Interval valued fuzzy weak bi-ideals of $\Gamma$-near-rings

V. Chinnadurai$^{a,*}$, K. Arulmozhi$^{a}$, S. Kadalarasi$^{a}$

$^a$Department of Mathematics, Annamalai University, Annamalainagar-608 002, India.

Received 11 May 2017; Revised 25 September 2017; Accepted 1 November 2017.

Abstract. In this paper, we introduce the concept of interval valued fuzzy weak bi-ideals of $\Gamma$-near-rings, which is a generalized concept of fuzzy weak bi-ideals of $\Gamma$-near-rings. We also characterize some properties and provide examples of interval valued fuzzy weak bi-ideals of $\Gamma$-near-rings.

$\copyright$ 2017 IAUCTB. All rights reserved.

Keywords: $\Gamma$-near-rings, fuzzy weak bi-ideals, interval valued fuzzy weak bi-ideals, homomorphism and anti-homomorphism.

2010 AMS Subject Classification: 16Y30, 03E72, 08A72.

1. Introduction


*Corresponding author.
E-mail address: kv.chinnadurai@yahoo.com (V. Chinnadurai); arulmozhie@gmail.com (K. Arulmozhi); kadalarasi89@gmail.com (S. Kadalarasi).
In this paper, we define a new notion of an interval valued fuzzy weak bi-ideals of \( \Gamma \)-near-rings, which is a generalized concept of interval valued fuzzy bi-ideals of \( \Gamma \)-near-rings. We also investigate some of its properties and illustrate with examples.

2. Preliminaries

In this section, we list some basic definitions.

Definition 2.1 [14] A near-ring is an algebraic system \((R, +, \cdot)\) consisting of a non empty set \(R\) together with two binary operations called + and \(\cdot\) such that \((R, +)\) is a group not necessarily abelian and \((R, \cdot)\) is a semigroup connected by the following distributive law: \((x + z) \cdot y = x \cdot y + z \cdot y\) valid for all \(x, y, z \in R\). We use the word \('near-ring \)' to mean \('right near-ring \)' . We denote \(xy\) instead of \(x \cdot y\).

Definition 2.2 [15] A \(\Gamma\)-near-ring is a triple \((M, +, \Gamma)\) where

(i) \((M, +)\) is a group,

(ii) \(\Gamma\) is a nonempty set of binary operators on \(M\) such that for each \(\alpha \in \Gamma\),

\((M, +, \alpha)\) is a near-ring,

(iii) \(x\alpha(y\beta z) = (x\alpha y)\beta z\) for all \(x, y, z \in M\) and \(\alpha, \beta \in \Gamma\).

Definition 2.3 [12] A \(\Gamma\)-near-ring \(M\) is said to be zero-symmetric if \(x\alpha 0 = 0\) for all \(x \in M\) and \(\alpha \in \Gamma\).

Throughout this paper \(M\) denotes a zero-symmetric right \(\Gamma\)-near-ring with atleast two elements.

Definition 2.4 [15] A subset \(A\) of a \(\Gamma\)-near-ring \(M\) is called a left (resp. right) ideal of \(M\) if

(i) \((A, +)\) is a normal subgroup of \((M, +)\) (i.e \(x - y \in A\) for all \(x, y \in A\) and \(y + x - y \in A\) for \(x \in A, y \in M\)),

(ii) \(u\alpha(x + v) - u\alpha v \in A\) (resp. \(x\alpha u \in A\)) for all \(x \in A, \alpha \in \Gamma\) and \(u, v \in M\).

Definition 2.5 [15] Let \(M\) be a \(\Gamma\)-near-ring. Given two subsets \(A\) and \(B\) of \(M\), we define \(A\Gamma B = \{aab \mid a \in A, b \in B\}\) and also define another operation \(\ast\) on the class of subset of \(M\) define by \(A\Gamma \ast B = \{a\gamma(a' + b) - a\gamma a' \mid a, a' \in A, \gamma \in \Gamma, b \in B\}\).

Definition 2.6 [16] A subgroup \(B\) of \((M, +)\) is called a bi-ideal of \(M\) if and only if \(B\Gamma M \Gamma B \subseteq B\).

Definition 2.7 [5] A subgroup \(H\) of \((M, +)\) is said to be a weak bi-ideal of \(M\) if \(H\Gamma H \subseteq H\).

The characteristic function of \(M\) is denoted by \(M\).

Definition 2.8 [22] If \(X\) be any set. A mapping \(\eta : X \to D[0, 1]\) is called an interval valued fuzzy subset (briefly, an i.v fuzzy subset) of \(X\), where \(D[0, 1]\) denotes the family of closed subintervals of \([0, 1]\) and \(\eta(x) = [\eta^-(x), \eta^+(x)]\) for all \(x \in X\), where \(\eta^-(x)\) and \(\eta^+(x)\) are fuzzy subsets of \(X\) such that \(\eta^-(x) \leq \eta^+(x)\) for all \(x \in X\).

Definition 2.9 [17] By an interval number \(\tilde{a}\), we mean an interval \([a^-, a^+]\) such that \(0 \leq a^- \leq a^+ \leq 1\) and where \(a^-\) and \(a^+\) are the lower and upper limits of \(\tilde{a}\) respectively. The set of all closed subintervals of \([0, 1]\) is denoted by \(D[0, 1]\). We also identify the interval \([a, a]\) by the number \(a \in [0, 1]\). For any interval numbers \(\tilde{a}_j = [a^+_j, a^-_j], \tilde{b}_j = [b^-_j, b^+_j] \in D[0, 1], j \in \Omega\) we define
Definition 2.10 [17] Let \( \tilde{\eta} \) be an i.v fuzzy subset of \( X \) and \( [t_1, t_2] \in D[0,1] \). Then the set \( \tilde{U}(\tilde{\eta} : [t_1, t_2]) = \{ x \in X | \tilde{\eta}(x) \geq [t_1, t_2] \} \) is called the upper level subset of \( \tilde{\eta} \).

Definition 2.11 [7, 13, 19] If \( \tilde{\eta} \) and \( \tilde{\lambda} \) are i.v fuzzy subsets of \( M \). Then \( \tilde{\eta} \cap \tilde{\lambda}, \tilde{\eta} \cup \tilde{\lambda}, \tilde{\eta} + \tilde{\lambda}, \) and \( \tilde{\eta} \ast \tilde{\lambda} \) are fuzzy subsets of \( M \) defined by:

\[
(\tilde{\eta} \cap \tilde{\lambda})(x) = \min^i\{\tilde{\eta}(x), \tilde{\lambda}(x)\}.
\]

\[
(\tilde{\eta} \cup \tilde{\lambda})(x) = \max^i\{\tilde{\eta}(x), \tilde{\lambda}(x)\}.
\]

\[
(\tilde{\eta} + \tilde{\lambda})(x) = \begin{cases} 
\sup^i_{x=y+z}\{\min^i\{\tilde{\eta}(y), \tilde{\lambda}(z)\}\} & \text{if } x \text{ is expressible as } x = y + z \\
0 & \text{otherwise}.
\end{cases}
\]

\[
(\tilde{\eta} \ast \tilde{\lambda})(x) = \begin{cases} 
\sup^i_{x=ya+z}\{\min^i\{\tilde{\eta}(y), \tilde{\lambda}(z)\}\} & \text{if } x \text{ is expressible as } x = ya + z \\
0 & \text{otherwise}.
\end{cases}
\]

for \( x, y, z \in M \).

Definition 2.12 [5] An i.v fuzzy subset \( \tilde{\eta} \) in a \( \Gamma \)-near-ring \( M \) is called an i.v fuzzy left (resp. right) ideal of \( M \) if

(i) \( \tilde{\eta} \) is an i.v fuzzy normal divisor with respect to the addition,

(ii) \( \tilde{\eta}(u\alpha(x + v) - u\alpha v) \geq \tilde{\eta}(x) \) (resp. \( \tilde{\eta}(x\alpha u) \geq \tilde{\eta}(x) \)) for all \( x, u, v \in M \) and \( \alpha \in \Gamma \).

The condition (i) of definition 2.12 means that \( \tilde{\eta} \) satisfies:

(i) \( \tilde{\eta}(x - y) \geq \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\} \),

(ii) \( \tilde{\eta}(y + x - y) \geq \tilde{\eta}(x) \), for all \( x, y \in M \)

Note that \( \tilde{\eta} \) is an i.v fuzzy left (resp. right) ideal of \( \Gamma \)-near-ring \( M \), then \( \tilde{\eta}(0) \geq \tilde{\eta}(x) \) for all \( x \in M \), where 0 is the zero element of \( M \).

Definition 2.13 [6] An i.v fuzzy subset \( \tilde{\eta} \) of \( M \) is called an i.v fuzzy bi-ideal of \( \Gamma \)-near-ring \( M \) if

(i) \( \tilde{\eta}(x - y) \geq \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\} \) for all \( x, y \in M \)

(ii) \( \tilde{\eta}(x\beta yz) \geq \min^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\} \) for all \( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \).

3. Interval valued fuzzy weak bi-ideals of \( \Gamma \)-near-rings

In this section, we introduce the notion of i.v fuzzy weak bi-ideal of \( M \) and discuss some of its properties.

Definition 3.1 An i.v fuzzy set \( \tilde{\eta} \) of \( M \) is called an i.v fuzzy weak bi-ideal of \( M \), if

(i) \( \tilde{\eta}(x - y) \geq \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\} \) for all \( x, y \in M \)

(ii) \( \tilde{\eta}(x\alpha y\beta z) \geq \min^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\} \) for all \( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \).

Example 3.2 Let \( M = \{0, a, b, c\} \) be a non-empty set with binary operation + and \( \Gamma = \{\alpha, \beta\} \) be a non-empty set of binary operations as shown in the following tables:
Let \( \tilde{\eta} : M \to D \) be an i.v fuzzy subset defined by \( \tilde{\eta}(0) = [0.8, 0.9], \tilde{\eta}(a) = [0.6, 0.7], \tilde{\eta}(b) = \tilde{\eta}(c) = [0.2, 0.3] \). Then \( \tilde{\eta} \) is an i.v fuzzy weak bi-ideal of \( M \).

**Theorem 3.3** Let \( \tilde{\eta} \) be an i.v fuzzy subgroup of \( M \). Then \( \tilde{\eta} \) is an i.v fuzzy weak bi-ideal of \( M \) if and only if \( \tilde{\eta} \ast \tilde{\eta} \ast \tilde{\eta} \subseteq \tilde{\eta} \).

**Proof.** Assume that \( \tilde{\eta} \) is an i.v fuzzy weak bi-ideal of \( M \). Let \( x, y, z, y_1, y_2 \in M \) and \( \alpha, \beta \in \Gamma \) such that \( x = y_1z \) and \( y = y_1\beta y_2 \). Then

\[
(\tilde{\eta} \ast \tilde{\eta} \ast \tilde{\eta})(x) = \sup_{x=y_1z} \{ \min \{ (\tilde{\eta} \ast \tilde{\eta})(y), \tilde{\eta}(z) \} \}
\]

\[
= \sup_{x=y_1z} \{ \min \{ \sup_{y=y_1, y_2} \{ \tilde{\eta}(y_1), \tilde{\eta}(y_2) \}, \tilde{\eta}(z) \} \}
\]

\[
= \sup_{x=y_1z} \sup_{y=y_1, y_2} \{ \min \{ \tilde{\eta}(y_1), \tilde{\eta}(y_2) \}, \tilde{\eta}(z) \} \}
\]

(since \( \tilde{\eta} \) is an i.v fuzzy weak bi-ideal of \( M \), \( \tilde{\eta}(y_1 \beta y_2 \alpha z) \geq \min \{ \tilde{\eta}(y_1), \tilde{\eta}(y_2), \tilde{\eta}(z) \} \))

\[
\leq \sup_{x=y_1z} \tilde{\eta}(y_1 \beta y_2 \alpha z)
\]

\[
= \tilde{\eta}(x).
\]

If \( x \) can not be expressed as \( x = y_1z \), then \( (\tilde{\eta} \ast \tilde{\eta} \ast \tilde{\eta})(x) = \tilde{\eta} \leq \tilde{\eta}(x) \). In both cases \( \tilde{\eta} \ast \tilde{\eta} \ast \tilde{\eta} \subseteq \tilde{\eta} \). Conversely, assume that \( \tilde{\eta} \ast \tilde{\eta} \ast \tilde{\eta} \subseteq \tilde{\eta} \). For \( x', x, y, z \in M \) and \( \alpha, \beta, \alpha_1, \beta_1 \in \Gamma \). Let \( x' \) be such that \( x' = x \alpha y \beta z \). Then

\[
\tilde{\eta}(x \alpha y \beta z) = \tilde{\eta}(x') \geq (\tilde{\eta} \ast \tilde{\eta} \ast \tilde{\eta})(x')
\]

\[
= \sup_{x'=p \alpha z} \{ \min_i \{ (\tilde{\eta} \ast \tilde{\eta})(p), \tilde{\eta}(q) \} \}
\]

\[
= \sup_{x'=p \alpha z} \{ \min_i \{ \sup_{p=p_1, p_2} \{ \tilde{\eta}(p_1), \tilde{\eta}(p_2) \}, \tilde{\eta}(q) \} \}
\]

\[
= \sup_{x'=p_1 \beta p_2 \alpha \alpha_1 q} \{ \min_i \{ \tilde{\eta}(p_1), \tilde{\eta}(p_2), \tilde{\eta}(q) \} \}
\]

\[
\geq \min \{ \tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z) \}.
\]

Hence \( \tilde{\eta}(x \alpha y \beta z) \geq \min \{ \tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z) \} \). \hfill \Box

**Lemma 3.4** Let \( \tilde{\eta} \) and \( \tilde{\lambda} \) be an i.v fuzzy weak bi-ideals of \( M \). Then the products \( \tilde{\eta} \ast \tilde{\lambda} \) and \( \tilde{\lambda} \ast \tilde{\eta} \) are also an i.v fuzzy weak bi-ideals of \( M \).
Theorem 3.6  
Every i.v fuzzy bi-ideal of $M$ and let $\alpha, \alpha_1, \alpha_2 \in \Gamma$. Now

$$(\bar{\eta} * \bar{\lambda})(x - y) = \sup_{x - y = ab} \min \{\bar{\eta}(a), \bar{\lambda}(b)\}$$

$${\geq} \sup_{x = a, b - b < (a_1 - a_2) (b_1 - b_2)} \min \{\bar{\eta}(a_1 - a_2), \bar{\lambda}(b_1 - b_2)\}$$

$${\geq} \sup \min \{\bar{\eta}(a), \bar{\lambda}(b)\}, \min \{\bar{\lambda}(b_1), \bar{\lambda}(b_2)\}$$

$${=} \sup \min \{\bar{\eta}(a), \bar{\lambda}(b)\}, \min \{\bar{\lambda}(a_2), \bar{\lambda}(b_2)\}$$

$${=} \min \{\bar{\eta}(a_1), \bar{\lambda}(b_1)\}, \sup \min \{\bar{\eta}(a_2), \bar{\lambda}(b_2)\}$$

$${=} \min \{\bar{\eta}(a), \bar{\lambda}(b)\}.$$ 

It follows that $\bar{\eta} * \bar{\lambda}$ is an i.v fuzzy subgroup of $M$. Further,

$$(\bar{\eta} * \bar{\lambda}) * (\bar{\eta} * \bar{\lambda}) * (\bar{\eta} * \bar{\lambda}) = \bar{\eta} * \bar{\lambda} * (\bar{\lambda} * \bar{\lambda} * \bar{\eta}) * \bar{\lambda}$$

$$\subset \bar{\eta} * \bar{\lambda} * (\bar{\lambda} * \bar{\lambda} * \bar{\eta}) * \bar{\lambda}, \text{since } \bar{\lambda} \text{ is an i.v fuzzy weak bi-ideal of } M$$

$$\subset \bar{\eta} * (\bar{\lambda} * \bar{\lambda} * \bar{\lambda}), \text{since } \bar{\lambda} \text{ is an i.v fuzzy weak bi-ideal of } M$$

$$\subset \bar{\eta} * \bar{\lambda}.$$

Therefore $\bar{\eta} * \bar{\lambda}$ is an i.v fuzzy weak bi-ideal of $M$. Similarly $\bar{\lambda} * \bar{\eta}$ is an i.v fuzzy weak bi-ideal of $M$.

Lemma 3.5 Every i.v fuzzy ideal of $M$ is an i.v fuzzy bi-ideal of $M$.

Proof. Let $\bar{\eta}$ be an i.v fuzzy ideal of $M$. Then

$$\bar{\eta} * M * \bar{\eta} \subset \bar{\eta} * M * \bar{\eta} \subset \bar{\eta} * M \subset \bar{\eta}$$

since $\bar{\eta}$ is an i.v fuzzy ideal of $M$. This implies that $\bar{\eta} * M * \bar{\eta} \subset \bar{\eta}$. Therefore $\bar{\eta}$ is an i.v fuzzy bi-ideal of $M$.

Theorem 3.6 Every i.v fuzzy bi-ideal of $M$ is an i.v fuzzy weak bi-ideal of $M$.

Proof. Assume that $\bar{\eta}$ is an i.v fuzzy bi-ideal of $M$. Then $\bar{\eta} * M * \bar{\eta} \subset \bar{\eta}$. We have $\bar{\eta} * \bar{\eta} * \bar{\eta} \subset \bar{\eta} * M * \bar{\eta}$. This implies that $\bar{\eta} * \bar{\eta} * \bar{\eta} \subset \bar{\eta} * M * \bar{\eta} \subset \bar{\eta}$. Therefore $\bar{\eta}$ is an i.v fuzzy weak bi-ideal of $M$.

Theorem 3.7 Every i.v fuzzy ideal of $M$ is an i.v fuzzy weak bi-ideal of $M$.

Proof. By Lemma 3.5, every i.v fuzzy ideal of $M$ is an i.v fuzzy bi-ideal of $M$. By Theorem 3.6, every i.v fuzzy bi-ideal of $M$ is an i.v fuzzy weak bi-ideal of $M$. Theorefore $\bar{\eta}$ is an i.v fuzzy weak bi-ideal of $M$.

However the converse of the Theorems 3.6 and 3.7 is not true in general which is demonstrated by the following example.

Example 3.8 Let $M = \{0, a, b, c\}$ be a non-empty set with binary operation $+$ and $\Gamma = \{\alpha, \beta\}$ be a nonempty set of binary operations as shown in the following tables:
Proof. Let \( \tilde{\eta} : M \to D[0,1] \) be an i.v fuzzy subset defined by \( \tilde{\eta}(0) = [0.7, 0.8], \tilde{\eta}(a) = [0.3, 0.4], \tilde{\eta}(b) = [0.5, 0.6] \). Then \( \tilde{\eta} \) is an i.v fuzzy weak bi-ideal of \( M \). But \( \tilde{\eta} \) is not an i.v fuzzy ideal and bi-ideal of \( M \), since \( \tilde{\eta}(bac) = \tilde{\eta}(b) = [0.3, 0.4] \leq [0.5, 0.6] = \tilde{\eta}(c), \tilde{\eta}(b\beta c) = \tilde{\eta}(b) = [0.3, 0.4] \leq [0.5, 0.6] = \tilde{\eta}(c) \) and \( \tilde{\eta}(c\alpha \beta c) = \tilde{\eta}(b) = [0.3, 0.4] \leq [0.5, 0.6] = \min\{\tilde{\eta}(c), \tilde{\eta}(c)\} \).

Theorem 3.9 Let \( \{\tilde{\eta}_i|i \in \Omega\} \) be family of i.v fuzzy weak bi-ideals of a \( \Gamma \)- near-ring \( M \), then \( \bigcap_{i \in \Omega} \tilde{\eta}_i \) is also an i.v fuzzy weak bi-ideal of \( M \), where \( \Omega \) is any index set.

Proof. Let \( \{\tilde{\eta}_i|i \in \Omega\} \) be a family of i.v fuzzy weak bi-ideals of \( M \). Let \( x, y, z \in M \), \( \alpha, \beta \in \Gamma \) and \( \tilde{\eta} = \bigcap_{i \in \Omega} \tilde{\eta}_i \). Then, \( \tilde{\eta}(x) = \bigcap_{i \in \Omega} \tilde{\eta}_i(x) = (\inf_{i \in \Omega} \tilde{\eta}_i)(x) = \inf_{i \in \Omega} \tilde{\eta}_i(x) \). Now,

\[
\tilde{\eta}(x - y) = \inf_{i \in \Omega} \tilde{\eta}_i(x - y) \\
\geq \inf_{i \in \Omega} \min_i \{\tilde{\eta}_i(x), \tilde{\eta}_i(y)\} \\
= \min_i \left\{\inf_{i \in \Omega} \tilde{\eta}_i(x), \inf_{i \in \Omega} \tilde{\eta}_i(y)\right\} \\
= \min_i \left\{\bigcap_{i \in \Omega} \tilde{\eta}_i(x), \bigcap_{i \in \Omega} \tilde{\eta}_i(y)\right\} \\
= \min_i \{\tilde{\eta}(x), \tilde{\eta}(y)\},
\]

and

\[
\tilde{\eta}(x\alpha y\beta z) = \inf_{i \in \Omega} \tilde{\eta}_i(x\alpha y\beta z) \\
\geq \inf_{i \in \Omega} \min_i \{\tilde{\eta}_i(x), \tilde{\eta}_i(y), \tilde{\eta}_i(z)\} \\
= \min_i \left\{\inf_{i \in \Omega} \tilde{\eta}_i(x), \inf_{i \in \Omega} \tilde{\eta}_i(y), \inf_{i \in \Omega} \tilde{\eta}_i(z)\right\} \\
= \min_i \left\{\bigcap_{i \in \Omega} \tilde{\eta}_i(x), \bigcap_{i \in \Omega} \tilde{\eta}_i(y), \bigcap_{i \in \Omega} \tilde{\eta}_i(z)\right\} \\
= \min_i \{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\}.
\]

Theorem 3.10 Let \( \tilde{\eta} \) be an i.v fuzzy subset of \( M \). Then \( \tilde{\eta} \) is an i.v fuzzy weak bi-ideal of \( M \) if and only if \( U(\tilde{\eta} : [t_1, t_2]) \) is a weak bi-ideal of \( M \), for all \( [t_1, t_2] \in D[0,1] \).

Proof. Assume that \( \tilde{\eta} \) is an i.v fuzzy weak bi-ideal of \( M \). Let \( [t_1, t_2] \in D[0,1] \) such that \( x, y \in U(\tilde{\eta} : [t_1, t_2]) \). Then \( \tilde{\eta}(x - y) \geq \inf_i \{\tilde{\eta}(x), \tilde{\eta}(y)\} \geq \inf_i \{[t_1, t_2], [t_1, t_2]\} = [t_1, t_2] \). Thus \( x - y \in U(\tilde{\eta} : [t_1, t_2]) \). Let \( x, y, z \in U(\tilde{\eta} : [t_1, t_2]) \) and \( \alpha, \beta \in \Gamma \). We have \( \tilde{\eta}(x\alpha y\beta z) \geq \inf_i \{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\} \geq \inf_i \{[t_1, t_2], [t_1, t_2]\} \geq [t_1, t_2] \). Therefore \( x\alpha y\beta z \in U(\tilde{\eta} : [t_1, t_2]) \). Hence \( U(\tilde{\eta} : [t_1, t_2]) \) is a weak bi-ideal of \( M \).
Conversely, assume \( \tilde{U}(\tilde{\eta} : [t_1, t_2]) \) is a weak bi-ideal of \( M \), for all \([t_1, t_2] \in D[0, 1]\). Let \( x, y \in M \). Suppose \( \tilde{\eta}(x - y) < \min\{\tilde{\eta}(x), \tilde{\eta}(y)\} \). Choose \([0, 0] < [t_1, t_2] \leq [1, 1]\) such that \( \tilde{\eta}(x - y) < [t_1, t_2] < \min\{\tilde{\eta}(x), \tilde{\eta}(y)\} \). This implies that \( \tilde{\eta}(x) > [t_1, t_2], \tilde{\eta}(y) > [t_1, t_2] \) and \( \tilde{\eta}(x - y) < [t_1, t_2] \). Then we have \( x, y \in \tilde{U}(\tilde{\eta} : [t_1, t_2]) \), but \( x - y \notin \tilde{U}(\tilde{\eta} : [t_1, t_2]) \) a contradiction. Thus, \( \tilde{\eta}(x - y) \geq \min\{\tilde{\eta}(x), \tilde{\eta}(y)\} \). If there exist \( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \) such that \( \tilde{\eta}(x \alpha y \beta z) < [t_1, t_2] < \min\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\} \). Choose \([t_1, t_2] \) such that \( \tilde{\eta}(x \alpha y \beta z) < [t_1, t_2] < \min\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\} \). Then \( \tilde{\eta}(x) > [t_1, t_2], \tilde{\eta}(y) > [t_1, t_2], \tilde{\eta}(z) > [t_1, t_2] \) and \( \tilde{\eta}(x \alpha y \beta z) < [t_1, t_2] \). So, \( x, y, z \in \tilde{U}(\tilde{\eta} : [t_1, t_2]) \), but \( x \alpha y \beta z \notin \tilde{U}(\tilde{\eta} : [t_1, t_2]) \), which is a contradiction. Hence \( \tilde{\eta}(x \alpha y \beta z) \geq \min\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\} \). Therefore, \( \tilde{\eta} \) is an i.v fuzzy weak bi-ideal of \( M \).

**Theorem 3.11** Let \( \tilde{\eta} = [\eta^-, \eta^+] \) be an i.v fuzzy subset of \( M \), then \( \tilde{\eta} \) is an i.v fuzzy weak bi-ideal of near-ring \( M \) if and only if \( \eta^-, \eta^+ \) are fuzzy weak bi-ideals of \( M \).

**Proof.** Assume that \( \tilde{\eta} \) is an i.v fuzzy weak bi-ideals of near-ring \( M \). For any \( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \). Now,

\[
[\eta^-(x - y), \eta^+(x - y)] = \tilde{\eta}(x - y)
\geq \min\{\tilde{\eta}(x), \tilde{\eta}(y)\}
= \min\{[\eta^-(x), \eta^+(x)], [\eta^-(y), \eta^+(y)]\}
= \min\{[\eta^-(x), \eta^-(y)], \min\{\eta^+(x), \eta^+(y)\}\}.
\]

It follows that \( \eta^-(x - y) \geq \min\{\eta^-(x), \eta^-(y)\} \) and \( \eta^+(x - y) \geq \min\{\eta^+(x), \eta^+(y)\} \).

\[
[\eta^-(x \alpha y \beta z), \eta^+(x \alpha y \beta z)] = \tilde{\eta}(x \alpha y \beta z)
\geq \min\{\tilde{\eta}(x, \tilde{\eta}(y), \tilde{\eta}(z)\}
= \min\{[\eta^-(x), \eta^+(x)], [\eta^-(y), \eta^+(y)], [\eta^-(z), \eta^+(z)]\}
= \min\{[\eta^-(x), \eta^-(y)], \eta^-(z)\}, \min\{\eta^+(x), \eta^+(y), \eta^+(z)\}}.
\]

It follows that \( \eta^-(x \alpha y \beta z) \geq \min\{\eta^-(x), \eta^-(y), \eta^-(z)\} \) and \( \eta^+(x \alpha y \beta z) \geq \min\{\eta^+(x), \eta^+(y), \eta^+(z)\} \). Conversely, assume that \( \eta^-, \eta^+ \) are fuzzy weak bi-ideals of near-ring \( M \). Let \( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \).

\[
\eta^-(x - y) = [\eta^-(x - y), \eta^+(x - y)]
\geq \min\{[\eta^-(x), \eta^-(y)], \min\{\eta^+(x), \eta^+(y)\}\}
= \min\{[\eta^-(x), \eta^+(x)], [\eta^-(y), \eta^+(y)]\}
= \min\{\tilde{\eta}(x), \tilde{\eta}(y)\}
\]

and

\[
\tilde{\eta}(x \alpha y \beta z) = [\eta^-(x \alpha y \beta z), \eta^+(x \alpha y \beta z)]
\geq \min\{[\eta^-(x), \eta^-(y), \eta^-(z)], \min\{\eta^+(x), \eta^+(y), \eta^+(z)\}\}
= \min\{[\eta^-(x), \eta^+(x)], [\eta^-(y), \eta^+(y)], [\eta^-(z), \eta^+(z)]\}
= \min\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\}.
\]
Therefore $\tilde{\eta}$ is an i.v fuzzy weak bi-ideals of near-ring $M$. ■

**Theorem 3.12** Let $I$ be a weak bi-ideal of near-ring $M$ then for any $[t_1, t_2] \in D[0, 1]$ with $[t_1, t_2] \neq [0, 0]$, there exists an i.v fuzzy weak bi-ideal $\tilde{\eta}$ of $M$ such that $\tilde{U}(\tilde{\eta} : [t_1, t_2]) = I$.

**Proof.** Let $I$ be a weak bi-ideal of $M$. Let $\tilde{\eta}$ be an i.v fuzzy subset of $M$ defined by

$$\tilde{\eta}(x) = \begin{cases} [t_1, t_2] & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases}$$

Then $\tilde{U}(\tilde{\eta} : [t_1, t_2]) = I$. Assume that $\tilde{\eta}(x - y) < \min\{\tilde{\eta}(x), \tilde{\eta}(y)\}$. This implies that $\tilde{\eta}(x - y) = 0$ and $\min\{\tilde{\eta}(x), \tilde{\eta}(y)\} = [t_1, t_2]$ so $x, y \in I$ and $\alpha, \beta \in \Gamma$ but $x - y \notin I$, which is a contradiction. Thus, $\tilde{\eta}(x - y) \geq \min\{\tilde{\eta}(x), \tilde{\eta}(y)\}$. Suppose that $\tilde{\eta}(x\alpha y\beta z) < \min\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\}$. Then $\tilde{\eta}(x\alpha y\beta z) = 0$ and $\min\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\} = [t_1, t_2]$ so $x, y, z \in I$ but $x\alpha y\beta z \notin I$ which is a contradiction. Hence $\tilde{\eta}(x\alpha y\beta z) \geq \min\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\}$.

**Theorem 3.13** Let $H$ be a nonempty subset of $M$ and $\tilde{s}$ be an i.v fuzzy subset of $M$ defined by

$$\tilde{s}(x) = \begin{cases} \tilde{s} & \text{if } x \in H \\ \tilde{t} & \text{otherwise} \end{cases}$$

for some $x \in M$, $\tilde{s}, \tilde{t} \in D[0, 1]$ and $\tilde{s} > \tilde{t}$. Then $H$ is a weak bi-ideal of $M$ if and only if $\tilde{s}$ is an i.v fuzzy weak bi-ideal of $H$.

**Proof.** Assume that $H$ is a weak bi-ideal of $M$. Let $x, y \in M$. We consider four Cases:

1. $x \in H$ and $y \in H$.
2. $x \in H$ and $y \notin H$.
3. $x \notin H$ and $y \in H$.
4. $x \notin H$ and $y \notin H$.

Case (1): If $x \in H$ and $y \in H$. Then $\eta(x) = \tilde{s} = \tilde{\eta}(y)$. Since $H$ is a weak bi-ideal of $M$, then $x - y \in H$. Thus, $\tilde{\eta}(x - y) = \tilde{s} = \min\{\tilde{s}, \tilde{s}\} = \min\{\tilde{\eta}(x), \tilde{\eta}(y)\}$. Case (2): If $x \in H$ and $y \notin H$. Then $\tilde{\eta}(x) = \tilde{s}$ and $\tilde{\eta}(y) = \tilde{t}$. So, $\min\{\tilde{\eta}(x), \tilde{\eta}(y)\} = \tilde{t}$. Now, $\tilde{\eta}(x - y) = \tilde{s}$ or $\tilde{t}$ according as $x - y \in H$ or $x - y \notin H$. By assumption, $\tilde{s} > \tilde{t}$, we have $\tilde{\eta}(x - y) \geq \min\{\tilde{\eta}(x), \tilde{\eta}(y)\}$. Similarly, we prove Case (3).

Case (4): $x, y \notin H$, we have, $\tilde{s}(x) = \tilde{t} = \tilde{\eta}(y)$. So, $\min\{\tilde{\eta}(x), \tilde{\eta}(y)\} = \tilde{t}$. Next, $\tilde{\eta}(x - y) = \tilde{s}$ or $\tilde{t}$, according as $x - y \in H$ or $x - y \notin H$. So, $\tilde{\eta}(x - y) \geq \min\{\tilde{\eta}(x), \tilde{\eta}(y)\}$. Now let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. We have the following eight Cases.

1. $x \in H, y \in H$ and $z \in H$.
2. $x \notin H, y \in H$ and $z \in H$.
3. $x \in H, y \notin H$ and $z \in H$.
4. $x \in H, y \notin H$ and $z \notin H$.
5. $x \notin H, y \in H$ and $z \notin H$.
6. $x \in H, y \notin H$ and $z \in H$.
7. $x \notin H, y \in H$ and $z \notin H$.
8. $x \notin H, y \notin H$ and $z \notin H$.

These cases can be proved by arguments similar to the fuzzy cases above. Hence, $\tilde{\eta}(x\alpha y\beta z) \geq \min\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\}$. Hence $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of $M$. Conversely, assume that $\tilde{s}$ is an i.v fuzzy weak bi-ideal of $M$. Let $x, y, z \in H$ and $\alpha, \beta \in \Gamma$ be such that $\tilde{\eta}(x) = \tilde{\eta}(y) = \tilde{\eta}(z) = \tilde{s}$. Since $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of $M$, we
have $\tilde{\eta}(x - y) \geq \min_i \{\tilde{\eta}(x), \tilde{\eta}(y)\} = \tilde{s}$ and $\tilde{\eta}(x\alpha y\beta z) \geq \min_i \{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\} = \tilde{s}$. So, $x - y, x\alpha y\beta z \in H$. Hence $H$ is a weak bi-ideal of $M$.

**Theorem 3.14** A nonempty subset $H$ of $M$ is a weak bi-ideal of $M$ if and only if the characteristic function $f_H$ is an i.v fuzzy weak bi-ideal of $M$.

**Proof.** The proof is straightforward.

**Theorem 3.15** Let $\tilde{\eta}$ be an i.v fuzzy weak bi-ideal of $M$ then the set $M_{\tilde{\eta}} = \{x \in M \mid \tilde{\eta}(x) = \tilde{\eta}(0)\}$ is weak bi-ideal of $M$.

**Proof.** Let $\tilde{\eta}$ be i.v fuzzy weak bi-ideal of $M$. Let $x, y \in M_{\tilde{\eta}}$. Then $\tilde{\eta}(x) = \tilde{\eta}(0), \tilde{\eta}(y) = \tilde{\eta}(0)$ and $\tilde{\eta}(x - y) \geq \min_i \{\tilde{\eta}(x), \tilde{\eta}(y)\} = \tilde{\eta}(0)$. So $\tilde{\eta}(x - y) = \tilde{\eta}(0)$. Thus $x - y \in M_{\tilde{\eta}}$. For every $x, y, z \in M_{\tilde{\eta}}$ and $\alpha, \beta \in \Gamma$ we have $\tilde{\eta}(x\alpha y\beta z) \geq \min_i \{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\} = \min_i \{\tilde{\eta}(0), \tilde{\eta}(0), \tilde{\eta}(0)\} = \tilde{\eta}(0)$. Thus $x\alpha y\beta z \in M_{\tilde{\eta}}$. Hence $M_{\tilde{\eta}}$ is a weak bi-ideal of $M$.

4. **Homomorphism of interval valued fuzzy weak bi-ideals of \(\Gamma\)-near-rings**

In this section, we characterize i.v fuzzy weak bi-ideals of \(\Gamma\)-near-rings using homomorphism.

**Definition 4.1** [9] Let $f$ be a mapping from a set $M$ to a set $S$. Let $\tilde{\eta}$ and $\tilde{\delta}$ be i.v fuzzy subsets of $M$ and $S$ respectively. Then $f(\tilde{\eta})$, the image of $\tilde{\eta}$ under $f$ is an i.v fuzzy subset of $S$ defined by

$$f(\tilde{\eta})(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \tilde{\eta}(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and the pre-image of $\tilde{\eta}$ under $f$ is an i.v fuzzy subset of $M$ defined by $f^{-1}(\tilde{\delta}(x)) = \tilde{\delta}(f(x))$, for all $x \in M$ and $f^{-1}(y) = \{x \in M \mid f(x) = y\}$.

**Definition 4.2** [9] Let $M$ and $S$ be $\Gamma$-near-rings. A map $\theta : M \to S$ is called a ($\Gamma$-near-ring)homomorphism if $\theta(x + y) = \theta(x) + \theta(y)$ and $\theta(x\alpha y) = \theta(x)\alpha\theta(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

**Theorem 4.3** Let $f : M \to S$ be a homomorphism between $\Gamma$-near-rings $M$ and $S$. If $\tilde{\delta}$ is an i.v fuzzy weak bi-ideal of $S$, then $f^{-1}(\tilde{\delta})$ is an i.v fuzzy weak bi-ideal of $M$.

**Proof.** Let $\tilde{\delta}$ is an i.v fuzzy weak bi-ideal of $S$. Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then

$$f^{-1}(\tilde{\delta})(x - y) = \tilde{\delta}(f(x - y)) = \tilde{\delta}(f(x) - f(y)) \geq \min_i \{\tilde{\delta}(f(x)), \tilde{\delta}(f(y))\} = \min_i \{f^{-1}(\tilde{\delta}(x)), f^{-1}(\tilde{\delta}(y))\}$$
and
\[
f^{-1}(\delta)(x\alpha y \beta z) = \tilde{\delta}(f(x)\alpha f(y)\beta f(z)) \\
= \tilde{\delta}(f(x)\alpha f(y)\beta f(z)) \\
\geq \min \{ \tilde{\delta}(f(x)), \tilde{\delta}(f(y)), \tilde{\delta}(f(z)) \} \\
= \min \{ f^{-1}(\tilde{\delta}(x)), f^{-1}(\tilde{\delta}(y)), f^{-1}(\tilde{\delta}(z)) \}.
\]
Therefore \( f^{-1}(\tilde{\delta}) \) is an i.v fuzzy weak bi-ideal of \( M \).

We can also state the converse of the Theorem 4.3 by strengthening the condition on \( f \) as follows.

**Theorem 4.4** Let \( f : M \to S \) be an onto homomorphism of \( \Gamma \)-near-rings \( M \) and \( S \). Let \( \delta \) be an i.v fuzzy subset of \( S \). If \( f^{-1}(\tilde{\delta}) \) is an i.v fuzzy weak bi-ideal of \( M \), then \( \tilde{\delta} \) is an i.v fuzzy weak bi-ideal of \( S \).

**Proof.** Let \( x, y, z \in S \). Then \( f(a) = x, f(b) = y \) and \( f(c) = z \) for some \( a, b, c \in M \) and \( \alpha, \beta \in \Gamma \). It follows that
\[
\tilde{\delta}(x - y) = \tilde{\delta}(f(a) - f(b)) \\
= \tilde{\delta}(f(a - b)) \\
= f^{-1}(\tilde{\delta})(a - b) \\
\geq \min \{ f^{-1}(\tilde{\delta})(a), f^{-1}(\tilde{\delta})(b) \} \\
= \min \{ \tilde{\delta}(f(a)), \tilde{\delta}(f(b)) \} \\
= \min \{ \tilde{\delta}(x), \tilde{\delta}(y) \}.
\]
and
\[
\tilde{\delta}(x\alpha y \beta z) = \tilde{\delta}(f(a)\alpha f(b)\beta f(c)) \\
= \tilde{\delta}(f(aab\beta c)) \\
= f^{-1}(\tilde{\delta})(aab\beta c) \\
\geq \min \{ f^{-1}(\tilde{\delta})(a), f^{-1}(\tilde{\delta})(b), f^{-1}(\tilde{\delta})(c) \} \\
= \min \{ \tilde{\delta}(f(a)), \tilde{\delta}(f(b)), \tilde{\delta}(f(c)) \} \\
= \min \{ \tilde{\delta}(x), \tilde{\delta}(y), \tilde{\delta}(z) \}.
\]
Hence \( \tilde{\delta} \) is an i.v fuzzy weak bi-ideal of \( S \).

**Theorem 4.5** Let \( f : M \to S \) be an onto \( \Gamma \)-near-ring homomorphism. If \( \tilde{\eta} \) is an i.v fuzzy weak bi-ideal of \( M \), then \( f(\tilde{\eta}) \) is an i.v fuzzy weak bi-ideal of \( S \).

**Proof.** Let \( \tilde{\eta} \) be an i.v fuzzy weak bi-ideal of \( M \). Since \( f(\tilde{\eta})(x') = \sup \{ f_{f(x')} = x'(\tilde{\eta}(x)) \} \) for \( x' \in S \) and hence \( f(\tilde{\eta}) \) is nonempty. Let \( x', y' \in S \) and \( \alpha, \beta \in \Gamma \). Then we have \( \{ x | x \in f^{-1}(x' - y') \} \supseteq \{ x - y | x \in f^{-1}(x') \) and \( y' \in f^{-1}(y') \} \) and \( \{ x | x \in f^{-1}(x'y') \} \supseteq \)
\{ x\alpha y | x \in f^{-1}(x') \text{ and } y \in f^{-1}(y') } \}.

\[
f(\tilde{\eta})(x' - y') = \sup_{f(z)=x'-y'} f(\tilde{\eta}(z)) \\
\geq \sup_{f(x)=x', f(y)=y'} f(\tilde{\eta}(x - y)) \\
\geq \sup_{f(x)=x', f(y)=y'} \{ \min^i \{ \tilde{\eta}(x), \tilde{\eta}(y) \} \} \\
= \min^i \{ \sup_{f(x)=x'} f(\tilde{\eta}(x)), \sup_{f(y)=y'} f(\tilde{\eta}(y)) \} \\
= \min^i \{ f(\tilde{\eta})(x'), f(\tilde{\eta})(y') \}.
\]

Next,

\[
f(\tilde{\eta})(x' \alpha y' \beta z') = \sup_{f(h)=x' \alpha y' \beta z'} f(\tilde{\eta}(h)) \\
\geq \sup_{f(x)=x', f(y)=y', f(z)=z'} f(\tilde{\eta}(x \alpha y \beta z)) \\
\geq \sup_{f(x)=x', f(y)=y', f(z)=z'} \{ \min^i \{ \tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z) \} \} \\
= \min^i \{ \sup_{f(x)=x'} f(\tilde{\eta}(x)), \sup_{f(y)=y'} f(\tilde{\eta}(y)), \sup_{f(z)=z'} f(\tilde{\eta}(z)) \} \\
= \min^i \{ f(\tilde{\eta})(x'), f(\tilde{\eta})(y'), f(\tilde{\eta})(z') \}.
\]

Therefore \( f(\tilde{\eta}) \) is an i.v fuzzy weak bi-ideal of \( S \).

\[\Box\]

5. \textbf{Anti-homomorphism of interval valued fuzzy weak bi-ideals of \( \Gamma \)-near-rings}

In this section, we characterize i.v fuzzy weak bi-ideals of \( \Gamma \)-near-rings using anti-homomorphism.

\textbf{Definition 5.1} [11] Let \( M \) and \( S \) be \( \Gamma \)-near-rings. A map \( \theta : M \to S \) is called a (\( \Gamma \)-near-ring)anti-homomorphism if \( \theta(x + y) = \theta(y) + \theta(x) \) and \( \theta(x \alpha y) = \theta(y) \alpha \theta(x) \) for all \( x, y \in M \) and \( \alpha \in \Gamma \).

\textbf{Theorem 5.2} Let \( f : M \to S \) be a anti-homomorphism between \( \Gamma \)-near-rings \( M \) and \( S \). If \( \tilde{\delta} \) is an i.v fuzzy weak bi-ideal of \( S \), then \( f^{-1}(\tilde{\delta}) \) is an i.v fuzzy weak bi-ideal of \( M \).

\textbf{Proof.} Let \( \tilde{\delta} \) be an i.v fuzzy weak bi-ideal of \( S \). Let \( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \). Then

\[
f^{-1}(\tilde{\delta})(x - y) = \tilde{\delta}(f(x - y)) \\
= \tilde{\delta}(f(y) - f(x)) \\
\geq \min^i \{ \tilde{\delta}(f(y)), \tilde{\delta}(f(x)) \} \\
= \min^i \{ \tilde{\delta}(f(x)), \tilde{\delta}(f(y)) \} \\
= \min^i \{ f^{-1}(\tilde{\delta}(x)), f^{-1}(\tilde{\delta}(y)) \}.
\]
and

\[
    f^{-1}(\tilde{\delta})(x\alpha y\beta z) = \tilde{\delta}(f(x\alpha y\beta z))
    = \tilde{\delta}(f(z)\alpha f(y)\beta f(x))
    \geq \min^i\{\tilde{\delta}(f(z)), \tilde{\delta}(f(y)), \tilde{\delta}(f(x))\}
    = \min^i\{\tilde{\delta}(f(x)), \tilde{\delta}(f(y)), \tilde{\delta}(f(z))\}
    = \min^i\{f^{-1}(\tilde{\delta}(x)), f^{-1}(\tilde{\delta}(y)), f^{-1}(\tilde{\delta}(z))\}.
\]

Therefore \( f^{-1}(\tilde{\delta}) \) is an i.v fuzzy weak bi-ideal of \( M \).

We can also state the converse of the Theorem 5.2 by strengthening the condition on \( f \) as follows.

**Theorem 5.3** Let \( f : M \to S \) be an onto anti-homomorphism of \( \Gamma \)-near-rings \( M \) and \( S \). Let \( \tilde{\delta} \) be an i.v fuzzy subset of \( S \). If \( f^{-1}(\tilde{\delta}) \) is an i.v fuzzy weak bi-ideal of \( M \), then \( \tilde{\delta} \) is an i.v fuzzy weak bi-ideal of \( S \).

**Proof.** Let \( x, y, z \in S \). Then \( f(a) = x, f(b) = y \) and \( f(c) = z \) for some \( a, b, c \in M \) and \( \alpha, \beta \in \Gamma \). It follows that

\[
    \tilde{\delta}(x - y) = \tilde{\delta}(f(a) - f(b))
    = \tilde{\delta}(f(b - a))
    = f^{-1}(\tilde{\delta})(b - a)
    \geq \min^i\{f^{-1}(\tilde{\delta})(b), f^{-1}(\tilde{\delta})(a)\}
    = \min^i\{\tilde{\delta}(f(a)), \tilde{\delta}(f(b))\}
    = \min^i\{\tilde{\delta}(x), \tilde{\delta}(y)\}
\]

and

\[
    \tilde{\delta}(x\alpha y\beta z) = \tilde{\delta}(f(a)\alpha f(b)\beta f(c))
    = \tilde{\delta}(f(cab\beta a))
    = f^{-1}(\tilde{\delta})(cab\beta a)
    \geq \min^i\{f^{-1}(\tilde{\delta})(c), f^{-1}(\tilde{\delta})(b), f^{-1}(\tilde{\delta})(a)\}
    = \min^i\{f^{-1}(\tilde{\delta})(a), f^{-1}(\tilde{\delta})(b), f^{-1}(\tilde{\delta})(c)\}
    = \min^i\{\tilde{\delta}(f(a)), \tilde{\delta}(f(b)), \tilde{\delta}(f(c))\}
    = \min^i\{\tilde{\delta}(x), \tilde{\delta}(y), \tilde{\delta}(z)\}.
\]

Hence \( \tilde{\delta} \) is an i.v fuzzy weak bi-ideal of \( S \).

**Theorem 5.4** Let \( f : M \to S \) be an onto \( \Gamma \)-near-ring anti-homomorphism. If \( \tilde{\eta} \) is an i.v fuzzy weak bi-ideal of \( M \), then \( f(\tilde{\eta}) \) is an i.v fuzzy weak bi-ideal of \( S \).

**Proof.** Let \( \tilde{\eta} \) be an i.v fuzzy weak bi-ideal of \( M \). Since
Next, 

\[
f(\bar{\eta})(x' \circ y' \circ z') = \sup_{f(h)=x' \circ y' \circ z'}^{i} \{\bar{\eta}(h)\}
\]

\[
\geq \sup_{f(x)=x', f(y)=y', f(z)=z'}^{i} \{\bar{\eta}(x \circ y \circ z)\}
\]

\[
\geq \sup_{f(x)=x', f(y)=y', f(z)=z'}^{i} \{\min^{i} \{\bar{\eta}(x), \bar{\eta}(y), \bar{\eta}(z)\}\}
\]

\[
= \min^{i} \{\sup_{f(x)=x'}^{i} \{\bar{\eta}(x)\}, \sup_{f(y)=y'}^{i} \{\bar{\eta}(y)\}, \sup_{f(z)=z'}^{i} \{\bar{\eta}(z)\}\}
\]

Therefore \(f(\bar{\eta})\) is an i.v fuzzy weak bi-ideal of \(S\). 

\[\blacksquare\]

Acknowledgements

The second author was supported in part by UGC-BSR Grant #F.25-1/2014-15(BSR)/7-254/2009(BSR) dated 20-01-2015 in India. The third author was supported in part by UGC-BSR Grant # F-4/1/2006(BSR)/7-254/2009(BSR) in India.

References