Implementing Basic Displacement Function to Analyze Free Vibration Rotation of Non-Prismatic Euler-Bernoulli Beams

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Received 2 October 2016, Accepted 18 December 2016

Abstract
Rotating beams have been considerably appealing to engineers and designers of complex structures i.e. aircraft’s propeller and windmill turbines. In this paper, a new flexibility-based method is proposed for the dynamic analysis of rotating non-prismatic Euler-Bernoulli beams. The flexibility basis of the method ensures the true satisfaction of equilibrium equations at any interior point of the elements. Following structural/mechanical principles, exact shape functions and consequently exact structural matrices i.e. consistent mass, geometric stiffness and flexural stiffness matrices are derived in terms of special so-called “Basic Displacement Functions”. The method is considered as the logical extension of conventional finite element method. Being straightforward formulated, the method can be incorporated into any standard finite element programs. The method poses no restrictions on either type of cross-section or variation of cross-sectional dimensions. The effects of rotational speed parameter and taper ratio on the variation of natural frequencies are studied and the results compare well with the other existing methods in the technical literature.

Keywords: Free vibration; Non-prismatic; Euler-Bernoulli beam; Basic Displacement Functions; Energy methods

1. Introduction

Free vibration analysis of non-rotating tapered Euler-Bernoulli beams have has been investigated by many researches researchers \cite{1-4}. The free analysis of rotating tapered beams have received great attention from designers of rotating machineries such as windmills, aircraft propellers due to its significant effect on their performance especially in presence of time-dependent loads. Obtaining natural frequencies of the system as an important component of free vibration analysis provides worthy insight for the designers in order to understand the response of the structural systems to dynamic loads. The presence of variable coefficients in the governing differential equation of motion introduced by varying cross-sectional area, moment of inertia and centrifugal forces is the troublesome point in the analysis of rotating tapered beams; hence it seems that derivation of a closed-form solution is impossible. Through years, many approximatenumerical techniques have been proposed such as dynamic stiffness method \cite{5-7}, finite element method \cite{8-9} and series solution of governing equation of motion \cite{10-11}. Understanding the fact that Hermite shape functions do not satisfy the homogeneous part of the static governing equation, Gunda and Ganguli \cite{12} proposed new rational shape functions which satisfy the homogeneous static part. They \cite{12} verified their method for both static and free vibration analyses of tapered beams. Using Frobenius method, Banerjee et al. \cite{7} derived the dynamic stiffness matrix for non-prismatic beams whose cross-sectional area and moment of inertia vary respectively along beam length with arbitrary integer powers \(n\) and \(n+2\). Considering the similar assumptions for cross-sectional area and moment of inertia as in Ref. \cite{7}, Ozgumus and Kaya \cite{13} carried out free vibration analysis for tapered rotating beams using differential transform method (DTM). Mei \cite{14} determined natural frequencies of rotating prismatic beam via application of DTM to the governing equation of motion.

In this paper, new set of functions namely Basic Displacement Functions (BDFs) are introduced
which hold mechanical interpretations. Considering basic structural/mechanical theorems, it is shown that exact shape functions are expressed in terms of BDFs; thus exact structural matrices are evaluated. The effects of centrifugal forces are introduced into the formulation by adding the geometric stiffness matrix to the flexural one. This method has been previously employed for the analysis of non-rotating tapered members by the first author [15-16]. The method is considered as the logical extension of the conventional finite elements method and can be easily incorporated into the existing finite element programs.

2. Basic Displacement Functions

In the following, four BDFs are introduced as,

\[ b_{\nu 1}(x) = \int_{0}^{\bar{x}} \frac{s(s - \bar{x})}{EI(s)} ds \]  

(1)

\[ b_{\theta 1}(x) = \int_{0}^{\bar{x}} -\frac{(s - \bar{x})}{EI(s)} ds \]  

(2)

where \( l \) and \( EI \) are length of the element and the flexural rigidity, respectively. On the basis of the reciprocal theorem [17], BDFs can equivalently be interpreted mechanically as,

\[ b_{\nu 1} \]: Transverse displacement at distance \( \bar{x} \) due to a unit load at the left node of a beam clamped at right as shown in Figure (1-a).

\[ b_{\theta 1} \]: Angle of rotation at distance \( \bar{x} \) due to a unit moment at the left node of a beam clamped at right as shown in Figure (1-b).

\[ b_{\nu 2} \]: Transverse displacement at distance \( \bar{x} \) due to a unit load at the right node of a beam clamped at left as shown in Figure (1-c).

\[ b_{\theta 2} \]: Angle of rotation at distance \( \bar{x} \) due to a unit moment at the right node of a beam clamped at left as shown in Figure (1-d).

Due to the structural definitions of BDFs, it is easily verified that each BDFs could be given via application of basic structural theorems such as the energy methods [17].

\[ b_{\nu 2}(\bar{x}) = \int_{0}^{\bar{x}} \frac{(l - s)(\bar{x} - s)}{EI(s)} ds \]  

(3)

\[ b_{\theta 2}(\bar{x}) = \int_{0}^{\bar{x}} -\frac{(s - \bar{x})}{EI(s)} ds \]  

(4)

Considering the equivalent definitions of BDFs, it is observed that both the first and second derivatives of BDFs have mechanical interpretations, as well, where they are respectively defined as the slope and curvature of the corresponding beam. The derivatives of BDFs are given in Table 1.
Figure 1. Mechanical illustrations of BDFs

Figure 2. Equivalent mechanical illustrations of BDFs
Table 1. First and second derivatives of BDFs

<table>
<thead>
<tr>
<th>Type of BDF</th>
<th>First Derivative</th>
<th>Second Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_{v1}$</td>
<td>$\int_{0}^{l} \frac{-s}{EI(s)} ds$</td>
<td>$\frac{x}{EI(x)}$</td>
</tr>
<tr>
<td>$b_{b1}$</td>
<td>$\int_{0}^{l} \frac{1}{EI(s)} ds$</td>
<td>$-\frac{1}{EI(x)}$</td>
</tr>
<tr>
<td>$b_{v2}$</td>
<td>$\int_{0}^{l} \frac{l-s}{EI(s)} ds$</td>
<td>$\frac{l-x}{EI(x)}$</td>
</tr>
<tr>
<td>$b_{b2}$</td>
<td>$\int_{0}^{l} \frac{1}{EI(s)} ds$</td>
<td>$\frac{1}{EI(x)}$</td>
</tr>
</tbody>
</table>

In which $F_{11}, F_{22}$ are nodal flexibility matrices of the first and second node, respectively.

3. Application of BDFs

Assume a general beam shown in Figure 3. In order to obtain the nodal reactions, the original structural system is decomposed into isostatic structures i.e. cantilever beams and the boundary conditions are imposed for the released support. Scrutinizing the definitions of equivalent BDFs, one can easily evaluate the nodal flexibility matrices as

The nodal displacements of point (2) in Figure (3-b) can be elegantly computed in terms of BDFs as,

$$\begin{align*}
\begin{bmatrix} w_2 \\ \theta_2 \end{bmatrix}^{(b)} &= \int_{0}^{l} q(x) \begin{bmatrix} b_{v2} \\ b_{b2} \end{bmatrix} d\bar{x} \\
\end{align*}
$$

(7)

In which $q(x)$ is the external distributed transverse loading. Apparently the nodal displacements of point (2) in Figure (3-c) are given as,

$$\begin{align*}
\begin{bmatrix} w_2 \\ \theta_2 \end{bmatrix}^{(c)} &= F_{22} \begin{bmatrix} V_2 \\ M_2 \end{bmatrix} \\
\end{align*}
$$

(8)

Imposing the boundary conditions at point (2)

$$\begin{align*}
\begin{bmatrix} w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} w_2 \\ \theta_2 \end{bmatrix}^{(b)} + \begin{bmatrix} w_2 \\ \theta_2 \end{bmatrix}^{(c)} = 0 \\
\end{align*}
$$

(9)

The nodal reactions are obtained by substituting Eqs. (7, 8) in Eq. (9)

$$\begin{align*}
\begin{bmatrix} V_2 \\ M_2 \end{bmatrix} = -K_{22} \int_{0}^{l} q(x) \begin{bmatrix} b_{v2} \\ b_{b2} \end{bmatrix} d\bar{x} \\
\end{align*}
$$

(10)

Similarly the nodal reactions of point (1) are obtained.

$$\begin{align*}
\begin{bmatrix} V_1 \\ M_1 \end{bmatrix} = -K_{11} \int_{0}^{l} q(x) \begin{bmatrix} b_{v1} \\ b_{b1} \end{bmatrix} d\bar{x} \\
\end{align*}
$$

(11)

Based on structural analysis, nodal equivalent forces are the negative of nodal reactions; thus

$$F = G \int_{0}^{l} q(x) b d\bar{x}$$

(12)

in which

$$G = \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix}$$

(13)

$$b = \begin{bmatrix} b_{v1} \\ b_{b1} \\ b_{v2} \\ b_{b2} \end{bmatrix}^{T}$$

(14)
\[ F = [V_1 M_1 V_2 M_2]^T \]  
\[ (15) \]

Comparing Eq. (12) with the relation proposed by finite element method

\[ F = \int_0^l q(x).N_x d\bar{x} \]  
\[ (16) \]

The shape functions \( N \) are obtained as,

\[ N = b^T \cdot G \]  
\[ (17) \]

Once shape functions are derived, the structural matrices can be evaluated as [1],

\[ M = \int_0^l N_x^T \rho A(x)N_x d\bar{x} \]  
\[ (18) \]

\[ K_g = \int_0^l N_x^T F_x N_x d\bar{x} \]  
\[ (19) \]

\[ K = \int_0^l N_x^T EI(x)N_x d\bar{x} \]  
\[ (20) \]

Where \( M, K_g \) and \( K \) are respectively consistent mass, geometric stiffness and flexural stiffness matrices, \( \rho \) and \( A \) are respectively mass per unit volume and cross-sectional area. \( F_x \) is the axial force i.e. centrifugal force which is given by

\[ F_x = \Omega^2 \int_0^L x\rho A(x)dx \]  
\[ (21) \]

where \( \Omega \) and \( L \) are respectively the rotational speed and whole length of the beam.

4. Numerical Examples and Discussion

The proposed method is employed to determine the natural frequencies of a beam whose cross-sectional area and moment of inertia vary respectively as,

\[ A(x) = A_0 \left(1 - \frac{x}{L}\right) \quad I(x) = I_0 \left(1 - \frac{x}{L}\right)^3 \]  
\[ (22) \]

Where \( A_0 \) and \( I_0 \) are respectively cross-sectional area and moment of inertia at the origin and \( c \) is the taper ratio. The following dimensionless parameters namely rotational speed parameter \( \eta \) and dimensionless natural frequency \( \mu \) are introduced to make comparisons with the literature.

\[ \eta^2 = \frac{\rho A_0 L^2 \Omega^2}{E I_0} \quad \mu^2 = \frac{\rho A_0 L^4 \omega^2}{E I_0} \]  
\[ (23) \]

The effect of rotational speed parameter on the variation of natural frequencies for a tapered beam with taper ratio \( c = 0.5 \) is tabulated in Table 2. As expected, the natural frequencies increase with the rotational speed due to the stiffening effect of centrifugal force.

In Table 3, the natural frequencies are given for different values of taper ratio. The natural frequencies except for the first mode decrease with the taper ratio due to the softening effect caused by the decrease in cross-sectional area and moment of inertia.

Figure 3. Decomposing the beam into two isostatic systems
Table 2 Effect of rotational speed on variation of dimensionless natural frequencies

<table>
<thead>
<tr>
<th>Taper Ratio=0.5</th>
<th>First Mode</th>
<th>Second Mode</th>
<th>Third Mode</th>
<th>Fourth Mode</th>
<th>Fifth Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.8216</td>
<td>3.82379</td>
<td>18.3073</td>
<td>18.3173</td>
<td>47.25</td>
</tr>
<tr>
<td>1</td>
<td>3.9845</td>
<td>3.98661</td>
<td>18.464</td>
<td>18.474</td>
<td>47.4024</td>
</tr>
<tr>
<td>2</td>
<td>4.4347</td>
<td>4.4368</td>
<td>18.9267</td>
<td>18.9366</td>
<td>47.8567</td>
</tr>
<tr>
<td>3</td>
<td>5.0906</td>
<td>5.09267</td>
<td>19.6741</td>
<td>19.6839</td>
<td>48.6041</td>
</tr>
<tr>
<td>4</td>
<td>5.8767</td>
<td>5.87877</td>
<td>20.6754</td>
<td>20.6851</td>
<td>49.6306</td>
</tr>
<tr>
<td>5</td>
<td>6.7412</td>
<td>6.7434</td>
<td>21.8957</td>
<td>21.9053</td>
<td>50.9187</td>
</tr>
<tr>
<td>6</td>
<td>7.6529</td>
<td>7.65514</td>
<td>23.2997</td>
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<td>52.4481</td>
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<tr>
<td>7</td>
<td>8.5932</td>
<td>8.59557</td>
<td>24.8552</td>
<td>23.3093</td>
<td>54.1972</td>
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<tr>
<td>10</td>
<td>11.4988</td>
<td>11.5015</td>
<td>30.1733</td>
<td>30.1827</td>
<td>60.5486</td>
</tr>
</tbody>
</table>

Table 3 Effect of taper ratio on variation of dimensionless natural frequencies

<table>
<thead>
<tr>
<th>Taper Ratio</th>
<th>( \eta=5 )</th>
<th>First Mode</th>
<th>Second Mode</th>
<th>Third Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>6.4891</td>
<td>6.49115</td>
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<tr>
<td>0.2</td>
<td>6.5370</td>
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<td>24.0961</td>
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<td>0.3</td>
<td>6.5931</td>
<td>6.59525</td>
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<td>23.3906</td>
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<tr>
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<td>22.6510</td>
<td>22.6612</td>
</tr>
<tr>
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<td>6.7434</td>
<td>21.8957</td>
<td>21.9053</td>
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<td>21.1207</td>
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<tr>
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<td>6.97848</td>
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<td>20.3086</td>
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<td>7.16281</td>
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</tr>
<tr>
<td>0.9</td>
<td>7.4411</td>
<td>7.44359</td>
<td>18.7364</td>
<td>18.7412</td>
</tr>
</tbody>
</table>

72
5. Conclusions

New shape functions are proposed for the analysis of rotating tapered beams which are obtained in terms of special structural functions namely BDFs. The superiority of the present method lies in proposing a mechanical solution rather than a mathematical one. The structural essence of BDFs let us obtain them on the basis of static deformations, it is observed that the method could be used in free vibration analysis and acceptable results could be expected even with a coarse mesh.

6. References


