Viscosity Approximation Methods for W-mappings in Hilbert space

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Abstract

We suggest an explicit viscosity iterative algorithm for finding a common element of the set of common fixed points for W-mappings which solves some variational inequality. Also, we prove a strong convergence theorem with some control conditions. Finally, we apply our results to solve the equilibrium problems. Finally, examples and numerical results are also given.

Key words: Nonexpansive mapping, equilibrium problems, strongly positive linear bounded operator, fixed point, Hilbert space, W-mapping.

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1 Introduction

Let $H$ be a real Hilbert space, and let $C$ be a nonempty closed convex subset of $H$. A mapping $T$ of $C$ into itself is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$, we denote $F(T)$ the set of fix points of $T$. The strong(weak) convergence of $\{x_n\}$ to $x$ is written by $x_n \to x$ ($x_n \rightharpoonup x$) as $n \to \infty$.

For any $x \in H$, there exists a unique nearest point in $C$, denoted it by $P_C(x)$ such that

$$\|x - P_Cx\| \leq \|x - y\|, \text{ for all } y \in C,$$

such that a mapping $P_C$ from $H$ onto $C$ is called the metric projection. Recall that $H$ satisfies the Opial’s condition [6] if for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \text{ holds for every } y \in H \text{ with } x \neq y.$$  

A self mapping $f : C \to C$ is a contraction if there exists $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$ for each $x, y \in C$.

An operator $A$ is said to be a strongly positive linear bounded operator on $H$, if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \text{ for all } x \in H.$$  

Let $F$ be a bifunction of $C \times C$ into $R$. The equilibrium problems for $C \times C \to C$, is to find $x \in C$ such that

$$F(x, y) \geq 0, \text{ for all } y \in C. \quad (1.1)$$  

The set of solution of Eq.(1.1) is denoted by $EP(F)$. Several problems in physics, optimization, and economics reduce to find a solution of Eq.(1.1)
[1], [4]. We consider the following iteration [10]

\[ U_{n+1} := I, \]
\[ U_n := \lambda_n S_n U_{n+1} + (1 - \lambda_n)I, \]
\[ U_{n-1} := \lambda_{n-1} S_{n-1} U_n + (1 - \lambda_{n-1})I, \]
\[ \vdots \]
\[ U_{k+1} := \lambda_k S_k U_{k+1} + (1 - \lambda_k)I, \]
\[ \vdots \]
\[ U_2 := \lambda_2 S_2 U_{n,3} + (1 - \lambda_2)I, \]
\[ W_n = U_{n,1} := \lambda_1 S_1 U_{n,2} + (1 - \lambda_1)I, \]

where \( \lambda_1, \lambda_2, \ldots \) are real numbers such that \( 0 \leq \lambda_n \leq 1 \), and \( S_1, S_2, \ldots \) be an infinite nonexpansive mappings. It is clear that nonexpansivity of each \( S_n \) ensure the nonexpansivity of \( W_n \). Such a mapping \( W_n \) is called \( W \)-mapping generated by \( S_n, S_{n-1}, \ldots, S_1 \) and \( \lambda_n, \lambda_{n-1}, \ldots, \lambda_1 \).

In this paper, by intuition from [7], a new iterative scheme is introduced. This scheme find a common solution of the equilibrium problem (EP) and fixed point problem for an infinite family of nonexpansive mappings. Also, we prove a strong convergence theorem.

The following lemmas will be useful for proving the main results of this article:

**Lemma 1.1** [8] Let \( C \) be a nonempty closed convex subset of a Banach space \( E \) and \( \{S_n\} : C \to C \) be a family of infinitely nonexpansive mappings such that \( \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset \), and \( \{\lambda_n\} \) be a sequence of positive numbers in \([0, b]\) for some \( b \in (0, 1) \). For any \( n \geq 1 \), let \( W_n \) be the \( W \)-mapping of \( C \) into itself generated by \( S_n, S_{n-1}, \ldots, S_1 \) and \( \lambda_n, \lambda_{n-1}, \ldots, \lambda_1 \). Then \( W_n \) is asymptotically regular and nonexpansive. Further, if \( E \) is strictly convex, then \( F(W_n) = \bigcap_{i=1}^{n} F(S_i) \).

**Lemma 1.2** [8] Let \( C \) be a nonempty closed convex subset of a strictly convex Banach space \( E \). Let \( \{S_n\} : C \to C \) be a family of infinitely nonexpansive mappings such that \( \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset \) and \( \{\lambda_n\} \) be a sequence of positive numbers in \([0, b]\) for some \( b \in (0, 1) \). Then, for every \( x \in C \)
and \( k \geq 1 \lim_{n \to \infty} U_{n,k} x \) exists.

**Lemma 1.3** [8] Let \( C \) be a nonempty closed convex subset of strictly convex Banach \( E \). Let \( \{S_n\} : C \to C \) be a family of infinitely nonexpansive mappings such that \( \bigcap_{n=1}^\infty F(S_n) \neq \emptyset \) and \( \{\lambda_n\} \) be a sequence of positive numbers in \( [0, b] \) for some \( b \in (0, 1) \). Then \( W \) is a nonexpansive mapping and \( F(W) = \bigcap_{n=1}^\infty F(S_n) \).

**Lemma 1.4** [2] Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \) and \( \{S_n\} : C \to C \) be a family of infinitely nonexpansive mappings such that \( \bigcap_{n=1}^\infty F(S_n) \neq \emptyset \) and \( \{\lambda_n\} \) be a sequence of positive numbers in \( [0, b] \) for some \( b \in (0, 1) \). If \( K \) is any bounded subset of \( C \), then
\[
\limsup_{n \to \infty} \|Wx - W_n x\| = 0.
\]

**Lemma 1.5** [5] Assume \( A \) is a strongly positive linear bounded operator on a Hilbert space \( H \) with coefficient \( \gamma > 0 \) and \( 0 < \rho < \|A\|^{-1} \). Then
\[
\|I - \rho A\| \leq I - \rho \gamma.
\]

**Lemma 1.6** [9] Let \( \{x_n\} \) and \( \{y_n\} \) be bounded sequences in a Banach space \( E \) and \( \{\beta_n\} \) be a sequence in \( [0, 1] \) with
\[
0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1.
\]
Suppose \( x_{n+1} = (1-\beta_n)y_n + \beta_n x_n \) for all integers \( n \geq 1 \) and \( \limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0 \). Then \( \lim_{n \to \infty} \|y_n - x_n\| = 0 \).

**Lemma 1.7** [2] Let \( H \) be a real Hilbert space. Then the following holds:

(a) \( \|x + y\|^2 \leq \|y\|^2 + 2 \langle x, x + y \rangle \) for all \( x, y \in H \),
(b) \( \|\alpha x + (1 - \alpha)y\|^2 = \alpha \|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \),
(c) \( \|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2 \langle x, y \rangle \).

**Lemma 1.8** [1] Let \( K \) be a nonempty closed convex subset of \( H \) and \( F \) be a bi-function of \( K \times K \) into \( \mathbb{R} \) satisfying the following conditions:

\begin{enumerate}
\item [(A1)] \( F(x, x) = 0 \) for all \( x \in K \),
\item [(A2)] \( F \) is monotone, that is, \( F(x, y) + F(y, x) \leq 0 \) for all \( x, y \in K \),
\item [(A3)] for each \( x, y, z \in K \)
\[
\lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y),
\]
\end{enumerate}
(A4) for each $x \in K$, $y \mapsto F(x, y)$ is convex and lower semi-continuous. Let $r > 0$ and $x \in H$.

Then, there exists $z \in K$ such that

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in K.$$ 

**Lemma 1.9** [3] Let $K$ be a nonempty closed convex subset of $H$ and let $F$ be a bifunction of $K \times K$ into $\mathbb{R}$ satisfying (A1) – (A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to K$ as follows:

$$T_r(x) = \{ z \in K : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in K \},$$

for all $x \in H$. Then the following hold

(i) $T_r$ is single valued map,

(ii) $T_r$ is firmly nonexpansive, that is, for any $x, y \in H$

$$\| T_r x - T_r y \|^2 \leq \langle T_r x - T_r y, x - y \rangle,$$

(iii) $F(T_r) = EP(F)$,

(iv) $EP(F)$ is closed and convex.

**Lemma 1.10** [11] Assume $\{ a_n \}$ be a sequence of nonnegative numbers such that

$$a_{n+1} \leq (1 - \alpha_n) a_n + \delta_n,$$

where $\{ \alpha_n \}$ is a sequence in $(0, 1)$ and $\{ \delta_n \}$ is a sequence in real number such that

(i) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n \to \infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$,

then $\lim_{n \to \infty} a_n = 0$. 

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2 Explicit viscosity iterative algorithm

In this section, a new iterative scheme for finding a common element of the set of solutions for an equilibrium problems and the set of common fixed point for an infinite family of mappings in Hilbert space, is introduced.

**Theorem 2.1** Let

- $C$ be a nonempty closed convex subset of a real Hilbert space $H$,
- $f$ be a $\rho$–contractive map on $C$,
- $J = \{1, 2, \ldots, k\}$ be a finite index set,
- For each $i \in J$, let $G_i$ be a bifunction from $C \times C$ into $R$ satisfying $(A1) - (A4)$,
- $A$ be a strongly positive linear bounded operator on $H$ with coefficient $\varpi > 0$,
- $\{S_n\} : H \to H$ be a family of infinite nonexpansive mappings,
- $\bigcap_{i=1}^{k} F(W) \cap EP(G_i) \neq \emptyset$ where $F(W) = \bigcap_{j=1}^{n} F(S_j)$,
- $\{x_n\}$ be the sequence generated as following:

$$
\begin{align*}
G_1(u_{n,1}, y) + \frac{1}{r_n} \langle y - u_{n,1}, u_{n,1} - x_n \rangle & \geq 0, \\
G_2(u_{n,2}, y) + \frac{1}{r_n} \langle y - u_{n,2}, u_{n,2} - x_n \rangle & \geq 0, \\
& \vdots \\
G_k(u_{n,k}, y) + \frac{1}{r_n} \langle y - u_{n,k}, u_{n,k} - x_n \rangle & \geq 0, \\
\theta_n & = \frac{1}{k} \sum_{i=1}^{k} u_{n,i}, \\
y_n & = \beta_n \gamma f(\theta_n) + (I - \beta_n A)\theta_n, \\
x_{n+1} & = \alpha_n x_n + (1 - \alpha_n)W_n y_n,
\end{align*}
$$

where $\{W_n\}$ is a sequence defined by Eq.(1.2). Also, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1], r_n \subset (0, \infty)$ and $0 < \gamma < \frac{\varpi}{\rho}$.

Suppose
(C1): \[ \lim_{n \to \infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty, \]

(C2): \[ \liminf_{n \to \infty} r_n > 0, \quad \lim_{n \to \infty} (r_{n+1} - r_n) = 0; \]

(C3): \[ 0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1, \]

(C4): for each \( i = 1, 2, \ldots, k \)
\[ 0 < \lambda_i \leq c < 1. \]

Then

(i) the sequence \( \{x_n\} \) is bounded.

(ii) \[ \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \]

(iii) \[ \lim_{n \to \infty} \|W_n y_n - y_n\| = 0. \]

**Proof.** From (C1), we may assume that \( \beta_n \leq \|A\|^{-1} \) for all \( n \geq 1 \). By Lemma 1.5, we obtain \( \|I - \beta_n A\| \leq 1 - \beta_n \varpi \). It is clear that
\[ P_{\bigcap_{i=1}^{k} F(W) \cap EP(G_i)} (I - A + \gamma f) \]

is a contraction of \( C \) into itself. Indeed, for all \( x, y \in C \)

\[
\begin{align*}
\| P_{\bigcap_{i=1}^{k} F(W) \cap EP(G_i)} (x) - P_{\bigcap_{i=1}^{k} F(W) \cap EP(G_i)} (y) \| & \leq \|(I - A + \gamma f)(x) - (I - A + \gamma f)(y)\| \\
& \leq \|I - A\| \|x - y\| + \|f(x) - f(y)\| \\
& \leq (1 - \varpi) \|x - y\| + \gamma \rho \|x - y\| \\
& = (1 - (\varpi - \gamma \rho)) \|x - y\|. \\
\end{align*}
\]

(i): Let \( x^* \in \bigcap_{i=1}^{k} F(W) \cap EP(G_i) \). Since \( u_{n,i} = T_{r_n,i} x_n \) and \( x^* = T_{r_n,i} x^* \), we see for any \( n \geq N \)

\[ \|u_{n,i} - x^*\| = \|T_{r_n,i} x_n - T_{r_n,i} x^*\| \leq \|x_n - x^*\|, \]

thus

\[ \|\theta_n - x^*\| \leq \|x_n - x^*\|. \quad (2.1) \]

Since \( f \) is \( \rho \)-contraction, we have
$$\|y_n - x^*\| = \|\beta_n \gamma f(\theta_n) + (I - \beta_n A)\theta_n - x^*\|
= \|\beta_n (\gamma f(\theta_n) - Ax^*) + (I - \beta_n A)(\theta_n - x^*)\|
\leq \beta_n \|\gamma f(\theta_n) - Ax^*\| + \|I - \beta_n A\|\|\theta_n - x^*\|
\leq \beta_n \gamma \|f(\theta_n) - f(x^*)\|
+ \beta_n \|\gamma f(x^*) - Ax^*\| + (1 - \beta_n) \varpi \|x_n - x^*\|.$$
where $M = \text{Sup}_{n \geq 1} \{ \|A\theta_n\| + \|\gamma(\theta_n)\| \}$.

Moreover, we have

$$G_i(u_{n+1,i}, u_{n,i}) + \frac{1}{r_{n+1}} \langle u_{n,i} - u_{n+1,i}, u_{n+1,i} - x_{n+1} \rangle \geq 0, \quad (2.4)$$

$$textbf{1} \geq i \geq k. \quad (2.5)$$

and

$$G_i(u_{n,i}, u_{n+1,i}) + \frac{1}{r_n} \langle u_{n+1,i} - u_{n,i}, u_{n,i} - x_i \rangle \geq 0. \quad (2.6)$$

Combining Eq.(2.5) and Eq.(2.6), we obtain

$$0 \leq r_{n+1} \{ G_i(u_{n+1,i}, u_{n,i}) + G_i(u_{n,i}, u_{n+1,i}) \} + \langle u_{n,i} - u_{n+1,i}, u_{n+1,i} - x_{n+1} - \frac{r_{n+1}}{r_n}(u_{n,i} - x_i) \rangle$$

$$\leq \langle u_{n,i} - u_{n+1,i}, u_{n+1,i} - x_{n+1} - \frac{r_{n+1}}{r_n}(u_{n,i} - x_i) \rangle,$$

from which it follows that

$$\langle u_{n,i} - u_{n+1,i}, u_{n,i} - u_{n+1,i} + x_{n+1} - x_n + x_n - u_{n,i} + \frac{r_{n+1}}{r_n}(u_{n,i} - x_i) \rangle \leq 0 \quad (2.7)$$

which implies that

$$\|u_{n+1,i} - u_{n,i}\| \leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_n}\|x_n - u_{n,i}\|. \quad (2.8)$$

Using the condition (C2) and noting that there exists $b > 0$ such that $r_n > b > 0$, we obtain

$$\|\theta_{n+1} - \theta_n\| \leq \frac{1}{k} \sum_{i=1}^{k} \|u_{n+1,i} - u_{n,i}\| \leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_n} M \quad (2.9)$$
\[ \| \theta_{n+1} - \theta_n \| \leq \| x_{n+1} - x_n \| + \frac{\hat{M}}{b} |r_{n+1} - r_n|, \quad (2.10) \]

where \( \hat{M} := \frac{1}{k} \sum_{i=1}^{k} \| x_n - u_{n,i} \| < \infty. \)

Moreover, we note that

\[ \| W_{n+1} y_n - W_n y_n \| = \| \lambda_1 S_1 U_{n+1,2} y_n + (1 - \lambda_1) y_n \\
- (\lambda_1 S_1 U_{n,2} y_n + (1 - \lambda_1) y_n) \| \leq \lambda_1 \| U_{n+1,2} y_n - U_{n,2} y_n \| \]

\[ \leq \lambda_1 \| \lambda_2 S_2 U_{n+1,3} y_n + (1 - \lambda_2) y_n \\
- (\lambda_2 S_2 U_{n,3} y_n + (1 - \lambda_2) y_n) \| \leq \lambda_1 \lambda_2 \| U_{n+1,3} y_n - U_{n,3} y_n \| \]

\[ \quad \vdots \]

\[ \leq \left( \prod_{m=1}^{n} \lambda_m \right) \| U_{n+1,n+1} y_n - U_{n,n+1} y_n \| \]

\[ = \left( \prod_{m=1}^{n} \lambda_m \right) \| \lambda_{n+1} S_{n+1} U_{n+1,n+2} y_n \\
+ (1 - \lambda_{n+1} y_n - y_n) \| \]

\[ = \left( \prod_{m=1}^{n} \lambda_m \right) \| \lambda_{n+1} S_{n+1} y_n - \lambda_n y_n \| \]

\[ = \left( \prod_{m=1}^{n+1} \lambda_m \right) \| S_{n+1} y_n - y_n \| \leq \hat{M} \left( \prod_{m=1}^{n+1} \lambda_m \right) \quad (2.11) \]

where \( \hat{M} := \text{Sup } \| S_{n+1} y_n - y_n \| \).

Combining Eq.(2.3), Eq.(2.10) and Eq.(2.11), we obtain
\[ \| W_{n+1}y_{n+1} - W_n y_n \| = \| W_{n+1}y_{n+1} - W_{n+1}y_n + W_{n+1}y_n - W_n y_n \| \]
\[ \leq \| y_{n+1} - y_n \| + \| W_{n+1}y_n - W_n y_n \| \]
\[ \leq \| \theta_{n+1}\theta_n \| + M|\beta_{n+1} - \beta_n| + \hat{M}\left( \prod_{m=1}^{n+1} \lambda_m \right) \]
\[ \leq \| x_{n+1} - x_n \| + \frac{\hat{M}}{b}|r_{n+1} - r_n| \]
\[ + M|\beta_{n+1} - \beta_n| + \hat{M}\left( \prod_{m=1}^{n+1} \lambda_m \right). \]

We have
\[
\limsup_{n \to \infty} (\| W_{n+1}y_{n+1} - W_n y_n \| - \| x_{n+1} - x_n \|) \leq 0.
\]

From Lemma 1.6, we see that
\[ \| W_n y_n - x_n \| \to 0 \quad \text{as} \quad n \to \infty. \quad (2.12) \]
which implies that
\[ \lim_{n \to \infty} \| x_{n+1} - x_n \| = \lim_{n \to \infty} (1 - \alpha_n)\| W_n y_n - x_n \| = 0 \]

(iii): We shall prove that \( \lim_{n \to \infty} \| x_n - z_n \| = 0. \)

Notic that
\[
\| u_{n,i} - x^* \|^2 \leq \langle T_{r_{n,i}}x_n - T_{r_{n,i}}x^*, x_n - x^* \rangle
\]
\[ = \frac{1}{2}\{ \| u_{n,i} - x^* \|^2 + \| x_n - x^* \|^2 - \| u_{n,i} - x_n \|^2 \} \]
thus
\[ \| u_{n,i} - x^* \|^2 \leq \| x_n - x^* \|^2 - \| u_{n,i} - x_n \|^2. \quad (2.13) \]
From Eq.(2.13), we get
\[
\|\theta_n - x^*\| = \left\| \sum_{i=1}^{k} \frac{1}{k} (u_{n,i} - x_n) \right\|^2 \\
\leq \|x_n - x^*\|^2 - \frac{1}{k} \sum_{i=1}^{k} \|u_{n,i} - x_n\|^2.
\] (2.14)

It follows from Eq.(2.14) that
\[
\|x_{n+1} - x^*\|^2 = \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \left\{ (1 - \beta_n) \|\theta_n - x^*\|^2 + \beta_n \|\gamma f(\theta_n) - Ax^*\|^2 \right\} \\
\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \|\theta_n - x^*\|^2 + \beta_n \|\gamma f(\theta_n) - Ax^*\|^2 \\
\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \|\theta_n - x^*\|^2 + \beta_n \|\gamma f(\theta_n) - Ax^*\|^2 \\
\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \|\theta_n - x^*\|^2 \\
- \frac{1}{k} \sum_{i=1}^{k} \|u_{n,i} - x_n\|^2 \\
+ \beta_n \|\gamma f(\theta_n) - Ax^*\|^2 \\
\leq \|x_n - x^*\|^2 - (1 - \alpha_n) \frac{1}{k} \sum_{i=1}^{k} \|u_{n,i} - x_n\|^2 \\
+ \beta_n \|\gamma f(\theta_n) - Ax^*\|^2.
\]

Thanks to the conditions of (C1)- (C4) and Eq.(2.13), we conclude that
\[
(1 - \alpha_n) \frac{1}{k} \sum_{i=1}^{k} \|u_{n,i} - x_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
+ \beta_n \|\gamma f(\theta_n) - Ax^*\|^2 \\
\leq \|x_{n+1} - x_n\| \left( \|x_{n+1} - x^*\| + \|x_n - x^*\| \right) \\
+ \beta_n \|\gamma f(\theta_n) - Ax^*\|^2 \\
\lim_{n \to \infty} \|u_{n,i} - x_n\| = 0, \text{ for each } i = 1,2,\ldots,k
also

$$\lim_{n \to \infty} \|\theta_n - x_n\| = \lim_{n \to \infty} \|u_{n,t} - x_n\| = 0$$  \hspace{1cm} (2.15)$$

$$\|y_n - \theta_n\| = \beta_n \|\gamma f(\theta_n) - A\theta_n\| \to 0 \text{ as } n \to \infty.$$ \hspace{1cm} (2.16)

Moreover, we know that

$$\|y_n - x_n\| \leq \|x_n - \theta_n\| + \|\theta_n - y_n\|$$

$$\|W_ny_n - y_n\| \leq \|W_ny_n - x_n\| + \|x_n - \theta_n\| + \|\theta_n - y_n\|.$$\hspace{1cm}

In view of Eq.(2.12), Eq.(2.15) and Eq.(2.16), we can obtain

$$\lim_{n \to \infty} \|y_n - x_n\| = 0$$ \hspace{1cm} (2.17)$$

$$\lim_{n \to \infty} \|W_ny_n - y_n\| = 0.$$ \hspace{1cm} (2.18)$$

\[\square\]

**Theorem 2.2** Suppose all assumptions of Theorem 2.1 are holds. Then the sequence \(\{x_n\}\) converge strongly to \(\hat{x}\), which solves the variational inequality

$$\langle (A - \gamma f)\hat{x}, \hat{x} - x_n \rangle \leq 0, \quad \hat{x} \in \bigcap_{i=1}^{k} F(W) \cap EP(G_i).$$

Equivalently, \(P_{\bigcap_{i=1}^{k} F(W) \cap EP(G_i)}(I - A - \gamma f)(\hat{x}) = \hat{x}.\)

**Proof.** We shall prove that

$$\limsup_{n \to \infty} \langle (A - \gamma f)x^*, y_n - x^* \rangle \leq 0,$$
where $x^* = P\cap_{i=1}^k F(W) \cap EP(G_i) f(x^*)$.

We choose a subsequence $\{y_{n_p}\}$ of $\{y_n\}$ such that

$$\limsup_{n \to \infty} \langle (A - \gamma f)x^*, y_n - x^* \rangle = \lim_{p \to \infty} \langle (A - \gamma f)x^*, y_{n_p} - x^* \rangle,$$

since $\{y_{n_p}\}$ is bounded, there exits a subsequence of $\{y_{n_p}\}$, we denote it by $\{y_{n_p}\}$ such that $y_{n_p} \rightharpoonup q$, $q \in C$.

We shall show that $q \in \bigcap_{i=1}^k F(W) \cap EP(G_i)$. On the contrary, suppose that $q \notin F(W)$. By Opial's condition

$$\liminf_{p \to \infty} \|y_{n_p} - q\| < \liminf_{p \to \infty} \|y_{n_p} - Wq\|$$

$$\leq \liminf_{p \to \infty} \{\|y_{n_p} - Wy_{n_p}\| + \|Wy_{n_p} - Wq\|\}$$

$$\leq \liminf_{p \to \infty} \{\|y_{n_p} - Wy_{n_p}\| + \|y_{n_p} - q\|\}.$$

By virtue of Lemma 1.4 and noticing Eq.(2.18)

$$\lim_{p \to \infty} \|Wy_{n_p} - y_{n_p}\| \leq \lim_{p \to \infty} \{\|Wy_{n_p} - W_{n_p}y_{n_p}\| + \|W_{n_p}y_{n_p} - y_{n_p}\|\}$$

$$\leq \lim_{p \to \infty} \{\sup_{x \in C} \|Wx - W_{n_p}x\|\}$$

$$+ \lim_{p \to \infty} \|W_{n_p}y_{n_p} - y_{n_p}\| = 0.$$

It follows that

$$\liminf_{p \to \infty} \|y_{n_p} - q\| < \liminf_{p \to \infty} \|y_{n_p} - q\|.$$

This is a contradiction. Therefore, we have $q \in F(W)$. Also, we prove $q \in \bigcap_{i=1}^k EP(G_i)$.

For each $i \in J = \{1, 2, \ldots, k\}$, since $G_i(u_{n_p}, y) + \frac{1}{r_{n_p}} \langle y, u_{n_p} - x_{n_p} \rangle \geq 0$, 28
from (A2), we see that
\[
\frac{1}{r_{np}} \langle y - u_{np}, u_{np} - x_{np} \rangle \geq G_i(u_{np}, y) + G_i(y, u_{np}) \\
+ \frac{1}{r_{np}} \langle y - u_{np}, u_{np} - x_{np} \rangle \\
\geq G_i(y, u_{np}),
\]
hence
\[
\langle y - u_{np}, \frac{u_{np} - x_{np}}{r_{np}} \rangle \geq G_i(y, u_{np}), \text{ for all } y \in C.
\]
Since \( \frac{|u_{np} - x_{np}|}{r_{np}} \to 0 \), \( u_{n,i} \rightharpoonup q \), in view of (A4), we conclude
\[
G_i(y, q) \leq 0, \text{ forall } y \in C.
\]
Let \( 0 < t \leq 1 \), \( y \in C \) and \( y_t = ty + (1 - t)q \). It is clear that \( G_i(y_t, q) \leq 0 \).
From (A1)-(A4), we obtain
\[
0 = G_i(y_t, y_t) \leq tG_i(y_t, y) + (1 - t)G_i(y_t, q) \leq tG_i(y_t, y), G_i(y, q) \geq 0,
\]
Thus \( q \in \cap_{i=1}^k EP(G_i) \).
From Eq.(2.19), we have
\[
\limsup_{n \to \infty} \langle (A - \gamma f)x^*, y_n - x^* \rangle = \lim_{p \to \infty} \langle (A - \gamma f)x^*, y_p - x^* \rangle \\
= \langle (A - \gamma f)x^*, x^* - q \rangle \leq 0.
\]
It follows from Eq.(2.17) and Eq.(2.19) that
\[
\limsup_{n \to \infty} \langle (A - \gamma f)x^*, x^* - x_n \rangle \leq \limsup_{n \to \infty} \langle (A - \gamma f)x^*, y_n - x_n \rangle \\
+ \limsup_{n \to \infty} \langle (A - \gamma f)x^*, x^* - y_n \rangle \leq 0.
\]
Finally, we prove that \( x_n \to q \) where \( x^* = P_{\cap_{i=1}^k F(W) \cap EP(G_i)} f(x^*) \).
By virtue of Lemma 1.7
\[ \| y_n - x^* \|^2 = \|(I - \beta_n A)(\theta_n - x^*) + \beta_n(\gamma f(\theta_n) - Ax^*)\|^2 \]
\[ \leq \|(I - \beta_n A)(\theta_n - x^*)\|^2 + 2\beta_n \langle \gamma f(\theta_n) - Ax^*, y_n - x^* \rangle \]
\[ \leq \|(I - \beta_n A)(\theta_n - x^*)\|^2 + 2\beta_n \gamma \| x_n - x^* \| \| y_n - x^* \| + 2\beta_n \langle \gamma f(\theta_n) - Ax^*, y_n - x^* \rangle \]
\[ \leq (1 - \beta_n \bar{\nu})^2 \| x_n - x^* \|^2 + \beta_n \gamma (\| x_n - x^* \|^2 + \| y_n - x^* \|^2 ) \]
\[ + 2\beta_n \langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle. \]

from which it follows that

\[
\| y_n - x^* \|^2 \leq \frac{(1 - \beta_n \bar{\nu})^2 + \beta_n \gamma}{1 - \beta_n \gamma} \| x_n - x^* \|^2 \\
+ \frac{2\beta_n}{1 - \beta_n \gamma} \langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle \\
\leq \{ 1 - \frac{2\beta_n (\bar{\omega} - \gamma \rho)}{1 - \beta_n \gamma} \} \| x_n - x^* \|^2 \\
+ \frac{2\beta_n (\bar{\omega} - \gamma \rho)}{1 - \beta_n \gamma} \{ \frac{1}{\bar{\omega} - \gamma \rho} \langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle \\
+ \frac{\beta_n \bar{\omega}^2}{2(\bar{\omega} - \gamma \rho)} L \},
\]

where \( L = \text{Sup}\{ \| x_n - x^* \| \} \).

Also

\[
\| x_{n+1} - x^* \|^2 \leq \alpha_n \| x_n - x^* \|^2 + (1 - \alpha_n) \| y_n - x^* \|^2 \tag{2.20}
\]

it follows from Eq.(2.20) that

\[
\| x_{n+1} - x^* \|^2 \leq \{ 1 - (1 - \alpha_n) \frac{2\beta_n (\bar{\omega} - \gamma \rho)}{1 - \beta_n \gamma} \} \| x_n - x^* \|^2 \\
+ (1 - \alpha_n) \frac{2\beta_n (\bar{\omega} - \gamma \rho)}{1 - \beta_n \gamma} \{ \frac{1}{\bar{\omega} - \gamma \rho} \langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle \\
+ \frac{\beta_n \bar{\omega}^2}{2(\bar{\omega} - \gamma \rho)} L \}. 
\]
Let $\xi_n := (1 - \alpha_n)\frac{2\beta_n(\varpi - \gamma \rho)}{1 - \beta_n \rho \gamma}$ and 

$$\varepsilon_n := \frac{1}{\varpi - \gamma \rho} \langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle + \frac{\beta_n \varpi^2}{2(\varpi - \gamma \rho)} L.$$ 

Therefore,

$$\|x_{n+1} - x^*\|^2 \leq (1 - \xi_n)\|x_n - x^*\|^2 + \xi_n \varepsilon_n. \quad (2.21)$$

Thanks to the condition (C1) and Eq. (2.21), we conclude that

$$\lim_{n \to \infty} \xi_n = 0, \sum_{n=1}^{\infty} \xi_n = \infty.$$

From Lemma 1.10 we can obtain $x_n \to x^*$. □

3 Numerical Example

In this section, we get one example is presented to guarantee the Theorem (2.2).

**Example 3.1** Let $H = \mathbb{R}$, $C = [-1, 1]$ and $G_1(x, y) = -3x^2 + xy + 2y^2$, $G_2(x, y) = -4x^2 + xy + 3y^2$ and $G_3(x, y) = -9x^2 + xy + 8y^2$. Also, we consider $S_n = I$, $f(x) = \frac{x}{5}$ and $A = I$ be a strongly positive linear bounded operator with coefficient $\gamma = 1$. It is easy to check that $A$ and $f$ satisfy all conditions in Theorem 2.2. For each $r > 0$ and $x \in C$, there exists $z \in C$ such that, for any $y \in C$,

$$G(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0$$

$$\iff -3z^2 + zy + 2y^2 + \frac{1}{r} (y - z)(z - x) \geq 0$$

$$\iff 2ry^2 + ((r + 1)z - x)y - 3rz^2 - z^2 + zx \geq 0$$

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Set \( G(y) = 2y^2 + ((r + 1)z + x)y - 3rz^2 - z^2 + zx. \) Then \( G(y) \) is a quadratic function of \( y \) with coefficients \( a = 2r; b = (r + 1)z - x \) and \( c = -3rz^2 - z^2 + zx. \) So

\[
\Delta = [(r + 1)z - x]^2 - 8r(zx - z^2 - 3rz^2) \\
= (r + 1)^2z^2 - 2(r + 1)xz + x^2 + 24r^2z^2 + 8rz^2 - 8rxz \\
= x^2 - 2(5rz + z)x + (25r^2z^2 + 10rz^2 + z^2) \\
= [(x - (5rz + z))]^2.
\]

Since \( G(y) \geq 0 \) for all \( y \in C, \) if and only if \( \Delta = [(x - (5rz + z))]^2 \leq 0. \) Therefore, \( z = \frac{x}{5r+1}, \) which yields \( T_{r_n,1} = u_n^{(1)} = \frac{x_n}{5r_n+1}. \)

By the same argument, for \( G_2 \) and \( G_3, \) one can conclude \( T_{r_n,2} = u_n^{(2)} = \frac{x_n}{5r_n+1} \)

and \( T_{r_n,3} = u_n^{(3)} = \frac{x_n}{5r_n+1}. \) Let \( r_n = \frac{n}{n+1}. \) Hence

\[
\theta_n = \frac{u_n^{(1)} + u_n^{(2)} + u_n^{(3)}}{3} = \frac{1}{3} \left( \frac{1}{280n^3} + \frac{344n^2}{3} + \frac{67n + 3}{3} + \frac{364n^3 + 300n^2 + 32n + 1}{x_n}. \right)
\]

Suppose that \( \alpha_n = \frac{2n-1}{10n-5}, \beta_n = \frac{1}{n} \) and \( \lambda_n = \varepsilon, \) we have

\[
W_1 = U_{11} = \lambda_1 S_1 U_{12} + (1 - \lambda_1)I, \\
W_2 = U_{21} = \lambda_1 S_1 U_{22} + (1 - \lambda_1)I \\
= \lambda_1 S_1 \{\lambda_2 S_2 U_{23} + (1 - \lambda_2)I\} + (1 - \lambda_1)I, \\
= \lambda_1 \lambda_2 S_1 S_2 + \lambda_1 (1 - \lambda_2) S_1 + (1 - \lambda_1)I, \\
W_3 = U_{31} = \lambda_1 S_1 U_{32} + (1 - \lambda_1)I \\
= \lambda_1 S_1 \{\lambda_2 S_2 U_{33} + (1 - \lambda_2)I\} + (1 - \lambda_1)I, \\
= \lambda_1 \lambda_2 S_1 S_2 U_{33} + \lambda_1 (1 - \lambda_2) S_1 + (1 - \lambda_1)I, \\
= \lambda_1 \lambda_2 S_1 S_2 \{\lambda_3 S_3 U_{34} + (1 - \lambda_3)I\} + \lambda_1 (1 - \lambda_2) S_1 + (1 - \lambda_1)I, \\
= \lambda_1 \lambda_2 \lambda_3 S_1 S_2 S_3 + \lambda_1 \lambda_2 (1 - \lambda_3) S_1 S_2 + \lambda_1 (1 - \lambda_2) S_1 + (1 - \lambda_1)I.
\]

By iteration this manner, we have

\[
W_n = U_{n1} = \lambda_1 \lambda_2 \cdots \lambda_n S_1 S_2 \cdots S_n + \lambda_1 \lambda_2 \cdots \lambda_{n-1} (1 - \lambda_n) S_1 S_2 \cdots S_{n-1} \\
+ \lambda_1 \lambda_2 \cdots \lambda_{n-2} (1 - \lambda_{n-1}) S_1 S_2 \cdots S_{n-2} + \cdots + \lambda_1 (1 - \lambda_2) S_1 + (1 - \lambda_1)I.
\]

Let \( T_n = I, \lambda_n = \varepsilon, \) we obtain

\[
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\]
\[ W_n = [\varepsilon^n + \varepsilon^{n-1}(1 - \varepsilon) + \cdots + \varepsilon(1 - \varepsilon) + (1 - \varepsilon)]I = I. \]

Hence

\[ y_n = \left( \frac{280n^3 + 344n^2 + 67n + 3}{864n^3 + 300n^2 + 32n + 1} \right) x_n. \]

We have the following algorithm for the sequence \( \{x_n\} \)

\[ x_{n+1} = \frac{2n - 1}{10n - 9} x_n + \frac{8n - 8}{10n - 9} y_n. \]

Choose \( x_1 = 1 \). By using MATLAB software, we obtain the following table and figure of the result.

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<th>( x_n )</th>
<th>( n )</th>
<th>( x_n )</th>
<th>( n )</th>
<th>( x_n )</th>
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<td>21</td>
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</tr>
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<td>0.0007885980277</td>
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<td>0.0000004609899888</td>
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<td>0.0003784282172</td>
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</tbody>
</table>
Fig. 1. The graph of \( \{x_n\} \) with initial value \( x_1 = 1 \).

References


