Stress Analysis of Rotational Shaft with a Constant Angular Speed using Strain Gradient Theory

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Abstract: In classical mechanics, considering Hook's law, stress is a linear function of strain. While in strain gradient theory, stress is a function of strain and strain differentials. In this paper, Novel formulation relating stress and strain and also new boundary conditions are derived based on minimum potential energy principle. In strain gradient theory a length coefficient parameter is defined. This statistical parameter shows that material behaviour in microscopic scale depends on material dimensions. In classical elasticity dependency of the material behaviour on material size could not be described due to the lack of length coefficient parameter. Here also a total stress tensor, different from the Cauchy's stress tensor, is defined which can be used as a total stress tensor in momentum equation. Using strain gradient theory, strain field for a rotational shaft with a constant angular speed is analytically studied. Knowing displacement field, total stress tensor can also be computed. A material constant is present in the derived displacement field in addition to the two Lame constants. Formulations based on strain gradient theory turn to those of classical mechanics if length coefficient is neglected. Results of stress analysis using strain gradient theory and those of classic mechanics are compared.

Keywords: Length Coefficient Parameter, Minimum Total Potential Energy Principle, Strain Gradient Theory, Total Stress Tensor

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1 INTRODUCTION

Kinetic quantities are functions of kinematic quantities. For instance stress (as a kinetic quantity) is dependent on strain (as a kinematic quantity). Thus kinetic quantities can be described as functions of kinematic quantities e.g. in simple linear theory they are linear functions of kinematic quantities. In classical mechanics, stress is a function of only strain and therefore density of the potential energy stored into the material due to the deformation can be described as a function of strain. In strain gradient theory in the higher order equations are considered. For instance, in this linear deformation theory, density of the potential energy is a function of both strain and strain differentials. This theory was first introduced by Mindlin [1] and Toupin [2].

In an elastic material, considering theory of elasticity Mindlin derived equilibrium equations, boundary conditions and constitutive equations. It was shown that density of the potential energy is a binomial in small strains with first and second orders of strain differentials. For isotropic materials Mindlin [3] developed the third order of strain gradient with 18 material constants and two Lame’s constants. It was shown that in an elastic continuum state if it is assumed that density of the potential energy in addition to strain, depends on revolution gradients; then elasticity theory yields a coupled stress tensor in addition to the general stress tensor. Higher orders continuum mechanics especially relies on nonlinear concepts and large deformation and non linear behaviours.

High order strain gradient theory which includes length coefficient shows that material behaviour in micro scale depends on material dimensions. This dependency could not be described in classical elasticity theory due to the lack of the length coefficient parameter. Fleck and Hutchinson [4] for the first time used coupled equations to introduce length coefficient parameter. To apply length coefficient parameter in equations, Cosserat theory can be used. The concept of the Cosserat continuum circumstance was first defined by Cosserat brothers [5] in 1990. As mentioned before in this circumstance in addition to force effects, the effect of couples on a physical element is also considered. This theory states that by applying force on material in addition to stress, moments will exist in material which should be considered in equilibrium equations of the particles.

However investigations on the numerous material-related parameters in the constitutive equation of higher order materials hinders the use of this theory. Therefore elastic strain gradient theory which considers a constant as a material parameter drew the attentions and showed to be useful in solution of vast variety of problems. Altan and Aifantis [6] suggested a new model for strain gradient theory. In this theory three elastic and two Lame’s constants are present in formulations. This model, due to its simplicity, was applied to analyze numerous problems.

Therefore, there was a growing need to completely formulate this simple elastic strain gradient theory based on the first principle of mechanics (minimum potential energy principle). To respond this demand a new modified formulation of strain gradient theory was introduced by Park and Gao [7]. In this paper a review on formulations in gradient theory was first performed. Then based on developed formulations displacement field was computed for a rotational shaft. Total stress tensor which depends on higher differentials of the strain was introduced. After computing displacement field, total stress tensor components are calculated and comparison was made between results of strain gradient theory and those of classical mechanics.

2 A REVIEW OF STRAIN GRADIENT ELASTICITY THEORY

Considering studies conducted by Mindlin and Eshel [8], for an isotropic material in strain gradient theory, strain energy density is a function of two parameters namely strain and strain gradient

$$W = W\left(\overset{<2>}{\varepsilon}, \overset{<3>}{\varepsilon}\right) \tag{1}$$

$\overset{<2>}{\varepsilon}$ and $\overset{<3>}{\varepsilon}$ are classic strain and strain gradient tensor components respectively which can be defined as follows.

$$\overset{<2>}{\varepsilon} = \frac{1}{2}\left(\nabla u + u\nabla\right) \tag{2}$$

$$\overset{<3>}{\varepsilon} = \nabla \overset{<2>}{\varepsilon} = \nabla\nabla u \tag{3}$$

where ’∇’ is gradient operator and ‘u’ is the displacement field. Therefore ‘∇u’ is strain and’$\nabla\nabla u$’ is strain gradient. In this condition total strain energy ‘U’ for a mass with the volume of ‘Ω’

with the assumption that it is linearly deformed can be defined as follows.

$$U = \int_{\Omega} w dv = \int_{\Omega} \left(\overset{<2>}{\tau} \cdot\cdot \nabla u + \overset{<3>}{\mu} \cdot\cdot\cdot \nabla\nabla u \right) dv \tag{4}$$

where $\overset{<2>}{\tau}$ and $\overset{<3>}{\mu}$ are Cauchy and double stress tensor components respectively. Each of these parameters can be calculated using equations introduced by Mindlin and Eshel [8] and Eq. (1).

$$\overset{<2>}{\tau} = \frac{\partial W}{\partial \overset{<2>}{\varepsilon}} = \tag{5}$$

$$\overset{<3>}{\mu} = \frac{\partial W}{\partial \nabla \overset{<2>}{\varepsilon}} = \frac{\partial W}{\partial \overset{<3>}{\varepsilon}} \tag{6}$$

Using the first principle of classical mechanics in a steady state equilibrium and fundamental subject of calculus of variation, the following equilibrium equations are obtained for gradient-dependent material body [1], [8]:

$$\nabla.\sigma + f = 0 \tag{7}$$

In Eq. (7) f and σ are body force and total stress tensor components respectively. They can be defined by Equations introduced by Altan and Aifantis [9] as follows:

$$\sigma = (1 - \ell^2 \nabla^2) \{ \lambda (\nabla.u) I + \mu [\nabla u + (u \nabla)^T] \} \tag{8}$$

In Eq. (8) 'λ' and 'μ' are Lame's constants and 'ℓ' is the length scale parameter. It is worth to mention that the dimension of length scale parameter is length square. ∇ , $\nabla.$ and ∇^2 are gradient, divergence and Laplasian operators and 'I' is the unity tensor of the second order. Taking divergence from both sides of the Eq. (8), Eq. (9) yields as:

$$\nabla.\sigma = (1 - \ell^2 \nabla^2)[(\lambda + \mu)\nabla(\nabla.u) + \mu \nabla^2 u] \tag{9}$$

Considering $\nabla^2 u = \nabla(\nabla.u) - \mathrm{curl}(\mathrm{curl}\, u)$ Eq. (9) changes to a simpler form which is given in Eq. (10):

$$\nabla.\sigma = (1 - \ell^2 \nabla^2)[(\lambda + 2\mu)\nabla(\nabla.u) - \mu \mathrm{curl}(\mathrm{curl}\, u)] \tag{10}$$

Substituting Eq. (7) by Eq. (10), Eq. (11) can be derived as:

$$(1 - \ell^2 \nabla^2)[(\lambda + 2\mu)\nabla(\nabla.u) - \mu \mathrm{curl}(\mathrm{curl}\, u)] + f = 0 \tag{11}$$

Eq. (11) is the equilibrium equation in the form of tensor and it depends on displacement field u . It is clear that the equilibrium equation introduced by strain gradient theory, in addition to two Lame's constants (λ, μ) there is a material-scale parameter. The boundary condition can be defined by Eq. (12):

$$\begin{aligned} &\sigma n - \nabla.(\mu(\mu + (\mu : n \otimes n)(\nabla.n) + (\nabla \mu : n \otimes n) n \\ &+ \mu : \{ n \otimes [(\nabla n) n] \} = \bar{t} \\ &\mu : (n \otimes n) = \overline{R} \\ &[[\mu : (m \otimes n)]] = 0 \end{aligned} \tag{12}$$

where

$$\mu = \ell^2 \nabla \left\{ \lambda (\nabla.u) I + \mu \left[\nabla u + (\nabla u)^T \right] \right\} \tag{13}$$

3 SOLUTION OF A ROTATIONAL SHAFT WITH A CONSTANT ANGULAR SPEED

Classic solution of the rotational shaft with a constant angular speed is introduced by Timoshenko and Goodier [10], even though in their solution there is no length parameter due to disability of classical mechanics to reveal the dependency of the material behaviour on material dimension.

In this paper solution of the rotational shaft is taken into account using strain gradient theory. Displacement field for this problem is obtained using mentioned formulations in the previous section. Contrary to the classic solution, here in addition to the two Lame's constants there is an extra parameter named length coefficient which can demonstrate the effect of material dimension on its behaviour. The problem is considered in the plane strain state. Fig. 1 shows assumed shaft with outer and inner radii of 'r_i' and 'r_o' respectively and constant angular velocity of 'ω' and density of 'ρ':

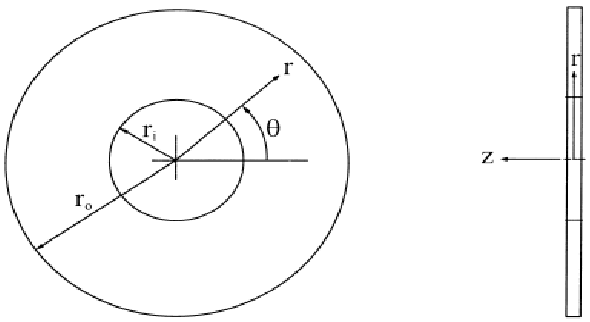

Fig. 1 Rotational shaft with a constant angular velocity

Cylindrical coordinate system was considered in development of the solution. Due to the symmetry of the geometry and loading conditions, displacement field can be defined as:

$$u = u(r)e_r \tag{14}$$

Noticing that displacement field depends on the radial component r and is independent of circumferential component θ ; this problem can be considered as a 1D problem even generally it is a 2D problem. Considering Eq. (14), Eq. (15) can be computed.

$$u = u(r)e_r \tag{15}$$

Inserting Eq. (15) in Eq. (11), the equilibrium equation can be simplified based on Eq. (10) as follows:

$$(1-\ell^2\nabla^2)[(\lambda+2\mu)\nabla(\nabla.u)] + f = 0 \tag{16}$$

Here, body force in the volume unit is $f = \rho r\omega^2$ and hence by substituting body force in Eq. (16), Equations (17-19) are calculated.

$$(1-\ell^2\nabla^2)[(\lambda+2\mu)\nabla(\nabla.u)] + \rho r\omega^2 = 0 \tag{17}$$

where

$$\nabla(\nabla.u) = \left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{1}{r^2}u\right)e_r \tag{18}$$

$$\nabla^2[\nabla(\nabla.u)] = \begin{pmatrix} \dfrac{d^4u}{dr^4} + \dfrac{2}{r}\dfrac{d^3u}{dr^3} - \dfrac{3}{r^2}\dfrac{d^2u}{dr^2} \\ + \dfrac{3}{r^3}\dfrac{du}{dr} - \dfrac{3}{r^4}u \end{pmatrix} e_r \tag{19}$$

Inserting Equations (18) and (19) into Eq. (17) yields:

$$\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{1}{r^2}u\right) - \ell^2 \begin{pmatrix} \dfrac{d^4u}{dr^4} + \dfrac{2}{r}\dfrac{d^3u}{dr^3} - \dfrac{3}{r^2}\dfrac{d^2u}{dr^2} \\ + \dfrac{3}{r^3}\dfrac{du}{dr} - \dfrac{3}{r^4}u \end{pmatrix} = -\frac{\rho r\omega^2}{(\lambda+2\mu)} \tag{20}$$

Ignoring 'ℓ' in the second term of Eq. (20) it can be simplified as presented as Eq. (21):

$$\frac{d^4u}{dr^4} + \frac{2}{r}\frac{d^3u}{dr^3} - \frac{3}{r^2}\frac{d^2u}{dr^2} + \frac{3}{r^3}\frac{du}{dr} - \frac{3}{r^4}u = \left(\frac{d}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right)\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{1}{r^2}u\right) \tag{21}$$

And linear differential operator L is defined by Eq. (22):

$$L \equiv \frac{d}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} \tag{22}$$

Substituting Equations (21) and (22) into Eq. (20), Eq. (23) can be written:

$$L(1-\ell^2L)u = -\frac{\rho r\omega^2}{(\lambda+2\mu)} \tag{23}$$

Differential equation of equilibrium in strain gradient theory ($\ell \neq 0$) is of the forth order. Therefore, 4 boundary conditions are needed to solve the equation. Boundary conditions in strain gradient theory are defined in two sections by using fundamental subject of the calculus of variations and principle of minimum total potential energy. When ($\ell = 0$) this equation can be simplified as Eq. (24):

$$Lu = -\frac{\rho r\omega^2}{(\lambda+2\mu)} \tag{24}$$

Eq. (24) is an Eulerian non-uniform differential equation of the second order. By solving this equation, Eq. (25) can be computed:

$$u = Ar + \frac{B}{r} - \frac{(1-2\nu)}{16(1-\nu)\mu}\rho\omega^2 r^3 \tag{25}$$

where 'A' and 'B' are constants which can be solved using boundary conditions. This solution is a classic solution which was introduced by Timoshenko and Goodier [10]. As mentioned earlier in classical solution there is no material-scale parameter. Eq. (23) is a non-uniform solution consisting of general and specific responses. Solving this equation yields displacement field in strain gradient theory.

$$u(r) = Ar + \frac{B}{r} + CI_1(\frac{1}{\ell}r) + DK_1(\frac{1}{\ell}r) - \frac{(1-2\nu)}{16(1-\nu)\mu}\rho\omega^2 r^3 \tag{26}$$

In Eq. (26), $I_1(0)$ and $K_1(0)$ are modified Bessel functions of first and second orders. A-D are constants known by boundary condition.

As it can be seen in Eq. (26) displacement field depends on the length scale parameter. Considering Equations (2) and (14), strain tensor is computed using Eq. (27).

$$\varepsilon(r) = \frac{du}{dr} e_r \otimes e_r + \frac{u}{r} e_\theta \otimes e_\theta \tag{27}$$

Substituting Eq. (27) in Eq. (5) Cauchy stress tensor components can be found based on Eq. (28).

$$\tau(r) = \left[(\lambda+2\mu)\frac{du}{dr} + \lambda\frac{u}{r}\right] e_r \otimes e_r + \left[\lambda\frac{du}{dr} + (\lambda+2\mu)\frac{u}{r}\right] e_\theta \otimes e_\theta + \left(\lambda\frac{du}{dr} + \lambda\frac{u}{r}\right) e_3 \otimes e_3 \tag{28}$$

Considering Eq. (6) total stress tensor 'σ' which depends on the Cauchy stress tensor is as defined in Eq. (29):

$$\begin{aligned}\sigma = &\left[\tau_{rr} - \ell^2\left(\tau''_{rr} + \frac{\tau'_{rr}}{r} - \frac{2\tau_{rr}}{r^2} + \frac{2\tau_{\theta\theta}}{r^2}\right)\right] e_r \otimes e_r \\ &+ \left[\tau_{\theta\theta} - \ell^2\left(\tau''_{rr} + \frac{\tau'_{\theta\theta}}{r} - \frac{2\tau_{\theta\theta}}{r^2} + \frac{2\tau_{rr}}{r^2}\right)\right] e_\theta \otimes e_\theta \\ &+ \left[\tau_{zz} - \ell^2\left(\tau''_{zz} + \frac{\tau'_{zz}}{r}\right)\right] e_3 \otimes e_3\end{aligned} \tag{29}$$

Substituting Equations (26) and (27) into Eq. (29), total stress tensor can be computed. Boundary conditions which were defined in Eq. (12) can be rewritten as below:

$$\begin{aligned}&\left\{\tau_{rr} - \ell^2\left[\begin{matrix}\tau''_{rr} + \frac{1}{r}(\tau'_{rr} - \tau'_{\theta\theta}) \\ -\frac{2}{r^2}(\tau_{rr} - \tau_{\theta\theta})\end{matrix}\right]\right\}_{r=r_i=a} e_r = 0 \\ &\left\{\tau_{rr} - \ell^2\left[\begin{matrix}\tau''_{rr} + \frac{1}{r}(\tau'_{rr} - \tau'_{\theta\theta}) \\ -\frac{2}{r^2}(\tau_{rr} - \tau_{\theta\theta})\end{matrix}\right]\right\}_{r=r_o=b} e_r = 0 \\ &\ell^2 \tau'_{rr}\Big|_{r=r_i=a} e_r = 0 \\ &\ell^2 \tau'_{rr}\Big|_{r=r_o=b} e_r = 0\end{aligned} \tag{30}$$

By substituting Equations (26) and (28) in Eq. (30), Eq. (31) can be computed:

$$\begin{aligned}&(-2\lambda - 2\mu)A + (\frac{2\mu}{r_i^2} + \frac{4\ell^2\mu}{r_i^4})B + \left[\frac{2\ell\mu}{r_i^2}K_0(\frac{r_i}{\ell}) - \left(\frac{\lambda}{r_i} - \frac{4\ell^2\mu}{r_i^3}\right)K_1(\frac{r_i}{\ell})\right]D = \frac{(1-2\nu)(2\lambda+3\mu)}{8(1-\nu)\mu}\left(r_i^2 - 2\ell^2\right)\rho\omega^2 \\ &(-2\lambda - 2\mu)A + (\frac{2\mu}{r_o^2} + \frac{4\ell^2\mu}{r_o^4})B + \left[\frac{2\ell\mu}{r_o^2}K_0(\frac{r_o}{\ell}) - \left(\frac{\lambda}{r_o} - \frac{4\ell^2\mu}{r_o^3}\right)K_1(\frac{r_o}{\ell})\right]D = \frac{(1-2\nu)(2\lambda+3\mu)}{8(1-\nu)\mu}\left(r_o^2 - 2\ell^2\right)\rho\omega^2 \\ &\frac{4\mu}{r_i^3}B + \left[\frac{2\mu}{\ell r_i}K_0(\frac{r_i}{\ell}) + (\frac{\lambda+2\mu}{\ell^2} + \frac{4\mu}{r_i^2})K_1(\frac{r_i}{\ell})\right]D = \frac{(1-2\nu)(2\lambda+3\mu)}{4(1-\nu)\mu}\rho\omega^2 r_i \\ &\frac{4\mu}{r_0^3}B + \left[\frac{2\mu}{\ell r_0}K_0(\frac{r_o}{\ell}) + (\frac{\lambda+2\mu}{\ell^2} + \frac{4\mu}{r_o^2})K_1(\frac{r_o}{\ell})\right]D = \frac{(1-2\nu)(2\lambda+3\mu)}{4(1-\nu)\mu}\rho\omega^2 r_o\end{aligned} \tag{31}$$

Solving linear system of equations presented in Eq. (31), A-D constants can be calculated. It can be inferred from Eq. 31 that these constants depend on Lame's constants (μ, λ), strain gradient coefficient 'ℓ' (length scale parameter), geometrical parameters as inner 'r_i' and outer 'r_o' radii, constant angular velocity 'ω' and density 'ρ'.

If these parameters ($\lambda, \mu, \ell, \rho, \omega, r_i, r_o$) are known, the constants can be computed using Eq. (31). Substituting these into constants Eq. 26, displacement field can be obtained. When displacement field is known, total stress components can be found using Equations (28) and (29).

4 RESULTS AND DISCUSSION

Results from analytical solution in strain gradient theory are illustrated in Figs. 2 and 3. In these figures total stress tensor components $(\sigma_{rr}, \sigma_{\theta\theta})$ are compared with those obtained from classical mechanics $(\tau_{rr}, \tau_{\theta\theta})$. Parameters considered in the analysis are $E = 71.7\,\text{GPa}$ and $\nu = 0.28$. To study the effects of length scale parameters on stress components, different amounts were considered in the analysis. Inner radius of $r_i = a = 1\mu m$ and outer radius of $r_o = b = 5\mu m$ were assumed. As ρ and ω were used in the dimensionless form, there is no need to specify angular speed and density.

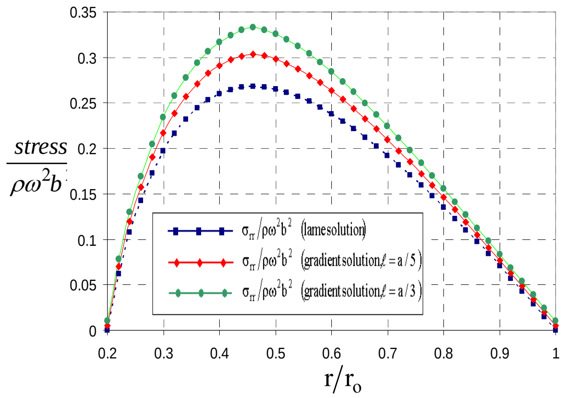

Fig. 2 Radial stress distributions along the radial shaft

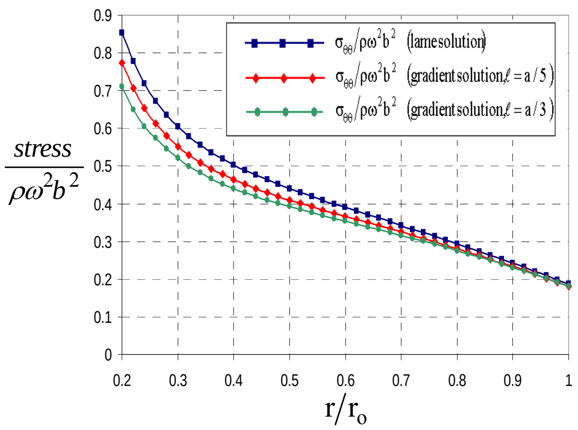

Fig. 3 Circumferential stress distributions along the radial shaft

As it can be seen from Figs. 3 and 4, results obtained from strain gradient theory discrepant those obtained from classic mechanics due to the existence of the strain gradients. Total stress tensor varies concurrent with the change in the length scale parameter. Results are depicted for $\ell = a/5$ and $\ell = a/3$. In Fig. 2 transverse stress distribution is shown. According to Fig. 3, with an increase in the length scale parameter, transverse stress increases and axial stress decreases.

5 CONCLUSION

Mentioned formulations were coded using Matlab. Solving equations derived by the use of strain gradient theory yields displacement field. It has been shown that in spite of classic mechanics, here in addition to two Lame's constants, displacement field depends on the material-scale parameter namely length coefficient. This parameter has a statistical basis which is appeared in nonlinear and large deformation formulations. A novel total stress tensor was introduced which can be used as total stress tensor in the momentum equation. The main objective of this paper was to compute total stress components for a rotational shaft with a constant angular speed; therefore displacement field was first computed using equations developed in strain gradient theory and then total stress components were calculated. Results using strain gradient theory and classic mechanics show discrepancies. These differences may be attributed to the existence of strain gradients in the strain gradient theory. It was observed that with the change in length coefficient, computed components of the total stress tensor change. Changes are depicted for both components of the total tensor in plane strain state.

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