Characterization of $\delta$-double derivations on rings and algebras

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Abstract. The main purpose of this article is to offer some characterizations of $\delta$-double derivations on rings and algebras. To reach this goal, we prove the following theorem:

Let $n > 1$ be an integer and let $R$ be an $n!$-torsion free ring with the identity element $1$. Suppose that there exist two additive mappings $d, \delta : R \to R$ such that

$$d(x^n) = \sum_{j=1}^{n} x^{n-j} d(x)x^{j-1} + 2 \sum_{k=0}^{n-2} \sum_{i=0}^{2-k} x^k \delta(x)x^i \delta(x)x^{n-2-k-i}$$

is fulfilled for all $x \in R$. If $\delta(1) = 0$, then $d$ is a Jordan $\delta$-double derivation. In particular, if $R$ is a semiprime algebra and further, $\delta^2(x^2) = \delta^2(x)x + x\delta^2(x) + 2(\delta(x))^2$ holds for all $x \in R$, then $d - \frac{1}{2} \delta^2$ is an ordinary derivation on $R$.

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1. Introduction and preliminaries

Throughout the paper, $R$ will represent an associative ring with the identity element $1$. We consider $x^0 = 1$ for all $x \in R$. The center of $R$ is

$$Z(R) = \{ x \in R \mid xy = yx \text{ for all } y \in R \}.$$
Given an integer \( n \geq 2 \), a ring \( \mathcal{R} \) is said to be \( n \)-torsion free, if for \( x \in \mathcal{R} \), \( nx = 0 \) implies \( x = 0 \). We denote the commutator \( xy - yx \) by \([x, y]\) for all \( x, y \in \mathcal{R} \). Recall that a ring \( \mathcal{R} \) is prime if for \( x, y \in \mathcal{R} \), \( x\mathcal{R}y = \{0\} \) implies \( x = 0 \) or \( y = 0 \), and is semiprime in case \( x\mathcal{R}x = \{0\} \) implies \( x = 0 \).

As well, the above-mentioned statements are considered for algebras. An additive mapping \( d : \mathcal{R} \to \mathcal{R} \), where \( \mathcal{R} \) is an arbitrary ring, is called a derivation (resp. Jordan derivation) if \( d(xy) = d(x)y + xd(y) \) (resp. \( d(x^2) = 2dx \)) holds for all \( x, y \in \mathcal{R} \). One can easily prove that every derivation is a Jordan derivation, but the converse is not true, in general. An additive mapping \( d : \mathcal{R} \to \mathcal{R} \) is called a left derivation (resp. Jordan left derivation) if \( d(xy) = xd(y) + yd(x) \) (resp. \( d(x^2) = 2xd(x) \)) holds for all \( x, y \in \mathcal{R} \). A classical result of Herstein [7] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein’s result can be found in [3]. Cusack [5] generalized Herstein’s result to 2-torsion free semiprime rings (see also [2] for an alternative proof). A series of results related to derivations on prime and semiprime rings can be found in [1–4, 8, 11–13].

M. Mirzavaziri and E. O. Tehrani [9] introduced the concept of a \((\delta, \varepsilon)\)-double derivation. Let \( \delta, \varepsilon : \mathcal{R} \to \mathcal{R} \) be additive mappings. An additive mapping \( D : \mathcal{R} \to \mathcal{R} \) is a \((\delta, \varepsilon)\)-double derivation if \( D(xy) = D(x)y + xD(y) + \delta(x)\varepsilon(y) + \varepsilon(x)\delta(y) \) is fulfilled for all \( x, y \in \mathcal{R} \). By a \( \delta \)-double derivation we mean a \((\delta, \delta)\)-double derivation, i.e.

\[
D(xy) = D(x)y + xD(y) + 2\delta(x)\delta(y),
\]

for all \( x, y \in \mathcal{R} \). Let \( \mathcal{A} \) be an algebra and let \( D : \mathcal{A} \to \mathcal{A} \) be a linear \((\delta, \delta)\)-double derivation. If \( d = \frac{1}{2}D \), then \( d(ab) = d(a)b + ad(b) + \delta(a)\delta(b) \) holds for all \( a, b \in \mathcal{A} \). In this study, we consider the additive mapping \( d \) as a \((\delta, \delta)\)-double derivation on a ring \( \mathcal{R} \). Indeed, an additive mapping \( d : \mathcal{R} \to \mathcal{R} \) is called a \((\delta, \delta)\)-double derivation if \( d(xy) = d(x)y + xd(y) + \delta(x)\delta(y) \) holds for all \( x, y \in \mathcal{R} \). It is clear that if \( \delta(x)\delta(y) = 0 \) for all \( x, y \in \mathcal{R} \), then \( d \) is an ordinary derivation. Here, we want to characterize such \( \delta \)-double derivations. Similar to Jordan derivations, an additive mapping \( d \) is called a Jordan \( \delta \)-double derivation if \( d(x^2) = d(x)x + xd(x) + (\delta(x))^2 \) holds for all \( x \in \mathcal{R} \). Let \( n > 1 \) be an integer and let \( d, \delta : \mathcal{R} \to \mathcal{R} \) be two additive maps satisfying

\[
d(x^n) = \sum_{j=1}^{n} x^{n-j}d(x)x^{j-1} + \sum_{k=0}^{n-2} \sum_{i=0}^{n-2-k} x^k\delta(x)x^i\delta(x)x^{n-2-k-i},
\]

for all \( x \in \mathcal{R} \). If \( \mathcal{R} \) is an \( n \)-torsion free ring with the identity element \( 1 \) and \( \delta(1) = 0 \), then \( d \) is a Jordan \( \delta \)-double derivation. In particular, if \( \mathcal{R} \) is a semiprime algebra and further,

\[
\delta^2(x^2) = \delta^2(x)x + x\delta^2(x) + 2(\delta(x))^2,
\]

for all \( x \in \mathcal{R} \), then \( d - \frac{1}{2}\delta^2 \) is an ordinary derivation on \( \mathcal{R} \). After defining a left \( \delta \)-double derivation, we present a characterization of such mappings on algebras.

At the end of the paper, by getting idea from a work of Vukman [10], we offer another characterization of \( \delta \)-double derivations on Banach algebras as follows. Let \( \mathcal{A} \) be a Banach algebra with the identity element \( 1 \) and \( \delta, d : \mathcal{A} \to \mathcal{A} \) be two additive maps satisfying \( d(a) = -ad(a^{-1})a - a\delta(a^{-1})\delta(a) \) for each invertible element \( a \in \mathcal{A} \). If \( \delta(a) = -a\delta(a^{-1})a \) holds for every invertible element \( a \), then \( d \) is a Jordan \( \delta \)-double derivation. In particular,
if $\mathcal{A}$ is semiprime and $\left(\delta(a)\right)^2 = \frac{1}{2}\left(\delta^2(a^2) - \delta^2(a)a - a\delta^2(a)\right)$ holds for all $a \in \mathcal{A}$, then $d - \frac{1}{2}\delta^2$ is a derivation on $\mathcal{A}$.

2. Main results

We begin with the following definition.

**Definition 2.1** Let $\mathcal{R}$ be a ring and let $\delta : \mathcal{R} \to \mathcal{R}$ be an additive mapping. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a $\delta$-double derivation if $d(xy) = d(x)y + xd(y) + \delta(x)\delta(y)$ for all $x, y \in \mathcal{R}$. The additive mapping $d$ is said to be a Jordan $\delta$-double derivation if $d(x^2) = d(x)x + xd(x) + (\delta(x))^2$ for all $x \in \mathcal{R}$.

The first main theorem reads as follows:

**Theorem 2.2** Let $n > 1$ be an integer and $\mathcal{R}$ be an $n!$-torsion free ring with the identity element 1. Suppose that $d, \delta : \mathcal{R} \to \mathcal{R}$ are two additive maps satisfying $d(x^n) = \sum_{j=1}^{n} x^{n-j}d(x)x^{j-1} + \sum_{k=0}^{n-2} \sum_{i=0}^{n-2-k} x^k\delta(x)x^i\delta(x)x^{n-2-k-i}$ for all $x \in \mathcal{R}$. If $\delta(1) = 0$, then $d$ is a Jordan $\delta$-double derivation. In particular, if $\mathcal{R}$ is a semiprime algebra and further, $\delta^2(x^2) = \delta^2(x)(x) + x\delta^2(x) + 2(\delta(x))^2$ holds for all $x \in \mathcal{R}$, then $d - \frac{1}{2}\delta^2$ is a derivation on $\mathcal{R}$.

**Proof.** Let $y$ be an element of $Z(\mathcal{R})$ such that both $d(y)$ and $\delta(y)$ are zero. Based on the above hypothesis, we have

$$d(x^n) = \sum_{j=1}^{n} x^{n-j}d(x)x^{j-1} + \sum_{k=0}^{n-2} \sum_{i=0}^{n-2-k} x^k\delta(x)x^i\delta(x)x^{n-2-k-i} \quad (1)$$

for all $x \in \mathcal{R}$. Putting $x + y$ instead of $x$ in equation (1), we have

$$d\left(\sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i\right) = \sum_{j=1}^{n} (x + y)^{n-j}d(x)(x + y)^{j-1}$$

$$+ \sum_{k=0}^{n-2} \sum_{i=0}^{n-2-k} (x + y)^k\delta(x)(x + y)^i\delta(x)(x + y)^{n-2-k-i}$$

$$= \sum_{k_1=0}^{n-1} \binom{n-1}{k_1} x^{n-1-k_1}y^{k_1}d(x)$$

$$+ \sum_{k_1=0}^{n-2} \binom{n-2}{k_1} x^{n-2-k_1}y^{k_1}d(x)(x + y)$$

$$+ \sum_{k_1=0}^{n-3} \binom{n-3}{k_1} x^{n-3-k_1}y^{k_1}d(x)(x + y)^2 \to$$
\[
+ \ldots + (x + y)^2 \delta(x) \sum_{k_1=0}^{n-3} \binom{n-3}{k_1} x^{n-3-k_1} y^{k_1} \\
+ (x + y) \delta(x) \sum_{k_1=0}^{n-2} \binom{n-2}{k_1} x^{n-2-k_1} y^{k_1} \\
+ d(x) \sum_{k_1=0}^{n-1} \binom{n-1}{k_1} x^{n-1-k_1} y^{k_1} \\
+ \sum_{i=0}^{n-2} \delta(x)(x + y)^i \delta(x)(x + y)^{n-2-i} \\
+ \sum_{i=0}^{n-3} (x + y) \delta(x)(x + y)^i \delta(x)(x + y)^{n-3-i} \\
+ \sum_{i=0}^{n-4} (x + y)^2 \delta(x)(x + y)^i \delta(x)(x + y)^{n-4-i} \\
+ \ldots + (x + y)^{n-2} (\delta(x))^2 = \sum_{k_1=0}^{n-1} \binom{n-1}{k_1} x^{n-1-k_1} y^{k_1} d(x) \\
+ \sum_{k_1=0}^{n-2} \binom{n-2}{k_1} x^{n-2-k_1} y^{k_1} d(x)(x + y) \\
+ \sum_{k_1=0}^{n-3} \binom{n-3}{k_1} x^{n-3-k_1} y^{k_1} d(x)(x + y)^2 \\
+ \ldots + (x + y)^2 d(x) \sum_{k_1=0}^{n-3} \binom{n-3}{k_1} x^{n-3-k_1} y^{k_1} \\
+ (x + y) \delta(x) \sum_{k_1=0}^{n-2} \binom{n-2}{k_1} x^{n-2-k_1} y^{k_1} \\
+ d(x) \sum_{k_1=0}^{n-1} \binom{n-1}{k_1} x^{n-1-k_1} y^{k_1} + \left[(\delta(x))^2 \sum_{k_1=0}^{n-2} \binom{n-2}{k_1} x^{n-2-k_1} y^{k_1} \\
+ \delta(x)(x + y) \delta(x) \sum_{k_1=0}^{n-3} \binom{n-3}{k_1} x^{n-3-k_1} y^{k_1} \\
+ \ldots + \delta(x) \sum_{k_1=0}^{n-2} \binom{n-2}{k_1} y^{k_1} x^{n-2-k_1} \delta(x) \right] \\
+ \left[(x + y)(\delta(x))^2 \sum_{k_2=0}^{n-3} \binom{n-3}{k_2} x^{n-3-k_2} y^{k_2} \\
+ (x + y) \delta(x)(x + y) \sum_{k_2=0}^{n-4} \binom{n-4}{k_2} x^{n-4-k_2} y^{k_2} \rightarrow \right]
\]
+ \ldots + (x + y)\delta(x) \sum_{k_2=0}^{n-3} \binom{n-3}{k_2} x^{n-3-k_2} y^{k_2} \delta(x) \\
+ \left[ (x + y)^2 \delta(x) \right]^2 \sum_{k_3=0}^{n-4} \binom{n-4}{k_3} x^{n-4-k_3} y^{k_3} \\
+ (x + y)^2 \delta(x) (x + y) \delta(x) \sum_{k_3=0}^{n-5} \binom{n-5}{k_3} x^{n-5-k_3} y^{k_3} \\
+ \ldots + (x + y)^2 \delta(x) \sum_{k_3=0}^{n-4} \binom{n-4}{k_3} x^{n-4-k_3} y^{k_3} \delta(x) \\
+ \ldots + \sum_{k_{n-1}=0}^{n-2} \binom{n-2}{k_{n-1}} x^{n-2-k_{n-1}} y^{k_{n-1}} \delta(x)^2.

Using (1) and collecting together terms of the above-mentioned relations involving the same number of factors of $y$, it can be obtained that

$$\sum_{i=1}^{n-1} \gamma_i(x, y) = 0, \quad x \in \mathcal{R},$$

(2)

where

$$\gamma_i(x, y) = \binom{n}{i} d(x^{n-i} y^i) - \sum_{l=1}^{n-1} \binom{n}{i} x^{n-i-l} y^l d(x) x^{l-1}$$

$$- \sum_{p=0}^{n-2-i} \sum_{q=0}^{n-2-i-p} \binom{n}{i} y^i x^p \delta(x) x^q \delta(x) x^{n-2-i-p-q}$$

Having replaced $y, 2y, 3y, \ldots, (n - 1)y$ instead of $y$ in (2), we obtain a system of $n - 1$ homogeneous equations as follows:

$$\begin{align*}
\sum_{i=1}^{n-1} \gamma_i(x, y) &= 0 \\
\sum_{i=1}^{n-1} \gamma_i(x, 2y) &= 0 \\
\sum_{i=1}^{n-1} \gamma_i(x, 3y) &= 0 \\
& \quad \vdots \\
\sum_{i=1}^{n-1} \gamma_i(x, (n - 1)y) &= 0
\end{align*}$$
It is observed that the coefficient matrix of the above system is:

\[
X = \begin{bmatrix}
\binom{n}{1} & 2\binom{n}{2} & 2^2\binom{n}{3} & \cdots & 2^{n-1}\binom{n}{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(n-1)\binom{n}{1} & (n-1)^2\binom{n}{2} & (n-1)^3\binom{n}{3} & \cdots & (n-1)^{n-1}\binom{n}{n-1}
\end{bmatrix}
\]

It is evident that

\[
\det X = \left(\prod_{k=1}^{n-1} \binom{n}{k}\right)(n-1)! \prod_{1 \leq i < j \leq n-1} (i-j).
\]

Since \(\det X \neq 0\), the above-mentioned system has only a trivial solution. In particular, \(\gamma_{n-2}(x, y) = 0\). Indeed,

\[
0 = \binom{n}{n-2}d(x^2y^{n-2}) - \sum_{l=1}^{2}\binom{n}{n-2}x^{2-l}y^{n-2}d(x)x^{l-1} - \sum_{p=0}^{0}\sum_{q=0}^{0}\binom{n}{n-2}y^{n-2}x^0\delta(x)x^0\delta(x)x^0
\]

\[
= \binom{n}{n-2}d(x^2y^{n-2}) - \binom{n}{n-2}xy^{n-2}d(x) - \binom{n}{n-2}y^{n-2}d(x)x - \binom{n}{n-2}(\delta(x))^2
\]

\[
(*)
\]

Since \(\delta(1) = 0\), we have \(d(1) = nd(1) + 0 = nd(1)\) and it demonstrates that \(d(1) = 0\). Substituting \(1\) instead of \(y\) in \((*)\), we achieve

\[
\binom{n}{n-2}d(x^2) - \binom{n}{n-2}xd(x) - \binom{n}{n-2}d(x)x - \binom{n}{n-2}(\delta(x))^2 = 0
\]

\[
(3)
\]

for all \(x \in \mathcal{R}\). Since \(\mathcal{R}\) is an \(n!\)-torsion free ring, it follows from equation \((3)\) that

\[
d(x^2) = xd(x) + d(x)x + (\delta(x))^2, \quad x \in \mathcal{R}.
\]

\[
(4)
\]

In other words, \(d\) is a Jordan \(\delta\)-double derivation. Now, assume that \(\mathcal{R}\) is a semiprime algebra and further, \(\delta^2(x^2) = \delta^2(x)(x) + x\delta^2(x) + 2(\delta(x))^2\) for all \(x \in \mathcal{R}\). This equation along with \((4)\) imply that \(d(x^2) = xd(x) + d(x)x + \frac{1}{2}(\delta^2(x^2) - \delta^2(x)x - x\delta^2(x)).\) Hence, \((d - \frac{1}{2}\delta^2)(x^2) = x(d - \frac{1}{2}\delta^2)(x) + (d - \frac{1}{2}\delta^2)(x)x.\) It means that \(d - \frac{1}{2}\delta^2\) is a Jordan derivation. It follows from Theorem 1 of \([2]\) that \(d - \frac{1}{2}\delta^2\) is an ordinary derivation on \(\mathcal{R}\). Thereby, our claim is achieved.

Using the above theorem, we obtain the following corollary:

**Corollary 2.3** Let \(n > 1\) be an integer and \(\mathcal{A}\) be a semiprime algebra with the identity element \(1\). Suppose that \(d, \delta : \mathcal{A} \rightarrow \mathcal{A}\) are two additive mappings such that

\[
d(a^n) = \sum_{j=1}^{n} a^{n-j}d(a)a^{j-1} + \sum_{k=0}^{n-2} \sum_{l=0}^{n-2-k} a^k\delta(a)a^l\delta(a)a^{n-2-k-l} \text{ for all } a \in \mathcal{A}.
\]

If \(\delta\) is a derivation, then \(d\) is a \(\delta\)-double derivation.

**Proof.** Previous theorem along with the assumption that \(\delta\) is a derivation imply that
\[ \Delta = d - \frac{1}{2}\delta^2 \] is a derivation. Therefore, we have

\[
d(ab) = \Delta(ab) + \frac{1}{2}\delta^2(ab) = \Delta(a)b + a\Delta(b) + \frac{1}{2}\left(\delta^2(a)b + a\delta^2(b) + 2\delta(a)\delta(b)\right)
\]

for all \( a, b \in A \). It means that \( d \) is a \( \delta \)-double derivation. \[ \blacksquare \]

**Definition 2.4** Let \( \mathcal{R} \) be a ring and let \( \delta : \mathcal{R} \to \mathcal{R} \) be an additive mapping. An additive mapping \( d : \mathcal{R} \to \mathcal{R} \) is called a left \( \delta \)-double derivation if \( d(xy) = xd(y) + yd(x) + \delta(x)\delta(y) \) holds for all \( x, y \in \mathcal{R} \). In addition, the additive mapping \( d \) is said to be a Jordan left \( \delta \)-double derivation if \( d(x^2) = 2xd(x) + (\delta(x))^2 \) is fulfilled for all \( x \in \mathcal{R} \).

Below, we provide a characterization of Jordan left \( \delta \)-double derivations.

**Theorem 2.5** Let \( n > 1 \) be an integer and \( \mathcal{R} \) be an \( n! \)-torsion free ring with the identity element \( 1 \). Suppose that \( d, \delta : \mathcal{R} \to \mathcal{R} \) are two additive maps satisfying

\[
d(x^n) = nx^{n-1}d(x) + \binom{n}{2} x^{n-2}(\delta(x))^2
\]

for all \( x \in \mathcal{R} \). If \( \delta(1) = 0 \), then \( d(x^2) = 2xd(x) + (\delta(x))^2 \). In particular, if \( \mathcal{R} \) is a semiprime algebra and further, \( \delta^2(x^2) = 2\left(x\delta^2(x) + (\delta(x))^2\right) \) holds for all \( x \in \mathcal{R} \), then \( d - \frac{1}{2}\delta^2 \) is a derivation mapping \( \mathcal{R} \) into \( Z(\mathcal{R}) \).

**Proof.** Similar to the presented argument in Theorem 2.2, let \( y \) be an element of \( Z(\mathcal{R}) \) such that both \( d(y) \) and \( \delta(y) \) are zero. According to the aforementioned assumption, we have

\[
d(x^n) = nx^{n-1}d(x) + \binom{n}{2} x^{n-2}(\delta(x))^2 \tag{5}
\]

for all \( x \in \mathcal{R} \). Having put \( x + y \) instead of \( x \) in the above equation, we have

\[
d\left(\sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i\right) = n(x + y)^{n-1}d(x) + \binom{n}{2} (x + y)^{n-2}(\delta(x))^2
\]

\[
= n \sum_{i=0}^{n-1} \binom{n-1}{i} x^{n-1-i}y^i d(x) + \binom{n}{2} \sum_{i=0}^{n-2} \binom{n-2}{i} x^{n-2-i}y^i(\delta(x))^2
\]

Therefore, we have

\[
d(x^n) + \binom{n}{1} d(x^{n-1}y) + \binom{n}{2} d(x^{n-2}y^2) + \ldots + \binom{n}{n-1} d(xy^{n-1})
\]

\[
= nx^{n-1}d(x) + n\binom{n-1}{1} x^{n-2}yd(x) + n\binom{n-1}{2} x^{n-3}y^2d(x) + \ldots + ny^{n-1}d(x)
\]

\[
+ \binom{n}{2} x^{n-2}(\delta(x))^2 + \binom{n}{2} \binom{n-2}{1} x^{n-3}y(\delta(x))^2 + \ldots + \binom{n}{2} y^{n-2}(\delta(x))^2
\]

Using (5) and collecting together terms of above-mentioned relations involving the same
number of factors of \( y \), we obtain

\[
\sum_{i=1}^{n-1} \lambda_i(x, y) = 0, \quad x \in \mathcal{R},
\]

where

\[
\lambda_i(x, y) = \binom{n}{i} d(x^{n-i} y^i) - n \binom{n-1}{i} x^{n-1-i} y^i d(x) - \binom{n-2}{i} x^{n-2-i} y^i (\delta(x))^2.
\]

Having replaced \( y, 2y, 3y, \ldots, (n-1)y \) instead of \( y \) in (6), we obtain a system of \( n-1 \) homogeneous equations as follows:

\[
\begin{align*}
\sum_{i=1}^{n-1} \lambda_i(x, y) &= 0 \\
\sum_{i=1}^{n-1} \lambda_i(x, 2y) &= 0 \\
\sum_{i=1}^{n-1} \lambda_i(x, 3y) &= 0 \\
& \quad \vdots \\
\sum_{i=1}^{n-1} \lambda_i(x, (n-1)y) &= 0
\end{align*}
\]

It is evident that the coefficient matrix of the above system is:

\[
Y = \begin{bmatrix}
\binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \cdots & \binom{n}{n-1} \\
2\binom{n}{1} & 2^2\binom{n}{2} & 2^3\binom{n}{3} & \cdots & 2^{n-1}\binom{n}{n-1} \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
(n-1)\binom{n}{1} & (n-1)^2\binom{n}{2} & (n-1)^3\binom{n}{3} & \cdots & (n-1)^{n-1}\binom{n}{n-1}
\end{bmatrix}
\]

Obviously,

\[
det Y = \left( \prod_{k=1}^{n-1} \binom{n}{k} \right) (n-1)! \prod_{1 \leq i < j \leq n-1} (i - j).
\]

Since \( det Y \neq 0 \), the above-mentioned system has only a trivial solution. In particular, \( \lambda_{n-2}(x, y) = 0 \), i.e.

\[
\binom{n}{n-2} d(x^2 y^{n-2}) - 2 \binom{n}{n-2} x y^{n-2} d(x) - \binom{n}{n-2} y^{n-2} (\delta(x))^2 = 0.
\]

Since \( \mathcal{R} \) is an \( n! \)-torsion free ring, we have

\[
d(x^2 y^{n-2}) - 2 x y^{n-2} d(x) - y^{n-2} (\delta(x))^2 = 0.
\]
\( d(1) = 0. \) Thus, we can put 1 instead of \( y \) in (7) to obtain
\[
d(x^2) = 2xd(x) + (\delta(x))^2, \tag{8}
\]
for all \( x \in \mathcal{A}. \) It means that \( d \) is a Jordan left \( \delta \)-double derivation. Now, assume that \( \mathcal{R} \) is a semiprime algebra and further, \( \delta^2(x^2) = 2 \left( x\delta^2(x) + (\delta(x))^2 \right) \) holds for all \( x \in \mathcal{R}. \) From this equation and equation (8), we arrive at
\[
d(x^2) = 2xd(x) + \frac{1}{2} \delta^2(x^2) - x\delta^2(x) \tag{9}
\]
Therefore, \( (d - \frac{1}{2}\delta^2)(x^2) = 2x(d - \frac{1}{2}\delta^2)(x) \), and it means that \( \Delta = d - \frac{1}{2}\delta^2 \) is a Jordan left derivation. At this moment, Theorem 2 of [10] is exactly what we need to complete the proof. ■

We are now ready to establish another characterization of \( \delta \)-double derivations on algebras.

**Corollary 2.6** Let \( n > 1 \) be an integer and \( \mathcal{A} \) be a semiprime algebra with the identity element 1. Suppose that \( d, \delta : \mathcal{A} \to \mathcal{A} \) are two additive maps satisfying
\[
d(a^n) = na^{n-1}d(a) + \binom{n}{2}a^{n-2}(\delta(a))^2
\]
for all \( a \in \mathcal{A}. \) If \( \delta \) is a left derivation, then \( d \) is a \( \delta \)-double derivation mapping \( \mathcal{A} \) into \( Z(\mathcal{A}). \)

**Proof.** It follows from Theorem 2 of [10] that \( \delta \) is a derivation mapping \( \mathcal{A} \) into \( Z(\mathcal{A}). \) Theorem 2.5 of the current study implies that \( \Delta(a) = d(a) - \frac{1}{2} \delta^2(a) \in Z(\mathcal{A}) \) for all \( a \in \mathcal{A}, \) and consequently, \( d(\mathcal{A}) \subseteq Z(\mathcal{A}). \) A straightforward verification shows that \( d \) is a \( \delta \)-double derivation. ■

The following theorem has been motivated by a work of Vukman [10].

**Theorem 2.7** Let \( \mathcal{A} \) be a Banach algebra with the identity element 1 and let
\[
d, \delta : \mathcal{A} \to \mathcal{A},
\]
be two additive maps satisfying
\[
d(a) = -ad(a^{-1})a - a\delta(a^{-1})\delta(a) \tag{10}
\]
for all invertible elements \( a \in \mathcal{A}. \) If \( \delta(a) = -a\delta(a^{-1})a \) for all invertible elements \( a, \) then \( d \) is a Jordan \( \delta \)-double derivation. In particular, if \( \mathcal{A} \) is semiprime and further, \( (\delta(x))^2 = \frac{1}{2} \left( \delta^2(x^2) - \delta^2(x)x - x\delta^2(x) \right) \) holds for all \( x \in \mathcal{A}, \) then \( d - \frac{1}{2} \delta^2 \) is a derivation.

**Proof.** Let \( x \) be an arbitrary element of \( \mathcal{A} \) and let \( n \) be a positive number so that \( \|x\|_{n-1} < 1. \) It is evident that \( \|x\|_n < 1, \) too. If we consider \( a = n1 + x, \) then we have \( \frac{a}{n} = 1 + \frac{x}{n} \). Since \( \|\frac{x}{n}\| < 1, \) it follows from Theorem 1.4.2 of [6] that \( 1 - \frac{x}{n} \) is invertible and consequently, \( a \) is invertible. Similarly, we can show that \( 1 - a \) is also an
invertible element of $\mathcal{A}$. In the following, we use the well-known Hua identity

$$a^2 = a - \left(a^{-1} + (1 - a)^{-1}\right)^{-1}.$$ 

Applying equation (10), we have

$$d(a^2) = d(a) - d\left((a^{-1} + (1 - a)^{-1})^{-1}\right)$$

$$= d(a) + (a^{-1} + (1 - a)^{-1})^{-1}d(a^{-1} + (1 - a)^{-1})(a^{-1} + (1 - a)^{-1})^{-1}$$

$$+ (a^{-1} + (1 - a)^{-1})^{-1}\delta(a^{-1} + (1 - a)^{-1})\delta((a^{-1} + (1 - a)^{-1})^{-1})$$

$$= d(a) + a(1 - a)(-a^{-1}d(a)a^{-1} - a^{-1}\delta(a)a^{-1}))a(1 - a)$$

$$+ a(1 - a)(-(1 - a)^{-1}d(1 - a)(1 - a)^{-1}) - \left((1 - a)^{-1}\delta(1 - a)\delta((1 - a)^{-1}\right)$$

$$\times a(1 - a)\right) + \left((a(1 - a)(-a^{-1}\delta(a)a^{-1})(1 - a)^{-1}\delta(1 - a)(1 - a)^{-1}\right)$$

$$\times ((-a^{-1} + (1 - a)^{-1})^{-1})\delta(a^{-1})) + (1 - a)^{-1}(a^{-1} + (1 - a)^{-1})^{-1}$$

$$= d(a) - a(1 - a)^{-1}d(a)a^{-1}a(1 - a) - a(1 - a)^{-1}\delta(a)a^{-1}a(1 - a)$$

$$+ a(1 - a)(1 - a)^{-1}d(a)(1 - a)^{-1}a(1 - a) + \left((a(1 - a)(1 - a)^{-1}\delta(a)\right)$$

$$\times \delta((1 - a)^{-1}a(1 - a)\right) + a(1 - a)a^{-1}\delta(a)a^{-1}(a\delta(a) + \delta(a)a - \delta(a))$$

$$- a(1 - a)(1 - a)^{-1}\delta(a)(1 - a)^{-1}(a\delta(a) + \delta(a)a - \delta(a))$$

$$= d(a) - (1 - a)d(a)(1 - a) - (1 - a)\delta(a)(1 - a)^{-1}a(1 - a) + ad(a)a$$

$$+ a\delta(a)\delta((1 - a) - (1 - a)(1 - a)^{-1}a(1 - a) + \delta(a)^{2}$$

$$- (1 - a)\delta(a)a^{-1}\delta(a) - a\delta(a)(1 - a)^{-1}a\delta(a) - a\delta(a)(1 - a)^{-1}\delta(a)$$

$$+ a\delta(a)(1 - a)^{-1}\delta(a)$$

$$= d(a)a + ad(a) - \delta(a)a^{-1}a + a\delta(a)a^{-1}a^{2} + a\delta(a)a^{-1}a$$

$$- a\delta(a)a^{-1}a^{2} + a\delta(a)(1 - a)^{-1}\delta(a)(1 - a)^{-1}a(1 - a) + \delta(a)^{2}$$

$$- a\delta(a)^{2} + \delta(a)a^{-1}\delta(a)a - a\delta(a)a^{-1}\delta(a)a - a\delta(a)a^{-1}\delta(a) + (a\delta(a)a^{-1}\right)$$

$$\times \delta(a)) - a\delta(a)(1 - a)^{-1}a\delta(a) - a\delta(a)(1 - a)^{-1}\delta(a)a + a\delta(a)(1 - a)^{-1}\delta(a)$$

$$= d(a)a + ad(a) + \delta(a)^{2} - a\delta(a)^{2} - a\delta(a)(1 - a)^{-1}a\delta(a)$$

$$+ a\delta(a)(1 - a)^{-1}\delta(a) \quad \text{(see \( \delta(a) = -a\delta(a^{-1})a \))}$$

$$= d(a)a + ad(a) + \delta(a)^{2} - a\delta(a)^{2} - a\delta(a)(1 - a)^{-1}a\delta(a)$$

$$+ a\delta(a)(1 - a)^{-1}\delta(a) = d(a)a + ad(a) + (\delta(a)^{2}).$$

Since $\delta(1) = -1$, $\delta'(1) = 0$ and it implies that $d(1) = 0$. We know that $d(a^2) = d(a)a + ad(a) + (\delta(a)^{2})$. Having put $a = n1 + x$ in the previous equation, we have
\[d(n^2 + 2nx + x^2) = d(x)(n1 + x) + (n1 + x)d(x) + (\delta(x))^2.\] Therefore,
\[d(x^2) = d(x)x + xd(x) + (\delta(x))^2,
\]
for all \(x \in \mathcal{A}\), i.e. \(d\) is a Jordan \(\delta\)-double derivation. Now, assume that \((\delta(x))^2 = \frac{1}{2}\left(\delta^2(x^2) - \delta^2(x)x - x\delta^2(x)\right)\) for all \(x \in \mathcal{A}\). Hence, \(d(x^2) = xd(x) + d(x)x + \frac{1}{2}\left(\delta^2(x^2) - \delta^2(x)x - x\delta^2(x)\right)\); equivalently we have, \((d - \frac{1}{2}\delta^2)(x^2) = x(d - \frac{1}{2}\delta^2)(x) + (d - \frac{1}{2}\delta^2)(x)x\). It means that \(d - \frac{1}{2}\delta^2\) is a Jordan derivation. Now, Theorem 1 of [2] completes our proof.

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