

Estimation of the Time- Dependent Heat Flux Using Temperature Distribution at a Point in a Three Layer System with None Homogeneous Boundary Conditions

Mohammad Mohammadiun*

Department of Mechanical Engineering, Shahrood branch,
Islamic Azad University, Shahrood, Iran

E-mail: mmohammadiun@yahoo.com

*Corresponding author

Received: 30 December 2015, Revised: 9 February 2016, Accepted: 16 March 2016

Abstract: In this paper, the conjugate gradient method coupled with adjoint problem is used in order to solve the inverse heat conduction problem and estimation of the time-dependent heat flux using the temperature distribution at a point in a three layer system with none homogeneous boundary conditions. Also, the effect of noisy data on final solution is studied. For solving this problem the general coordinate method is used. The inverse heat conduction problem of estimating the transient heat flux, applied on part of the boundary of an irregular region is solved in this paper. The present formulation is general and can be applied to the solution of boundary inverse heat conduction problems over any region that can be mapped into a rectangle. The obtained results for few selected examples show the good accuracy of the presented method. Also the solutions have good stability even if the input data includes noise. Applications of this model are in the thermal protect systems (t.p.s.) and heat shield systems.

Keywords: Adjoint problem, Conjugate gradient method, Time-dependent heat flux, Three layer system

Reference: Mohammadiun, M., "Estimation of the time-dependent heat flux using the temperature distribution at a point in a three layer system", *Int J of Advanced Design and Manufacturing Technology*, Vol. 9/No. 3, 2016, pp. 93–102.

Biographical notes: **M. Mohammadiun** is an assistant professor of Mechanical Engineering at Islamic Azad University, Shahrood branch, Shahrood, Iran. He received his PhD in 2011 from Ferdowsi University of Mashhad, Iran, and MSc in 2004 from Amir Kabir University, Tehran, Iran, and BSc in 2001 from Khajeh Nasireddin Toosi University, Iran, all in mechanical engineering. His scientific interests include Heat transfer and Fluid Dynamics.

1 INTRODUCTION

Direct heat conduction problems are concerned with the determination of temperature at interior points of a region when the initial and boundary conditions, thermophysical properties, and heat generation are specified [1]. In contrast to the direct problems, inverse heat conduction problems (IHCP) are defined as the estimation of initial/boundary conditions, properties of the system/material, sources or sink terms, shape, and governing equations from transient temperature measurements at one or several interior locations [2]. Solution of inverse problems is much more difficult in comparison with direct problems due to instability in solution where these problems are called mathematically ill-posed.

With the improvement of computer capability, inverse techniques have become a popular means of resolving heat transfer problems in the last decade. Important applications for inverse heat conduction problem solutions include, for example, controlled cooling of electronic components, estimation of jet-flow rate of cooling in machining or quenching, determination of conditions at the interface between the mold and metal during metal casting or rolling process[3], heat flux estimation in the surface of a wall subjected to fire or the inside surface of a combustion chamber [4] and also in surfaces where ablation takes place or in surfaces going through welding process [5].

Some other applications of the IHCP are prediction of the inner wall temperature of a reactor, determination of the heat transfer coefficient and outer surface conditions in the re-entry of a space vehicle and modeling of the temperature or heat flux at the tool-work interface of machine cutting [6] and also in the transpiration cooling control [7]. Estimation of the unknown time-dependent heat flux and temperature distributions for the system composed of a multi-layer composite strip and semi-infinite foundation, from the knowledge of temperature measurements taken within the strip [8].

A regularization method for determining a moving boundary from Cauchy data in one-dimensional heat equation with a multilayer domain [9]. Estimation of the boundary thermal behavior of a furnace with two layer walls [10]. An inverse method, an input estimation method, to recursively estimates both the time varied heat flux and the inner wall temperature in the chamber [11]. Computation of the temperature field in multi-dimensional, multi-layer bodies [12]. There are many different methods for solving the inverse heat conduction problems. Some of these methods will be listed here. For instance, the exact solution technique, the inversion of Duhamel's integral, Laplace transformation techniques, the control volume method, the use of Helmholtz equation, the finite difference method, the finite element approaches, the digital filtering method, Tikhonov regularization method, Alifannov iterative regularization, the conjugate gradient method [13,18],etc.

Jiang et al. [14] obtained the time-dependent boundary heat flux applied on a solid bar by using the conjugate gradient method with adjoint equation and the zeroth-order Tikhonov regularization to stabilize the inverse solution. They used finite difference method to solve their problem. San guay chen et al. [19] calculate the heat flux and temperature distribution of the quenching surface with use of inverse method. They make use of conjugate gradient method to improve the estimation of the distribution of the surface temperature and heat flux for a 2D cylindrical coordinate problem and solve the governing equation using finite element method.

Liu [21] developed a hybrid method to identify simultaneously the fluid thermal conductivity and heat capacity for a transient inverse heat transfer problem. The proposed method is a combination of the modified genetic algorithm and Levenberg–Marquardt method. Tai et al. [22] investigated the workpiece temperature in minimum quantity lubrication (MQL) deep hole drilling. An inverse heat transfer method is developed to estimate the spatial and temporal change of heat flux on the drilled hole wall surfaces based on the workpiece temperature measured using embedded thermocouples and analyzed using the finite element method.

The inverse method was validated experimentally in both dry and MQL deep-hole drilling conditions and their results showed good agreement with the experimental temperature measurements. In the study done by Hong et al. [23], a test facility which can control the liquid level and the evaporation saturation pressure is developed to characterize the water evaporation/boiling on copper particle sintered porous wick under the conditions of capillary feeding at reduced pressures, the main factor determining the performance of a vapor chamber. The effective heat transfer coefficient is obtained using an inverse heat transfer method. Wu et al. [24] applied an inverse algorithm based on the conjugate gradient method and the discrepancy principle to solve the inverse hyperbolic heat conduction problem with the dual-phase-lag heat transfer model in estimating the unknown boundary pulse heat flux in an infinitely long solid cylinder from the temperature measurements taken within the medium.

The inverse solutions were justified based on the numerical experiments in which two different heat flux distributions are to be determined. Their results showed that an excellent estimation on the time-dependent pulse heat flux can be obtained for the test cases. J. Beck et al. [25] presented a study of the inverse heat Conduction Problem (IHCP) which is the estimation of the surface heat flux history of a heat conducting body. Transient temperature measurements inside the body are utilized in the calculation procedure.

Mohammadiun et al. [26] estimated the unknown heat flux using temperature at a point. They used a finite difference technique for solving the governing equations. Rahimi et al. [27] estimated the strength of the unknown heat source using temperature distribution at a point in a three layer

system. They used finite difference and finite volume methods to solve the governing equations. Mohammadiun et al. [28] developed and verified the sequential function specification method (SFSM) for the identification of heat flux at the receding surface of decomposing materials. They solved an inverse problem to anticipate the front-surface heating condition. Mohammadiun et al. [29] applied the sequential function specification method (SFSM) to the problem of estimating the effect of severe thermal environments on the ablative structures. They used an inverse method for obtaining heat flux in the moving interface.

In this research, I use the conjugate gradient method coupled with adjoint equation approach to solve the inverse heat conduction problem and estimation of the time-dependent heat flux using the temperature distribution at a point in a three layer system with none homogeneous boundary conditions. The problem is solved in axisymmetric case and the general coordinate method is used. The irregular region in the physical domain (r, z) is transformed into a rectangle in the computational domain (ξ, η). Z axis is the symmetric axis. By revolving the model around the z axis; we obtain the three-dimensional model (semi spherical crust). Applications of this model are in the thermal analysis of missile nose, thermal protect systems (t.p.s) and heat shield systems. The present formulation is general and can be applied to the solution of boundary inverse heat conduction problems over any region that can be mapped into a rectangle. The governing equations are solved by employing the finite difference method. The obtained results show that the applied method causes high stability even if the input data includes considerable noise.

2 PROBLEM FORMULATION AND SOLUTION

The geometry of this problem is presented in Fig.1. I aim to obtain the unknown heat flux $q(t)$ in the outer surface for the time $0 \leq t \leq t_f$ using the temperature field at a point. The input data could include noise. In the numerical solution the general coordinate method is applied. The calculations have been done in the rectangular coordinate system (ξ, η) initially, and then the results transfer to physical coordinate system (r - z). The computational plane and corresponding boundary conditions are shown in Fig. 2. The value of q_0 is $10^5 \frac{W}{m^2}$. The heat conduction equation in cylindrical coordinate system in the axisymmetric case with the initial and boundary conditions are as follows:

$$\frac{1}{r} \frac{\partial}{\partial r} (kr \frac{\partial T}{\partial r}) + \frac{\partial}{\partial z} (k \frac{\partial T}{\partial z}) = \rho C_p \frac{\partial T}{\partial t} \tag{1}$$

$$T_z = \frac{1}{J} (r_\eta T_\xi - r_\xi T_\eta) \tag{2}$$

$$T_r = \frac{1}{J} (-z_\eta T_\xi + z_\xi T_\eta) \tag{3}$$

$$\nabla^2 T = \frac{1}{J^2} [\alpha T_{\xi\xi} - 2\beta T_{\xi\eta} + \gamma T_{\eta\eta}] + \tag{4}$$

$$[(\nabla^2 \xi) T_\xi + (\nabla^2 \eta) T_\eta] \tag{5}$$

$$\alpha = z_\eta^2 + r_\eta^2 \tag{5}$$

$$\beta = z_\xi z_\eta + r_\xi r_\eta \tag{6}$$

$$\gamma = z_\xi^2 + r_\xi^2 \tag{7}$$

$$\nabla^2 \xi = \frac{k_1 (r_{\xi\xi} z_\eta - z_{\xi\xi} r_\eta) + k_2 (r_{\xi\eta} z_\eta - z_{\xi\eta} r_\eta)}{J} + \tag{8}$$

$$\frac{k_3 (r_{\eta\eta} z_\eta - z_{\eta\eta} r_\eta)}{J}$$

$$\nabla^2 \eta = \frac{k_1 (z_{\xi\xi} r_\xi - r_{\xi\xi} z_\xi) + k_2 (z_{\xi\eta} r_\xi - r_{\xi\eta} z_\xi)}{J} + \tag{9}$$

$$\frac{k_3 (z_{\eta\eta} r_\xi - r_{\eta\eta} z_\xi)}{J}$$

$$k_1 = \frac{1}{J^2} (z_\eta^2 + r_\eta^2) \tag{10}$$

$$k_2 = \frac{-2}{J^2} (z_\xi z_\eta + r_\xi r_\eta) \tag{11}$$

$$k_3 = \frac{1}{J^2} (z_\xi^2 + r_\xi^2) \tag{12}$$

$$\xi_z = \frac{1}{J} r_\eta \tag{13}$$

$$\xi_r = -\frac{1}{J} z_\eta \tag{14}$$

$$\eta_z = -\frac{1}{J} r_\xi \tag{15}$$

$$\eta_r = \frac{1}{J} z_\xi \tag{16}$$

$$J = z_\xi r_\eta - r_\xi z_\eta \tag{17}$$

Where the subscripts denote differentiation with respect to the variable considered.

$$\frac{\partial T}{\partial n} = 0 \quad R = R_i, t > 0 \tag{18}$$

$$K \frac{\partial T}{\partial n} = q(t) \quad R = R_o, t > 0 \tag{19}$$

$$\frac{\partial T}{\partial n} = \frac{q}{k} \quad \text{other boundary, } t > 0 \tag{20}$$

In the above relation, q is equal to zero on z axis and q_0 on r axis.

$$T(r, z, 0) = 300K \quad t = 0 \tag{21}$$

In the above relations $T, t, q(t), \rho, k, C_p, R_i, R_o, n$ are temperature, time, time-dependent heat flux, density, thermal conductivity, specified thermal capacity, inner

radius, outer radius and normal vector to the surface respectively.

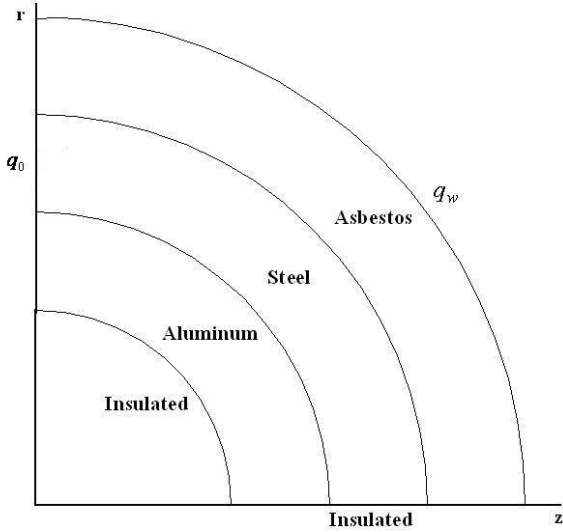


Fig. 1 Geometry of the problem and boundary conditions

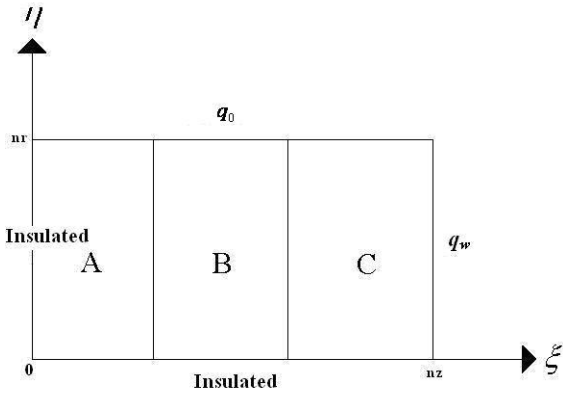


Fig. 2 Boundary conditions in computational plane

In the interface of materials, the following relations are used:

$$q_{\xi in} + q_{\eta in} = q_{\xi out} + q_{\eta out} \tag{22}$$

$$k_A(T_{i,j} - T_{i-1,j}) + \frac{2k_A k_B}{k_A + k_B}(T_{i,j} + T_{i-1,j}) = \tag{23}$$

$$k_B(T_{i+1,j} - T_{i,j}) + \frac{2k_A k_B}{k_A + k_B}(T_{i,j+1} + T_{i,j}) =$$

$$k_B(T_{i,j} - T_{i-1,j}) + \frac{2k_C k_B}{k_C + k_B}(T_{i,j} + T_{i-1,j}) = \tag{24}$$

$$k_C(T_{i+1,j} - T_{i,j}) + \frac{2k_C k_B}{k_C + k_B}(T_{i,j+1} + T_{i,j}) =$$

As it is shown in Fig. 3, by considering a boundary element in physical plane and applying the energy equation, the boundary conditions are calculated as follows:

$$k_{i,j} ds_1 \frac{T_{nz-1,j}^{n-1} - T_{nz,j}^{n-1}}{ds_2} + k_{i,j} \frac{ds_2}{2} \frac{T_{nz,j-1}^{n-1} - T_{nz,j}^{n-1}}{ds_1} + k_{i,j} \frac{ds_2}{2} \frac{T_{nz,j+1}^{n-1} - T_{nz,j}^{n-1}}{ds_1} + q_w ds_1 = \tag{25}$$

$$\rho_{i,j} C_{i,j} ds_1 \frac{ds_2}{2} \left(\frac{T_{nz,j}^n - T_{nz,j}^{n-1}}{\Delta t} \right)$$

Where $T_{nz,j}^n$ in the above relation is as:

$$T_{nz,j}^n = \frac{1}{F(T_{nz,j}^{n-1} + A1 + A2 + (2q_w \Delta t / \rho_{i,j} C_{i,j} ds_2))} \tag{26}$$

$$F = 1 + \frac{2\alpha_{i,j} \Delta t}{ds_{1nz,j}^2} + \frac{2\alpha_{i,j} \Delta t}{ds_{2nz,j}^2} \tag{27}$$

$$A_1 = \frac{2\alpha_{i,j} \Delta t T_{nz-1,j}^{n-1}}{ds_{2nz,j}^2} \tag{28}$$

$$A_2 = \frac{\alpha_{i,j} \Delta t (T_{nz,j+1}^{n-1} + T_{nz,j-1}^{n-1})}{ds_{1nz,j}^2} \tag{29}$$

$$\alpha_{i,j} = \frac{k_{i,j}}{C_{i,j}} \tag{30}$$

With similar method for other boundary conditions, we have:

$$T_{1,j}^n = \frac{1}{F(T_{1,j}^{n-1} + A1 + A2)} \tag{31}$$

$$F = 1 + \frac{2\alpha_{i,j} \Delta t}{ds_{11,j}^2} + \frac{2\alpha_{i,j} \Delta t}{ds_{21,j}^2} \tag{32}$$

$$A_1 = \frac{2\alpha_{i,j} \Delta t T_{2,j}^{n-1}}{ds_{21,j}^2} \tag{33}$$

$$A_2 = \frac{\alpha_{i,j} \Delta t (T_{1,j+1}^{n-1} + T_{1,j-1}^{n-1})}{ds_{11,j}^2} \tag{34}$$

$$\alpha_{i,j} = \frac{k_{i,j}}{C_{i,j}} \tag{35}$$

$$T_{i,1}^n = \frac{1}{F(T_{i,1}^{n-1} + A1 + A2)} \tag{36}$$

$$F = 1 + \frac{2\alpha_{i,j} \Delta t}{ds_{i,1}^2} + \frac{2\alpha_{i,j} \Delta t}{ds_{2i,1}^2} \tag{37}$$

$$A_1 = \frac{2\alpha_{i,j} \Delta t T_{i,2}^{n-1}}{ds_{i,1}^2} \tag{38}$$

$$A_2 = \frac{\alpha_{i,j} \Delta t (T_{i+1,1}^{n-1} + T_{i-1,1}^{n-1})}{ds_{2i,1}^2} \tag{39}$$

$$\alpha_{i,j} = \frac{k_{i,j}}{C_{i,j}} \quad (40)$$

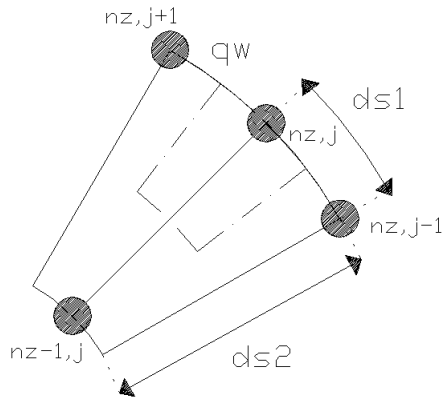


Fig. 3 Boundary element in physical plane

3 INVERSE PROBLEM

In inverse problem, the time dependent heat flux using measured transient temperatures is estimated with a sensor positioned at a point. The inverse problem should be solved as the following function is minimized [20]:

$$S[q(t)] = \frac{1}{2} \int_0^{t_f} \sum_{m=1}^{NS} [T(\xi_m, \eta_m, t; q) - Y_m(t)]^2 dt \quad (41)$$

In the above relation, $T(\xi_m, \eta_m, t; q)$, $Y_m(t)$ are the estimated temperatures and the measured temperature, respectively. Also, number of sensors NS is equal to 1. The above equation will be minimized using the conjugate gradient method based on iterative processes. In the conjugate algorithm, the direction of seeking the unknown heat flux depends on the gradient of the error function which will be solved with adjoint equation [15, 16]:

Adjoint problem

$$\frac{1}{r} \frac{\partial}{\partial r} (k r \frac{\partial \lambda}{\partial r}) + \frac{\partial}{\partial z} (k \frac{\partial \lambda}{\partial z}) + \quad (42)$$

$$\sum_{m=1}^{NS} [T(\xi, \eta, t; q) - Y_m(t)] \delta(\eta - \eta_m) \delta(\xi - \xi_m) = \rho C_p \frac{\partial \lambda}{\partial t}$$

$$\frac{\partial \lambda}{\partial n} = 0 \quad R = R_i, R = R_o, t > 0 \quad (43)$$

$$\text{other boundary, } t > 0 \quad (44)$$

$$\lambda(r, z, t_f) = 0 \quad t = t_f \quad (45)$$

In the interface of materials, the following relations are used:

$$k_A (\lambda_{i,j} - \lambda_{i-1,j}) + \frac{2k_A k_B}{k_A + k_B} (\lambda_{i,j} + \lambda_{i-1,j}) = \quad (46)$$

$$k_B (\lambda_{i+1,j} - \lambda_{i,j}) + \frac{2k_A k_B}{k_A + k_B} (\lambda_{i,j+1} + \lambda_{i,j})$$

$$k_B (\lambda_{i,j} - \lambda_{i-1,j}) + \frac{2k_C k_B}{k_C + k_B} (\lambda_{i,j} + \lambda_{i-1,j}) = \quad (47)$$

$$k_C (\lambda_{i+1,j} - \lambda_{i,j}) + \frac{2k_C k_B}{k_C + k_B} (\lambda_{i,j+1} + \lambda_{i,j})$$

By considering a boundary element in physical plane and applying the energy equation, the boundary conditions are calculated as follows:

$$k_{i,j} ds_1 \frac{\lambda_{nz-1,j}^{n-1} - \lambda_{nz,j}^{n-1}}{ds_2} + k_{i,j} \frac{ds_2}{2} \frac{\lambda_{nz,j-1}^{n-1} - \lambda_{nz,j}^{n-1}}{ds_1}$$

$$+ k_{i,j} \frac{ds_2}{2} \frac{\lambda_{nz,j+1}^{n-1} - \lambda_{nz,j}^{n-1}}{ds_1} = \quad (48)$$

$$\rho_{i,j} C_{i,j} ds_1 \frac{ds_2}{2} \left(\frac{\lambda_{nz,j}^n - \lambda_{nz,j}^{n-1}}{\Delta t} \right)$$

Where $\lambda_{nz,j}^n$ in the above relation is as:

$$\lambda_{nz,j}^n = \frac{1}{F(\lambda_{nz,j}^{n-1} + A1 + A2)} \quad (49)$$

$$F = 1 + \frac{2\alpha_{i,j} \Delta t}{ds_{1nz,j}^2} + \frac{2\alpha_{i,j} \Delta t}{ds_{2nz,j}^2} \quad (50)$$

$$A_1 = \frac{2\alpha_{i,j} \Delta t \lambda_{nz-1,j}^{n-1}}{ds_{2nz,j}^2} \quad (51)$$

$$A_2 = \frac{\alpha_{i,j} \Delta t (\lambda_{nz,j+1}^{n-1} + \lambda_{nz,j-1}^{n-1})}{ds_{1nz,j}^2} \quad (52)$$

$$\alpha_{i,j} = \frac{k_{i,j}}{C_{i,j}} \quad (53)$$

With similar method for other boundary conditions, we have:

$$\lambda_{1,j}^n = \frac{1}{F(\lambda_{1,j}^{n-1} + A1 + A2)} \quad (54)$$

$$F = 1 + \frac{2\alpha_{i,j} \Delta t}{ds_{11,j}^2} + \frac{2\alpha_{i,j} \Delta t}{ds_{21,j}^2} \quad (55)$$

$$A_1 = \frac{2\alpha_{i,j} \Delta t \lambda_{2,j}^{n-1}}{ds_{21,j}^2} \quad (56)$$

$$A_2 = \frac{\alpha_{i,j} \Delta t (\lambda_{1,j+1}^{n-1} + \lambda_{1,j-1}^{n-1})}{ds_{11,j}^2} \quad (57)$$

$$\alpha_{i,j} = \frac{k_{i,j}}{C_{i,j}} \tag{58}$$

$$\lambda_{i,1}^n = \frac{1}{F(\lambda_{i,1}^{n-1} + A1 + A2)} \tag{59}$$

$$F = 1 + \frac{2\alpha_{i,j}\Delta t}{ds_{1i,1}^2} + \frac{2\alpha_{i,j}\Delta t}{ds_{2i,1}^2} \tag{60}$$

$$A_1 = \frac{2\alpha_{i,j}\Delta t\lambda_{i,1}^{n-1}}{ds_{1i,1}^2} \tag{61}$$

$$A_2 = \frac{\alpha_{i,j}\Delta t(\lambda_{i+1,1}^{n-1} + \lambda_{i-1,1}^{n-1})}{ds_{2i,1}^2} \tag{62}$$

$$\alpha_{i,j} = \frac{k_{i,j}}{C_{i,j}} \tag{63}$$

Where the λ parameter is adjoint temperature and δ is dirac delta function. The optimum step size can be obtained based on the sensitivity problem which is defined as in Ref. [16]:

Sensitivity problem

To obtain the sensitivity equation, it is assumed that perturbing $q(t)$ by $\Delta q(t)$ would change $T(r, z, t)$ by $\Delta T(r, z, t)$. Thus, in direct problem, the quantities $[T(r, z, t) + \Delta T(r, z, t)]$ and $[q(t) + \Delta q(t)]$ are replaced by $T(r, z, t)$ and $q(t)$ and the resulting expression is subtracted from the direct problem. In this way, the sensitivity equation is obtained as:

$$\frac{1}{r} \frac{\partial}{\partial r} (kr \frac{\partial \Delta T}{\partial r}) + \frac{\partial}{\partial z} (k \frac{\partial \Delta T}{\partial z}) = \rho C_p \frac{\partial \Delta T}{\partial t} \tag{64}$$

$$\frac{\partial \Delta T}{\partial n} = 0 \quad R = R_i, t > 0 \tag{65}$$

$$K \frac{\partial \Delta T}{\partial n} = \Delta q \quad R = R_o, t > 0 \tag{66}$$

$$\frac{\partial \Delta T}{\partial n} = 0 \quad \text{other boundary, } t > 0 \tag{67}$$

$$\Delta T(r, z, 0) = 0 \quad t = 0 \tag{68}$$

In the interface of materials, the following relations are used:

$$k_A(\Delta T_{i,j} - \Delta T_{i-1,j}) + \frac{2k_A k_B}{k_A + k_B}(\Delta T_{i,j} + \Delta T_{i-1,j}) = \tag{69}$$

$$k_B(\Delta T_{i+1,j} - \Delta T_{i,j}) + \frac{2k_A k_B}{k_A + k_B}(\Delta T_{i,j+1} + \Delta T_{i,j})$$

$$k_B(\Delta T_{i,j} - \Delta T_{i-1,j}) + \frac{2k_C k_B}{k_C + k_B}(\Delta T_{i,j} + \Delta T_{i-1,j}) = \tag{70}$$

$$k_C(\Delta T_{i+1,j} - \Delta T_{i,j}) + \frac{2k_C k_B}{k_C + k_B}(\Delta T_{i,j+1} + \Delta T_{i,j})$$

As explained before, by considering a boundary element in physical plane and applying the energy balance relations, the boundary conditions are calculated as follows:

$$k_{i,j} ds_1 \frac{\Delta T_{nz-1,j}^{n-1} - \Delta T_{nz,j}^{n-1}}{ds_2} + k_{i,j} \frac{ds_2}{2} \frac{\Delta T_{nz,j-1}^{n-1} - \Delta T_{nz,j}^{n-1}}{ds_1} + k_{i,j} \frac{ds_2}{2} \frac{\Delta T_{nz,j+1}^{n-1} - \Delta T_{nz,j}^{n-1}}{ds_1} + \Delta q ds_1 = \rho_{i,j} C_{i,j} ds_1 \frac{ds_2}{2} \left(\frac{\Delta T_{nz,j}^{n-1} - \Delta T_{nz,j}^{n-1}}{\Delta t} \right) \tag{71}$$

From which $\Delta T_{nz,j}^n$ is calculated as below:

$$\Delta T_{nz,j}^n = \frac{1}{F(\Delta T_{nz,j}^{n-1} + A1 + A2 + (\frac{2\Delta q \Delta t}{\rho_{i,j} C_{i,j} ds_2}))} \tag{72}$$

$$F = 1 + \frac{2\alpha_{i,j}\Delta t}{ds_{1nz,j}^2} + \frac{2\alpha_{i,j}\Delta t}{ds_{2nz,j}^2} \tag{73}$$

$$A_1 = \frac{2\alpha_{i,j}\Delta t \Delta T_{nz-1,j}^{n-1}}{ds_{2nz,j}^2} \tag{74}$$

$$A_2 = \frac{\alpha_{i,j}\Delta t(\Delta T_{nz,j+1}^{n-1} + \Delta T_{nz,j-1}^{n-1})}{ds_{1nz,j}^2} \tag{75}$$

$$\alpha_{i,j} = \frac{k_{i,j}}{C_{i,j}} \tag{76}$$

Other boundary conditions are obtained in similar manner as:

$$\Delta T_{1,j}^n = \frac{1}{F(\Delta T_{1,j}^{n-1} + A1 + A2)} \tag{77}$$

$$F = 1 + \frac{2\alpha_{i,j}\Delta t}{ds_{11,j}^2} + \frac{2\alpha_{i,j}\Delta t}{ds_{21,j}^2} \tag{78}$$

$$A_1 = \frac{2\alpha_{i,j}\Delta t \Delta T_{2,j}^{n-1}}{ds_{21,j}^2} \tag{79}$$

$$A_2 = \frac{\alpha_{i,j}\Delta t(\Delta T_{1,j+1}^{n-1} + \Delta T_{1,j-1}^{n-1})}{ds_{11,j}^2} \tag{80}$$

$$\alpha_{i,j} = \frac{k_{i,j}}{C_{i,j}} \tag{81}$$

$$\Delta T_{i,1}^n = \frac{1}{F(\Delta T_{i,1}^{n-1} + A1 + A2)} \tag{82}$$

$$F = 1 + \frac{2\alpha_{i,j}\Delta t}{ds_{1i,1}^2} + \frac{2\alpha_{i,j}\Delta t}{ds_{2i,1}^2} \tag{83}$$

$$A_1 = \frac{2\alpha_{i,j} \Delta t \Delta T^{n-1}_{i,2}}{ds_{1i,1}^2} \quad (84)$$

$$A_2 = \frac{\alpha_{i,j} \Delta t (\Delta T^{n-1}_{i+1,1} + \Delta T^{n-1}_{i-1,1})}{ds_{2i,1}^2} \quad (85)$$

$$\alpha_{i,j} = \frac{k_{i,j}}{C_{i,j}} \quad (86)$$

Where ΔT is the sensitivity temperature. The transient heat flux $q(t)$ which is an unknown function can be estimated by minimizing the function $S[q(t)]$ in the equation (41). The iterative equation for estimating the $q(t)$ is as below [15, 17]:

$$q^{k+1}(t) = q^k(t) - \beta^k d^k(t) \quad (87)$$

In which k is the number of iteration. The direction of descent $d^k(t)$ is determined as in References [15, 17]:

$$d^k(t) = \nabla S[q^k(t)] + \gamma^k d^{k-1}(t) \quad (88)$$

Here, γ^k is the conjugate coefficient, as in references [16, 18] which are calculated by:

$$\gamma^k = \frac{\int_{t=0}^{t_f} \{ \nabla S[q^k(t)] \}^2 dt}{\int_{t=0}^{t_f} \{ \nabla S[q^{k-1}(t)] \}^2 dt} \quad (89)$$

Where γ^0 is assumed zero. To calculate $\nabla S[q^k(t)]$, the following relation is used:

$$\nabla S[q(t)] = \lambda(\xi, nr, t) \quad (90)$$

The above equality depends on the position of unknown function. The search step-size β^k , is obtained by minimizing $S[q^{k+1}(t)]$ with respect to β^k as follows: [16, 18]

$$\beta^k = \frac{\int_{t=0}^{t_f} \sum_{m=1}^{NS} [T(\xi_m, \eta_m, t; q^k) - Y_s(t)] \Delta T(\xi_m, \eta_m, t; d^k) dt}{\int_{t=0}^{t_f} \sum_{m=1}^{NS} [\Delta T(\xi_m, \eta_m, t; d^k)]^2 dt} \quad (91)$$

Where $\Delta T(\xi_m, \eta_m, t; d^k)$ is obtained from the sensitivity problem by considering $\Delta q^k(t) = d^k(t)$. By checking the equation (90), it is determined that the gradient equation in final time (t_f) is equal to zero, therefore the initial guess used for $q(t)$ in $t = t_f$ does not change with iterative process in conjugate gradient method. When the initial guess is very far from exact solution, the estimated function in the neighborhood of t_f can be deviated from the exact solution. This solution can be eliminated easily by use of a

larger value of final time. Thus the effect of initial guess on the actual time of the problem is not significant. The iterative procedure mentioned above, continues until the stopping criterion is satisfied. The stopping criterion is defined as follows:

$$S[q(t)] \leq \varepsilon \quad (92)$$

In the above relation, $S[q(t)]$ is obtained from equation (58). The value of ε should be selected such that, if there were errors in the measured data, the accuracy of the results would be satisfactory.

Computational algorithm

The computational procedure for obtaining the unknown heat flux can be summarized as follows [20]:

- 1- Choose an initial guess for example $q^0(t)$ for the function $q(t)$ and set $k = 0$.
- 2- Solve the direct problem to obtain $T(z, r, t)$ based on $q^k(t)$ (Eqs. 1-24).
- 3- Check the stopping criterion and continue if not satisfied (Eq.92).
- 4- Solve the adjoint equation and compute the $\lambda(\xi, nr, t)$ by knowing $T(\xi_m, \eta_m, t)$ and the measured temperature $Y_m(t)$ (Eqs. 42-47).
- 5- Knowing $\lambda(\xi, nr, t)$, compute $\nabla S[q^k(t)]$ from Eq. (90).
- 6- Knowing $\nabla S[q^k(t)]$, compute γ^k from Eq. (89) and $d^k(t)$ from Eq. (88).
- 7- Set $\Delta q^k(t) = d^k(t)$ and solve the sensitivity problem to obtain $\Delta T(\xi_m, \eta_m, t; d^k)$ (Eqs. 64-70).
- 8- Knowing $\Delta T(\xi_m, \eta_m, t; d^k)$, Compute β^k from Eq. (91).
- 9- Knowing β^k and $d^k(t)$, compute $q^{k+1}(t)$ and return to step 2. (Eq. 87).

4 RESULTS AND DISCUSSIONS

I aim to estimate the unknown heat flux in a three layer system using conjugate gradient method when there is no information about unknown function. It should be noted that in conjugated gradient method the initial guess for unknown function is arbitrary; in other word the method is independent of initial guess. Here, initial estimation for heat flux is assumed zero. The governing equations were discretized by the finite-difference method and the mesh size used in numerical is a uniform 35x35, 45x45, 55x55 (z-direction x r-direction, respectively), which all of them show that the problem is independent of mesh size, but by

noting the calculation time, I choose the 35x35 mesh size. The final time $t_f = 10$ and time step $\Delta t = 0.01$ are considered. In this work, by measuring the temperature at a point only, the heat flux is estimated and the sensitivity of the problem for a noisy data is investigated. In Figure 4, the mesh used and the position of sensor is shown. To investigate the accuracy of the presented solution, a step function is considered as:

$$q(t) = \begin{cases} 10^7 & \text{for } 4 < t < 8 \\ 0 & \text{for } t \leq 4 \text{ and } t \geq 8 \end{cases}$$

One should note that the discontinuous and sharp corner functions are well known for being highly ill-posed. Therefore, these functions can be used to evaluate the accuracy of the solutions. In the next example, a combination of sine and cosine functions is considered for the heat flux as:

$$q(t) = 10^7 \sin(0.1t) + 10^7 \cos(2t)$$

As the last example a triangle function is considered for the heat flux. In this part, the inverse solution with noisy data is presented. In practice, there are errors in measured data; therefore noisy data are used to simulate the errors and using a data with 3% noise.

The effect of noisy data can be seen in Figs. 8, 9 and 10 in comparison to noiseless cases (Figs. 5, 6 and 7). It is found that despite of a noise existing in data, results have very good stability. The mesh study has been done for $q(t) = 10^7 \sin(0.1t) + 10^7 \cos(2t)$ using three mesh sizes 35x35, 45x45, 55x55. As can be seen, the exact heat flux is recovered by the inverse solution using all of the mesh sizes, thus the results are independent of mesh size.

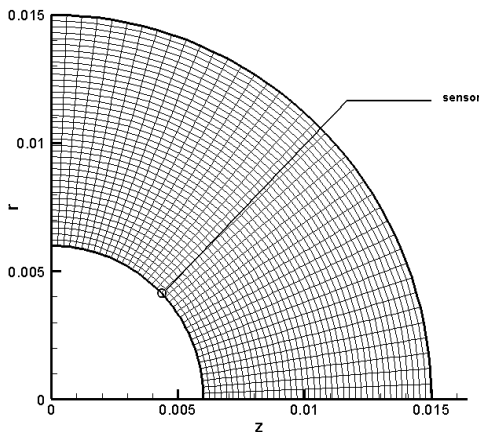


Fig. 4 Using grid in solution of problem and sensor position

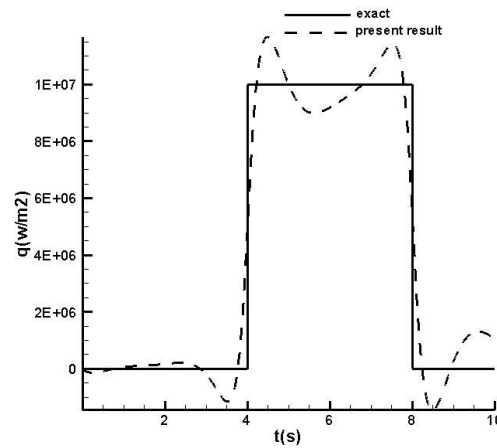


Fig. 5 Estimated heat flux in comparison with exact function for Step-function

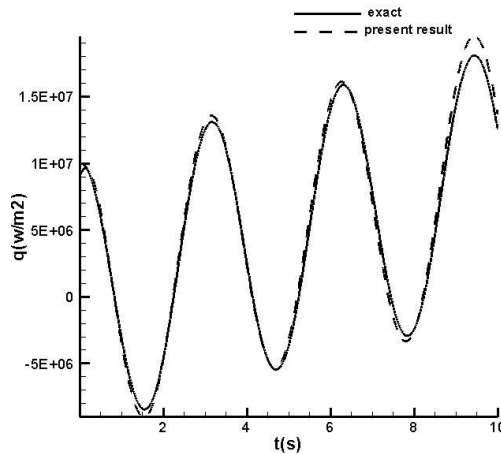


Fig. 6 Estimated heat flux in comparison with exact function for a combination of sine and cosine functions

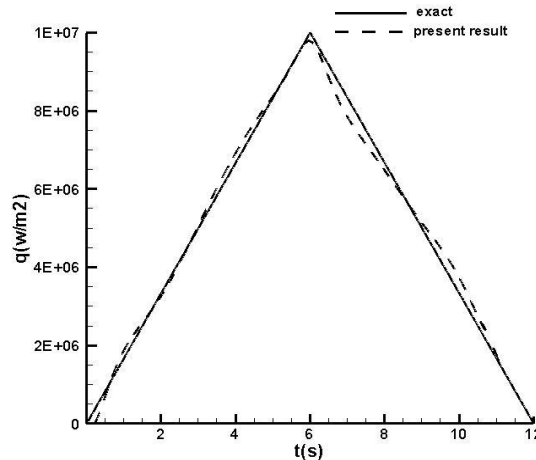


Fig. 7 Estimated heat flux in comparison with exact function for triangle function

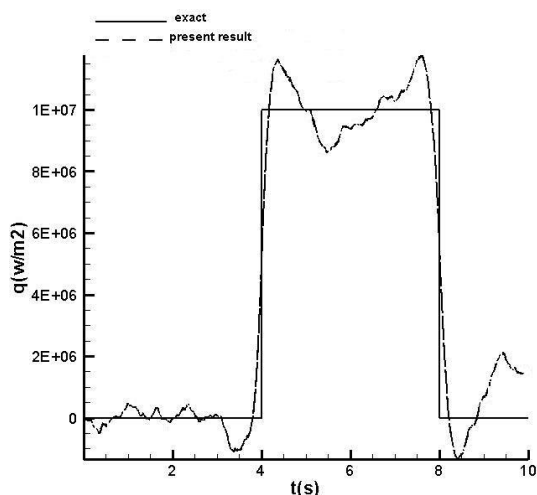


Fig. 8 Estimated heat flux with noisy data in comparison with Exact- function for step-function

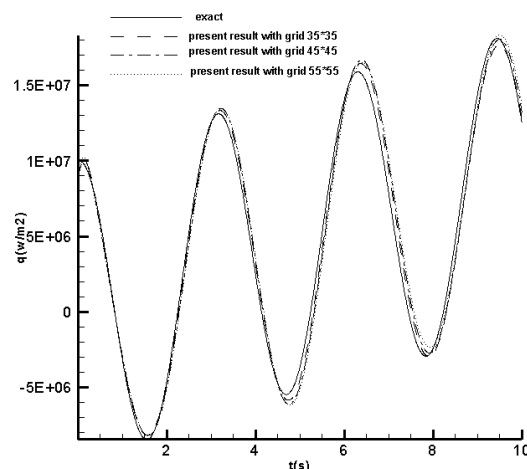


Fig. 11 Mesh size study on obtained results

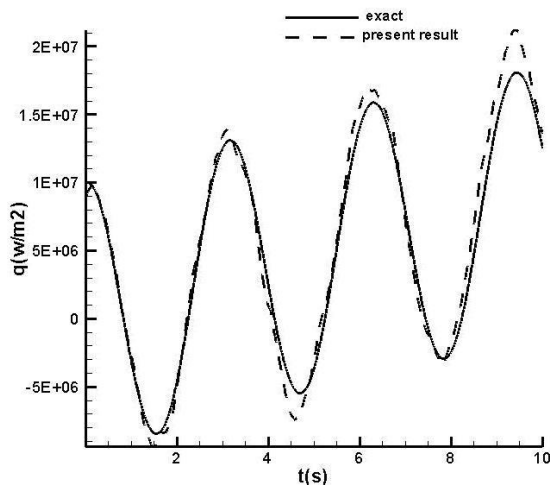


Fig. 9 Estimated heat flux with noisy data in comparison with exact function for a combination of sine and cosine functions

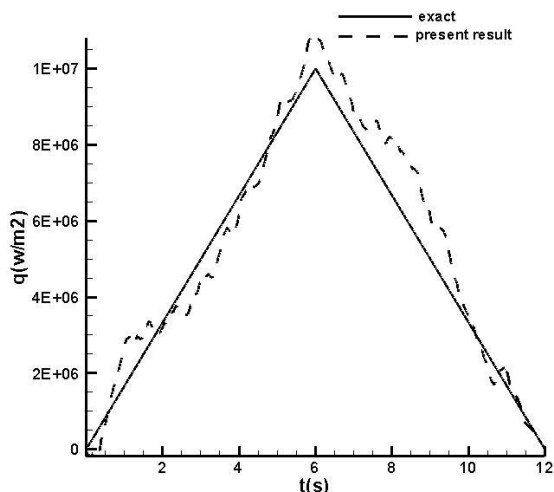


Fig. 10 Estimated heat flux with noisy data in comparison with exact function for triangle function

5 CONCLUSIONS

The conjugate gradient method with adjoint problem has been successfully applied for the solution of inverse heat conduction to estimate the unknown time-dependent heat flux using the temperature distribution at a point in a three layer system with none homogeneous boundary conditions and the general coordinate method is used. The present formulation is general and can be applied to the solution of boundary inverse heat conduction problems over any region that can be mapped into a rectangle. In this paper the discontinuous and sharp corner functions that are well known for being highly ill-posed were used for illustrating the good accuracy of the presented method. The obtained results show that the presented solution has good stability when there is a noise in input data up to 3%. Therefore the presented method is a good method for estimating the time-dependent unknown heat flux in multi-layer systems.

REFERENCES

- [1] Huang, C.H., Wang, P., "A three-dimensional inverse heat conduction problem in estimating surface heat flux by conjugate gradient method," International Journal of Heat and Mass Transfer, 1999, pp. 3387-3403.
- [2] Shigumori, E.H., Harter, F.P., Campos Velho, H.F., da Silva, J.D.S., "Estimation of boundary conditions in conduction heat transfer by neural networks," Tendências em Matemática Aplicada e Computacional, Vol. 3, No. 2, 2002, pp. 189-195.
- [3] Volle, F., Maillet, D., Gradeck, M., Kouachi, A., Lebouché, M., "Practical application of inverse heat conduction for wall condition estimation on a rotating cylinder," International Journal of Heat and Mass Transfer, Vol. 52, 2009, pp. 210-221.
- [4] Golbahar Haghighi, M.R., Eghtesad, M., Malekzadeh, P., Neculescu, D.S., "Three-dimensional inverse transient heat

- transfer analysis of thick functionally graded plates," *Energy Conversion and Management*, Vol. 50, 2009, pp. 450-457.
- [5] Su, J., Neto, A., "Two dimensional inverse heat conduction problem of source strength estimation in cylindrical rods," *Applied Mathematical Modeling*, Vol. 25, 2001, pp. 861- 872.
- [6] Hsu, P.T., "Estimating the boundary condition in a 3D inverse hyperbolic heat conduction problem," *Applied Mathematics and Computation*, Vol. 177, 2006, pp. 453-464.
- [7] Shi, J., Wang, J., "Inverse problem of estimating space and time dependent hot surface heat flux in transient transpiration cooling process," *International Journal of Thermal Sciences*, Vol. 48, 2009, pp. 1398-1404.
- [8] Yang, Y. C., Chu, S. S., Chang, W. J., Wu, T. S., "Estimation of heat flux and temperature distributions in a composite strip and homogeneous foundation," *International Communications in Heat and Mass Transfer*, Vol.37, 2010, pp. 495-500.
- [9] Wei, T., Li, Y.S., "An inverse boundary problem for one-dimensional heat equation with a multilayer domain," *Engineering Analysis with Boundary Elements*, Vol. 33, 2009, pp. 225-232.
- [10] Chen, C. K., Su, C. R., "Inverse estimation for temperatures of outer surface and geometry of inner surface of furnace with two layer walls," *Energy Conversion and Management*, Vol. 49, 2008, pp. 301-310.
- [11] Chen, T. C., Liu, C. C., Jang, H.Y., Tuan, P. C., "Inverse estimation of heat flux and temperature in multi-layer gun barrel," *International Journal of Heat and Mass Transfer*, Vol. 50, 2007, pp. 2060-2068.
- [12] Haji-Sheikh, A., Beck, J.V., "Temperature solution in multi-dimensional multi-layer bodies," *International Journal of Heat and Mass Transfer*, Vol. 45, 2002, pp. 1865-1877.
- [13] Ling, X., Atluril, S. N., "Stability analysis for inverse heat conduction problems," *Tech Science Press., CMES*, vol.13, No.3, 2006, pp. 219-228.
- [14] Jiang, B.H., Nguyen, T.H., Prud'homme, M., "Control of the boundary heat flux during the heating process of a solid material," *International Communications in Heat and Mass Transfer*, 2005, Vol. 32, pp. 728-738.
- [15] Jarny, Y., Ozisik, M.N., and Bardon, J.P., "A General Optimization Method Using Adjoint Equation for Solving Multidimensional Inverse Heat Conduction," *Journal of Heat and Mass Transfer*, Vol. 34, 1991, pp. 2911-2919.
- [16] Daniel, J.W., *Approximate Minimization of Functionals*, Prentice-Hall Inc. Englewood Cliffs, 1971.
- [17] Ozisik, M.N., *Heat Conduction*, 2nd ed., Wiley, New York, 1993.
- [18] Alifanov, O.M., *Inverse Heat Transfer Problems*, Springer-Verlag, New York, 1994.
- [19] Chen, S. G., Weng, C.I., Lin, J., "Inverse estimation of transient temperature distribution in the end quenching test," *Journal of Materials Processing Technology*, Vol. 86, 1999, pp. 257-263.
- [20] Ozisik, M. N., Orlando, R.B., *Inverse Heat Transfer*, New York, Taylor & Francis, 2000.
- [21] Liu, F. B., "A hybrid method for the inverse heat transfer of estimating fluid thermal conductivity and heat capacity," *International Journal of Thermal Sciences*, Vol. 50 (5), 2011, pp. 718-724.
- [22] Tai, B. L., Stephenson D. A., Shih, A. J., "An Inverse Heat Transfer Method for Determining Work piece Temperature in Minimum Quantity Lubrication Deep Hole Drilling," *Journal of Manufacturing Science and Engineering*, Vol.134 (2), 2012.
- [23] Hong, F.J., Cheng, P., Wu, H.Y., Sun, Z., "Evaporation/boiling heat transfer on capillary feed copper particle sintered porous wick at reduced pressure," *International Journal of Heat and Mass Transfer*, Vol. 63, 2013, pp. 389-400.
- [24] Wu, T. S., Lee, H. L., Chang, W. J., Yang, Y. C., "An inverse hyperbolic heat conduction problem in estimating pulse heat flux with a dual-phase-lag model," *International Communications in Heat and Mass Transfer*, Vol. 60, 2015, pp.1-8.
- [25] Beck, J., Blackwell, B., Clair, C. St., *Inverse Heat Conduction*, J. Wiley, New York, 1985.
- [26] Mohammadiun, M., Rahimi, A.B., Khazaei, I., "Estimation of the time-dependent heat flux using temperature distribution at a point by conjugate gradient method," *International Journal of Thermal Sciences*, Vol.50, 2011, pp. 2443-2450.
- [27] Rahimi, A.B., Mohammadiun, M., "Estimation of the Strength of the Time-dependent Heat Source Using Temperature Distribution at a Point in a Three Layer System," *International Journal of Engineering*, Vol. 25, 2012, pp. 343-351.
- [28] Mohammadiun, H., Molavi, H., Talesh Bahrami, H.R., Mohammadiun, M., "Real-Time Evaluation of Severe Heat Load Over Moving Interface of Decomposing Composites," *Journal of Heat Transfer*, Vol. 134, 2012, pp. 111202,1-111202-7.
- [29] Mohammadiun, M., Molavi, H., Talesh Bahrami, H.R., Mohammadiun, H., "Application of Sequential Function Specification Method in Heat Flux Monitoring of Receding Solid Surfaces," *Heat Transfer Engineering*, Vol. 35(10), 2014, pp. 933-941.