbers \( \alpha \) and \( \beta \) can be found, for example, in following manner. Let

\[ W_f = PMP^*, \quad M = \text{diag}(\mu_1, \mu_2), \]

be the spectral decomposition of \( W \). Set

\[ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = P \begin{pmatrix} T \\ K \end{pmatrix}, \]  

(15)

and recast (12) as

\[ \mu_1|t|^2 + \mu_2|\sigma|^2 = 0, \quad |t|^2 + |\sigma|^2 = 1. \]  

(16)
The pair
\[
\left( \frac{\mu_2}{|\mu_1| + |\mu_2|} \right)^{\frac{1}{2}}, \quad \left( \frac{\mu_1}{|\mu_1| + |\mu_2|} \right)^{\frac{1}{2}}
\]
is a solution to system (16). (Recall again that \( \mu_1 \) and \( \mu_2 \) are numbers of different signs.)

Using (15), we obtain the corresponding pair \( \alpha, \beta \). In the example above with matrix \( G \), this technique yields
\[
\alpha = -0.74759578100550, \quad \beta = 0.66415400941556
\]

For the corresponding singular vectors (9) and (10), we have
\[
u_1^*Nv_2 = -2.238877486182567 \times 10^{-17}
\]
The matrix \( \Delta \) constructed from these vectors has the norm
\[
7.60093681855830
\]
which is in very good agreement with \( \sigma^* \). Finally, we found
\[
U^*U = \begin{pmatrix} 0.50000003728108 & 0.17160917645601 & 0.4999996271892 \\ 0.17160917645601 & 0.4999996271892 & 0.17160917645601 \\ 0.4999996271892 & 0.17160917645601 & 0.50000003728108 \end{pmatrix},
\]
\[
V^*V = \begin{pmatrix} 0.4999996271892 & 0.17160917645601 & 0.17160917645601 \\ 0.17160917645601 & 0.50000003728108 & 0.4999996271892 \\ 0.17160917645601 & 0.4999996271892 & 0.50000003728108 \end{pmatrix}
\]
it follows that \( U^*U \simeq V^*V \) and the eigenvalues of matrix \( G \) as follows:
\[
\begin{align*}
\text{The eigenvalues of matrix } G &= \left\{ 38.23444569157454, \\
&\phantom{=} 8.7751531971971, \\
&\phantom{=} 6.20796134531682, \\
&\phantom{=} 0.0000000420761, \\
&\phantom{=} -0.0000113313609 \right\}.
\end{align*}
\]

References