

Airy Equation with Memory Involvement Via Liouville Differential Operator

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Abstract In this work, a non-integer order Airy equation involving Liouville differential operator is considered. Proposing an undetermined integral solution to the left fractional Airy differential equation, we utilize some basic fractional calculus tools to clarify the closed form. A similar suggestion to the right FADE, converts it into an equation in the Laplace domain. An illustration to the approximation and asymptotic behavior of the integral solution to the left FADE with respect to the existing parameters is presented.

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1. Introduction

Airy equation, whether in its original form which is given by

$$D_x^2 y - xy = 0, \quad (1)$$

or in more general eigenvalue-form as

$$D_x^2 y - \lambda r(x)y = 0, \quad (2)$$

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has appeared in diverse areas including optics and quantum mechanics as well. The mathematical beauty of both mentioned forms comes from the existence of a turning point that merges two exponential and oscillatory branches of a solution. For a comprehensive reference, including the history, a full discussion of the solutions and physical applications see [12].

A modern appearance of Airy function is linked to the concept of the Airy beams. The (potential free) Schrödinger's equation

$$i \frac{\partial \psi}{\partial z} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} = 0, \quad (3)$$

which has a solution in terms of Airy function $\text{Ai}(x; z)$, can be converted via a coordinate transform into the Airy equation. The solution to the equation (3) formulates a non-diffracting waveform that is called Airy beams. This unusual non-diffracting property of the waves, has interested experts in atomic physics and optics [2, 3, 8]. However, for solutions to some nonlinear forms of (3) which satisfy a certain integral criterion, we encounter solitons [6].

There are different approaches pertaining to the solutions of (1.2). In [4], section 4, an asymptotic studies of solutions which are given by assuming an integral on a contour in the complex plane (which has to be known), is suggested. As a homotopy analysis based approach, [10] obtains a series which was comparable with asymptotic and approximate based solutions and satisfy the expected behaviour of the Airy function.

Since it is stated in [9] (and also discussed in section 2.1.3 [11]), non-local fractional calculus in its very nature, suggests the notion of the memory, in which the memory effect is given by the convolution kernel, that is $\delta(t - \tau)$ is replaced by $(t - \tau)^{-\alpha}$, where α is a positive non-integer. Due to this stickiness in time, which is interpreted as a memory effect, in dynamic equations, all the moments in a finite (Riemann-Liouville and Caputo FC) or infinite (Liouville FC) time domain must be taken into account, since the differentiation process via an integro-differential operator with the convolution kernel is non-localized.

There are a few contributions to study Airy DE in the local or non-local fractional case. In [1], using Laplace transform method, solutions to the Airy FDE in the Caputo sense of the form

$${}^C D_{0+}^{2\alpha} y - (\lambda + \mu x)y = 0 \quad (4)$$

is derived.

An eigenvalue value problem

$$\begin{aligned} D^\alpha y(x) - \lambda x y(x) &= 0, \quad 1 < \alpha < 2 \\ y(0) = A, y'(0) &= B, \quad -\infty < x, y < +\infty \end{aligned} \quad (5)$$

where D^α is the conformable differential operator which is a local differential operator, has been studied in [5] and this is the first local non-integer order study in the literature which involves a local derivative.

This work, studies the left and right FADE in the sense of Liouville of the form

$$\begin{cases} D^{2\alpha}\omega(\lambda) - \lambda^\alpha\omega(\lambda) = 0 \\ D_+^{2\alpha}\omega(\lambda) - \lambda^\alpha\omega(\lambda) = 0, \end{cases} \quad (6)$$

in which $\alpha, \lambda \in \mathbb{C}$ with $\Re(\alpha) \geq 0$ and $\Re(\lambda) > 0$. To the best knowledge of authors, there is no other contribution to the Airy equation which involves non-local fractional derivative. By suggesting an integral solution on a finite domain, and implementing a fractional integration by parts, an asymptotic behaviour of the derived solution (with respect to the main variable and the upper limit of the integral) will be illustrated. An interesting aspect of this method is to include arbitrary order α and it provides a mathematical view to study the solutions of the equations (6) while α grows rapidly.

2. Calculus in the Liouville Sense

We start off with some preliminary definitions:

Definition 2.1. ([7], Sec 2.3) Suppose $z \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. The left and right sided fractional integrals on the half-axis in the Liouville sense are defined (respectively) by

$$I_+^\alpha \psi(z) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^z \frac{\psi(t)}{(z-t)^{1-\alpha}} dt. \quad (7)$$

$$I_-^\alpha \psi(z) = \frac{1}{\Gamma(\alpha)} \int_z^\infty \frac{\psi(t)}{(t-z)^{1-\alpha}} dt. \quad (8)$$

Correspondingly, the left and right sided fractional derivative in the Liouville sense are defined (respectively) by

$$D_+^\alpha \psi(z) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dz}\right)^n \int_{-\infty}^z \frac{\psi(t)}{(z-t)^{\alpha-n+1}} dt. \quad (9)$$

$$D_-^\alpha \psi(z) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dz}\right)^n \int_z^\infty \frac{\psi(t)}{(t-z)^{\alpha-n+1}} dt. \quad (10)$$

where $\Re(\alpha) \geq 0$ and $n = [\Re(\alpha)] + 1$.

By [7], Property 2.5(c) and Property 2.11(b), if $\Re(\lambda) > 0$, $\Re(\alpha) \geq 0$, then

$$D_-^\alpha e^{-\lambda z} = \lambda^\alpha e^{-\lambda z} \quad (\alpha \geq 0), \quad (11)$$

$$D_+^\alpha e^{\lambda z} = \lambda^\alpha e^{\lambda z} \quad (\alpha \geq 0). \quad (12)$$

Suppose $\Omega \subset \mathbb{R}$ be a domain and let $I_+^\alpha(L^p(\Omega))$ ($I_-^\alpha(L^p(\Omega))$) denotes the set of all functions $f : \Omega \rightarrow \mathbb{R}$ for which, there exists $\phi \in L^p(\Omega)$ such that $f = I_+^\alpha \phi$ ($f = I_-^\alpha \phi$).

Fractional integration by parts (which will be utilized later) is given in Property 2.14 [7]:

If $\alpha > 0$, then the relation

$$\int_{-\infty}^{\infty} f(z)(D_+^\alpha g)(z)dz = \int_{-\infty}^{\infty} g(z)(D_-^\alpha f)(z)dz \quad (13)$$

holds for $f \in I_-^\alpha(L^p(\mathbb{R}))$ and $g \in I_+^\alpha(L^q(\mathbb{R}))$ provided that $p > 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$.

3. Main Result

For the initiation, consider the second equation of (6), which is

$$D_+^{2\alpha}\omega(\lambda) - \lambda^\alpha\omega(\lambda) = 0, \quad (14)$$

with $\mathcal{R}(\alpha) \geq 0$, $\mathcal{R}(\lambda) > 0$. Setting

$$\omega(\lambda) = \int_0^T \psi(z)e^{\lambda z} dz, \quad (15)$$

with $T > 0$ and fractional differentiating of order 2α , provided ψ vanishes outside $(0, T)$, gives rise to

$$\begin{aligned} \partial_+^{2\alpha}\omega(\lambda) &= \frac{1}{\Gamma([\mathcal{R}(2\alpha)] - 2\alpha + 1)} \left(\frac{\partial}{\partial \lambda}\right)^n \int_{-\infty}^{\lambda} \int_0^T \frac{\psi(z)e^{\mu z}}{(\lambda - \mu)^{2\alpha - [\mathcal{R}(2\alpha)] + 2}} dz d\mu \\ &= \int_0^T \frac{\psi(z)}{\Gamma([\mathcal{R}(2\alpha)] - 2\alpha + 1)} \left(\frac{\partial}{\partial \lambda}\right)^n \int_{-\infty}^{\lambda} \frac{e^{\mu z}}{(\lambda - \mu)^{2\alpha - [\mathcal{R}(2\alpha)] + 2}} d\mu dz \\ &= \int_0^T \psi(z) \partial_+^{2\alpha} e^{\lambda z} dz = \int_0^T z^{2\alpha} \psi(z) e^{\lambda z} dz. \end{aligned} \quad (16)$$

Taking $\psi(z)$ and $e^{\mu z}$ in (13) formally as f and g , we have:

$$\int_{-\infty}^{\infty} f(z) D_+^\alpha g(z) dz = \int_0^T \psi(z) \partial_+^\alpha e^{\lambda z} dz = \lambda^\alpha \omega(\lambda). \quad (17)$$

To satisfy the sufficient conditions to utilize fractional integration by parts, we should have $\psi \in I_-^\alpha(L^p(0, T))$, which is equivalent to the existence of $\phi \in L^p(0, T)$ with $I_-^\alpha(\phi) = \psi$ and it gives $\phi = D_-^\alpha \psi \in L^p(0, T)$. On the other hand, $e^{\lambda z}$ should satisfy $e^{\lambda z} \in I_+^\alpha(L^q(0, T))$, with $I_+^\alpha \phi_1 = e^{\lambda z}$ and it gives

$$\phi_1 = D_+^\alpha e^{\lambda z} = \lambda^\alpha e^{\lambda z} \in L^q(0, T) \quad (18)$$

Now, implementing (13) and using (16) together with (14), one obtains

$$\int_0^T e^{\lambda z} \left\{ z^{2\alpha} \psi(z) - D_-^\alpha \psi(z) \right\} dz = 0. \quad (19)$$

Assuming the continuity of $D_-^\alpha \psi(z)$, we infer that

$$D_-^\alpha \psi(z) = z^{2\alpha} \psi(z), \quad (20)$$

that is

$$\frac{1}{\Gamma(n - \alpha)} \left(-\frac{d}{dz}\right)^n \int_z^T \frac{\psi(t)}{(t - z)^{\alpha - n + 1}} dt = z^{2\alpha} \psi(z). \tag{21}$$

where $n = [\mathcal{R}(\alpha)] + 1$ and we have

$${}_z D_T^\alpha \psi(z) = z^{2\alpha} \psi(z). \tag{22}$$

where ${}_z D_T^\alpha$ is the Riemann-Liouville (right) differential operator. In favour of assuming $\psi \in I_-^\alpha(L^p(0, T))$ and from Lemma 2.6(a) [7], equation (22) can be converted into an integral equation:

$$\psi(z) = \frac{1}{\Gamma(\alpha)} \int_z^T \frac{t^{2\alpha} \psi(t)}{(t - z)^{1 - \alpha}} dt. \tag{23}$$

The approximate solution of equation (23) using Adomian decomposition method(ADM), with the help of *Mathematica* for $\alpha = \frac{2}{3}$, we find

$$\begin{aligned} \psi(z) \simeq & \frac{T^2}{2\Gamma(\frac{2}{3})} + \frac{Tx}{3\Gamma(\frac{2}{3})} - \frac{x^2}{2\Gamma(\frac{2}{3})} - \frac{\pi x^2}{9\sqrt{3}\Gamma(\frac{2}{3})} - \frac{2x^2 \log(x)}{9\Gamma(\frac{2}{3})} + \\ & \frac{14x^3}{81T\Gamma(\frac{2}{3})} {}_3F_2\left(1, 1, \frac{10}{3}; 2, 4; \frac{x}{T}\right) + \frac{2x^2 \log(T)}{9\Gamma(\frac{2}{3})} + \frac{x^2 \log(3)}{3\Gamma(\frac{2}{3})}, \end{aligned} \tag{24}$$

where ${}_3F_2(a; b; c)$ is the generalized hypergeometric function. Thus, for (14)

$$\begin{aligned} w(\lambda) \simeq & -\frac{1}{\lambda} + \frac{e^{\lambda T}}{\lambda} - \frac{3640}{729\lambda^6 T^3 \Gamma(\frac{2}{3})} + \frac{3640e^{\lambda T}}{729\lambda^6 T^3 \Gamma(\frac{2}{3})} + \frac{140}{81\lambda^5 T^2 \Gamma(\frac{2}{3})} - \frac{4900e^{\lambda T}}{729\lambda^5 T^2 \Gamma(\frac{2}{3})} - \\ & \frac{28}{27\lambda^4 T \Gamma(\frac{2}{3})} + \frac{3836e^{\lambda T}}{729\lambda^4 T \Gamma(\frac{2}{3})} + \frac{5}{3\lambda^3 \Gamma(\frac{2}{3})} - \frac{8165e^{\lambda T}}{2187\lambda^3 \Gamma(\frac{2}{3})} - \frac{4\gamma}{9\lambda^3 \Gamma(\frac{2}{3})} + \frac{2\pi}{9\sqrt{3}\lambda^3 \Gamma(\frac{2}{3})} - \\ & \frac{2\pi e^{\lambda T}}{9\sqrt{3}\lambda^3 \Gamma(\frac{2}{3})} + \frac{T}{3\lambda^2 \Gamma(\frac{2}{3})} + \frac{3677Te^{\lambda T}}{2187\lambda^2 \Gamma(\frac{2}{3})} + \frac{2\pi Te^{\lambda T}}{9\sqrt{3}\lambda^2 \Gamma(\frac{2}{3})} - \frac{T^2}{2\lambda \Gamma(\frac{2}{3})} + \frac{205T^2 e^{\lambda T}}{4374\lambda \Gamma(\frac{2}{3})} - \\ & - \frac{2G_{2,3}^{3,0}\left(-T\lambda \left| \begin{matrix} 1, 1 \\ 0, 0, 3 \end{matrix} \right.\right)}{9\lambda^3 \Gamma(\frac{2}{3})} + \frac{T^2 \log(3)e^{\lambda T}}{3\lambda \Gamma(\frac{2}{3})} - \frac{4 \log(\lambda(-T))}{9\lambda^3 \Gamma(\frac{2}{3})} + \frac{2T \log(3)e^{\lambda T}}{3\lambda^2 \Gamma(\frac{2}{3})}. \end{aligned} \tag{25}$$

where $G_{m,n}^{p,q}\left(z \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right.\right)$ is the Meijer G function and γ is Euler's constant.

Assuming $\lambda \in \mathbb{R}^+$, the plot in Fig.1, shows the approximate solution expressed by (25) when $\alpha = \frac{2}{3}$. The plot in Fig.2 shows the asymptotic behavior of (25), when $\lambda \rightarrow \infty$ and T is fixed. The plot in Fig.3 shows the asymptotic behavior of (25), when $T \rightarrow \infty$ and λ is fixed.

For the first equation of (6), let

$$\omega(\lambda) = \int_0^\infty \psi(z) e^{-\lambda z} dz. \tag{26}$$

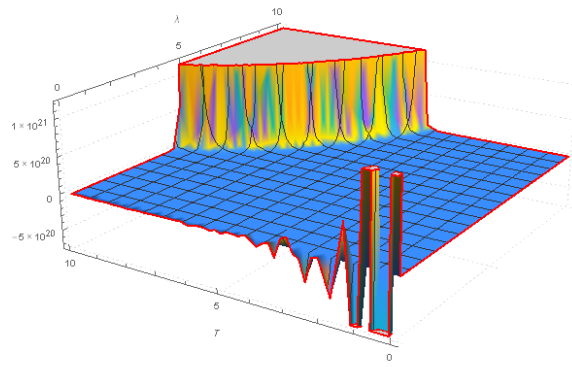


Figure 1. The approximate solution in the case $\alpha = \frac{2}{3}$ for Eq. (25).

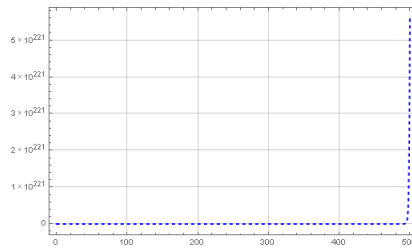


Figure 2. The asymptotic behavior of (25), when $\lambda \rightarrow \infty$ and T is fixed.

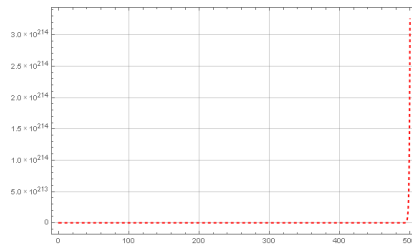


Figure 3. The asymptotic behavior of (25), when $T \rightarrow \infty$ and λ is fixed.

with the assumption $\psi \equiv 0$ outside the half axis $[0, \infty)$. Then the equation reads

$$D_-^{2\alpha} \hat{\psi}(\lambda) - \lambda^\alpha \hat{\psi}(\lambda) = 0. \tag{27}$$

which is the same equation in the Laplace domain. Similar to (16)-(22), the following integral equation is obtained

$$\psi(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{t^{2\alpha} \psi(t)}{(z-t)^{1-\alpha}} dt, \tag{28}$$

provided $\psi \in I_-(L^p(0, \infty))$.

4. Conclusion

In this paper, using fractional integration by parts in the Liouville sense, the left and right Airy fractional differential equations have been studied and an approximation of the integral solution to the left FADE has been illustrated.

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