The cosine method to Gardner equation and (2+1)- dimensional breaking soliton system

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Received 02 May 2016; accepted 4 February 2017

Abstract

In this letter, we established a traveling wave solution by using cosine function algorithm for Gardner equation and (2+1)-dimensional breaking soliton system. The cosine method is used to obtain the exact solution.

Key words: Gardner equation; cosine function method; exact solution; (2+1)dimensional breaking soliton system

2010 AMS Mathematics Subject Classification : 65L11; 46E22; 65F25.

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1 introduction

Gardner equation known as the mixed $kdv - mkdv$ equation is very widely studied in various areas of physics that includes plasma physics, Fluid dynamic, Quantum Field Theory, solid state physics and other [3] the Gardner equation is solved by sin-cosine function method [2]. the breaking soliton system was used to describes the (2+1)-dimensional interaction of Riemann wave propagated along the $y$-axis with long wave propagated along the $x$- axis and it seems to have been investigated extensively where over lapping solutions have been derived. this system is solved to Generalized jacobi elliptic function method [4] and the $(G'/G)$-Expansion method [5].

2 The cosine-function method

Consider the nonlinear partial differential equation in the form

$$F(u, u_t, u_x, u_{xx}, u_{xxt}) = 0, \quad (1)$$

Where $u(x,t)$ is the solution of nonlinear partial differential equation Eq.(1). we use the

$$u(x,t) = f(\xi), \quad (2)$$

Where $\xi = x - ct$. This enables us to use the following changes:

$$\frac{\partial}{\partial t} (\ldots) = -c \frac{d}{d\xi}, \quad \frac{\partial}{\partial x} (\ldots) = \frac{d}{d\xi} (\ldots), \quad \frac{\partial^2}{\partial x^2} (\ldots) = \frac{d^2}{d\xi^2} (\ldots), \quad \ldots \quad (3)$$

Using Eq.(3) to transfer the nonlinear partial differential equation Eq.(??) to nonlinear ordinary differential equation

$$G(f, f', f'', f''', \ldots) = 0. \quad (4)$$
The solution of Eq.(4) can be expressed in the form:

\[ f(\xi) = \lambda \cos^\beta(\mu \xi), \quad |\xi| \leq \frac{\pi}{2\mu}, \]  

(5)

Where \( \lambda, \beta \) and \( \mu \) are unknown parameters which will be determined. Then we have:

\[ f' = \frac{df(\xi)}{d\xi} = \lambda \beta \mu \cos^{\beta-1}(\mu \xi) \sin(\mu \xi), \]
\[ f'' = \frac{d^2f(\xi)}{d\xi^2} = -\lambda \beta \mu^2 \cos^\beta(\mu \xi) + \lambda \mu^2 \beta(\beta - 1) \cos^{\beta-2}(\mu \xi) \]
\[ -\lambda \mu^2 \beta(\beta - 1) \cos^\beta(\mu \xi). \]  

(6)

Substituting Eq.(6) into the nonlinear ordinary differential equation Eq.(4) gives a trigonometric of terms. To determine the parameters first balancing the exponents of each pair of cosine to determine \( \alpha \). Then we collect all terms with the same power in \( \cos^\beta(\mu \xi) \) and put to zero their coefficients to get a system of algebraic equations among the unknown \( \beta, \lambda \) and \( \mu \). Now, the problem is reduced to a system of algebraic equations that can be solved to obtain the unknown parameters \( \beta, \lambda \) and \( \mu \). Hence, the solution considered in Eq.(5) is obtained [1].

3 Application

Example 1

Gardner equation

\[ u_t - 6(u + \epsilon^2 u^2)u_x + u_{xxx} = 0. \]  

(7)

By using the wave variable \( \xi = x - ct \) and \( u(x, t) = f(\xi) \) then equation becomes

\[ -\frac{df(\xi)}{d\xi} - 6 \left( f(\xi + \epsilon^2 f^2(\xi)) \right) \frac{df(\xi)}{d\xi} + \frac{d^3f(\xi)}{d\xi^3} = 0, \]  

(8)
Integrating Eq.(8) gives

\[-cf(\xi) - \frac{f^2(\xi)}{2} - 6e^2f^3(\xi) + \frac{d^2 f(\xi)}{d\xi^2} = 0.\]  \hspace{1cm} (9)

Substituting Eq.(6) into Eq.(9) gives:

\[-\lambda \text{cos}^\beta(\mu \xi) - 3\lambda^2 \text{cos}^{2\beta}(\mu \xi) - 2\lambda^2 \text{cos}^{3\beta}(\mu \xi) - \lambda \beta \mu^2 \text{cos}^\beta(\mu \xi) + \lambda \beta (\beta - 1) \mu^2 \text{cos}^\beta(\mu \xi) - \lambda \beta (\beta - 1) \mu^2 \text{cos}^\beta(\mu \xi) = 0,\]  \hspace{1cm} (10)

By equating the exponents and the coefficients of each pair of the cosine function we obtain the following system of algebraic equations:

\[
\begin{cases}
- c\lambda - \lambda \beta \mu^2 - \lambda \beta (\beta - 1) \mu^2 = 0, \\
- 2\lambda^2 \lambda^2 + \lambda \beta (\beta - 1) \mu^2 = 0, \\
3\beta = \beta - 2 \rightarrow \beta = 1.
\end{cases}
\]  \hspace{1cm} (11)

By using maple for solving the system Eq(11):

\[\beta = 1-, \mu = \pm i\sqrt{c}, \lambda = \pm \frac{\sqrt{c}}{\epsilon}\]  \hspace{1cm} (12)

Substituting Eq.(12) into Eq.(5) gives: (figure1)

\[u(x, t) = \pm \frac{\sqrt{c}}{\cos}(\pm i\sqrt{c}(x - ct)),\]  \hspace{1cm} (13)

\[u(x, t) = \pm \frac{\sqrt{c}}{\cosh}(\sqrt{c}(x - ct)).\]

**Example 2**

(2+1)-dimensional breaking soliton system

\[u_t + 4buv_x + 4bu_x v + bv_{xx} = 0,\]  \hspace{1cm} (14)

\[v_x - u_y = 0.\]

Suppose \(u(x, y, t) = f(\xi)\) and \(v(x, y, t) = g(\xi)\) and \(\xi = x + ky - ct\) then
Fig. 1. shows the soliton solution for Gardner equation with increase time. the (2+1)-dimensional soliton system becomes

\[-c \frac{df(\xi)}{d\xi} + 4bf(\xi) \frac{dg(\xi)}{d\xi} + 4b \frac{df(\xi)}{d\xi} g(\xi) + bk \frac{d^3 f(\xi)}{d\xi^3} = 0,\]  \hspace{1cm} (15)

\[\frac{dg(\xi)}{d\xi} - k \frac{df(\xi)}{d\xi} = 0,\]  \hspace{1cm} (16)

Integrating Eq.(15), Eq.(16)gives:

\[-cf(\xi) + 4bf(\xi)g(\xi) + bk \frac{d^2 f(\xi)}{d\xi^2} = 0,\]  \hspace{1cm} (17)

\[g(\xi) - kf(\xi) = 0,\]  \hspace{1cm} (18)

From Eq.(18)

\[g(\xi) = kf(\xi) = 0,\]  \hspace{1cm} (19)

Substituting Eq.(19) into Eq.(17) gives:

\[-cf(\xi) + 4bk f^2(\xi) + bk \frac{d^2 f(\xi)}{d\xi^2} = 0.\]  \hspace{1cm} (20)

Substituting Eq.(6) into Eq.(20) gives:

\[-c\lambda \cos^\beta(\mu \xi) + 4bk \lambda^2 \cos^{2\beta}(\mu \xi) + bk [-\lambda \beta \mu^2 \cos^\beta(\mu \xi)
+ \lambda \mu^2 \beta(\beta - 1) \cos^{\beta-2}(\mu \xi)
- \lambda \mu^2 \beta(\beta - 1) \cos^{\beta}(\mu \xi)] = 0,\]  \hspace{1cm} (21)
By equating the exponents and the coefficients of each pair of the cosine function we obtain the following system of algebraic equations:

\[
\begin{aligned}
-c\lambda - bk\lambda \mu^2 \beta - bk\lambda \mu^2 \beta (\beta - 1) &= 0, \\
4bk\lambda^2 + bk\lambda \mu^2 \beta (\beta - 1) &= 0.
\end{aligned}
\] (22)

\[2\beta = \beta - 2 \rightarrow \beta = 2.\]

By using Maple for solving the system Eq.(22) we get:

\[\beta = -2, \quad \mu = \pm \frac{i}{2} \sqrt{\frac{c}{bk}}, \quad \lambda = \frac{3}{8} \frac{c}{bk}\] (23)

Then by substituting Eq.(23) into Eq.(5) then, the exact soliton solutions of the (2+1)-dimensional breaking soliton system can be written in the form: (figure 2, 3)

\[
\begin{align*}
    u(x, y, t) &= \frac{3}{8} \frac{c}{bk} \cos^{-2} \left( \pm \frac{i}{2} \sqrt{\frac{c}{bk}} (x + ky - ct) \right), \\
    u(x, y, t) &= \frac{3}{8} \frac{c}{bk} \sec^2 \left( \frac{1}{2} \sqrt{\frac{c}{bk}} (x + ky - ct) \right), \\
    v(x, y, t) &= \frac{3}{8} \frac{c}{bk} \sec^2 \left( \frac{1}{2} \sqrt{\frac{c}{bk}} (x + ky - ct) \right).
\end{align*}
\] (24) (25)

Figures 3 and 3 show soliton solutions \(u, v\) of the (2+1)-dimensional breaking soliton system at \(y = 0\) to \(b = 1, k = 2\)

4 Conclusion

In the letter, the cosine function method has been successfully applied to find the solution for two nonlinear partial differential equations such as Gardner and (2+1)-dimensional breaking soliton system. The cosine function method is used to find a new exact solution.
References


[2] Anwar jaafar mohamad et. al., The sine-cosine function method for the
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