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# Order reduction and $\mu$-conservation law for the non-isospectral KdV type equation with variable-coefficients

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**Abstract**

The goal of this paper is to calculate of order reduction of the KdV type equation and the non-isospectral KdV type equation using the $\mu$-symmetry method. Moreover we obtain $\mu$-conservation law of the non-isospectral KdV type equation using the variational problem method.

*Key words:* Symmetry, $\mu$-symmetry, $\mu$-conservation law, variational problem, order reduction.

## 1 Introduction

Partial differential equations (PDEs) have been a most important subject of study in all areas of mathematical physics, engineering sciences

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and other technical arena. At present time, different methods are being established to order reduction and conservation law of nonlinear PDEs, such as, the symmetries method [15], the direct method [16], the general theorem [16], the Noether theorem [6].

The Korteweg-de Vries (KdV) type equation

$$KdV \ : \ u_t + \alpha u_x + \beta u u_x + \gamma u_{xxx} = 0\,,$$

where $\alpha$, $\beta$ and $\gamma$ are real constants, is one of the most popular equations by Korteweg and de Vries known as water waves equations. This equation used to model waves as long wavelength in liquids, hydro-magnetic waves in cold plasma, acoustic-gravity waves in compressible fluids, acoustic waves in anharmonic crystals, etc.

The non-isospectral KdV type equation with variable-coefficients (nvcKdV) can be show as follows:

$$nvcKdV \ : \ u_t + \alpha(t)(u_{xxx} + 6uu_x) + 4\beta(t)u_x - \gamma(t)(xu_x + 2u) = 0, \quad (1.1)$$

where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are arbitrary functions of time $t$.

The nvcKdV is important as this equation often model realistic situations in certain case. However, literature on the nvcKdV is rather limited compared to the constant coefficient counterparts. Due to the physical applications and mathematical properties of the nvcKdV, we have been motivated to obtain $\mu$-symmetry and $\mu$-conservation law, etc.

Many researchers studied of the KdV type equation and the nvcKdV for obtaining solutions, stabilization of global solutions, numerical solution, Lie symmetry analysis using different methods. But, to the best of our knowledge, the nvcKdV is not investigated via the $\mu$-symmetry method to order reduction and the variational problem method to $\mu$-conservation law. In this article, we calculate an order reduction of the nvcKdV using the $\mu$-symmetry method. Moreover we calculate $\mu$-conservation law of the nvcKdV using the variational problem method.

Because the nvcKdV is of odd order; therefore, it is no variational problem, but this equation is of even order in the potential form; hence, the variational problem can be accepted. At first, we obtain $\mu$-conservation law of the nvcKdV in potential forms and using this technique, then we can obtain $\mu$-conservation law of the nvcKdV.

The outline of this paper is as follows. Firstly, $\mu$-symmetry and reduced equations for the nvcKdV is provided. Secondly, lagrangian for the nvcKdV is shown in potential form. Finally, $\mu$-conservation law for the nvcKdV is described.

## 2 Background

In 2001, Muriel and Romero introduced a new method to order reduction of ordinary differential equations (ODEs), and they called it as $\lambda$-symmetries method to order reduction of ODEs. In 2004, Gaeta and Morando expanded $\lambda$-symmetries method of ODEs to $\mu$-symmetries method of the partial differential equations (PDEs) frame with $p$ independent variables $x = (x^1, ..., x^p)$ and $q$ dependent variables $u = (u^1, ..., u^q)$, where $\mu = \lambda_i dx^i$ is a horizontal one-form on first order jet space $(J^{(1)}M, \pi, M)$ and also $\mu$ is a compatible, i.e. $D_i\lambda_j - D_j\lambda_i = 0$.

In 2006, Muriel, Romero and Olver have expanded the concept of variational problem and conservation law in the case of symmetries to the case of $\lambda$-symmetries of ODEs. They have presented an adapted formulation of the Nother's theorem for $\lambda$-symmetry of ODEs. In 2007, Cicogna and Gaeta have generalized the results obtained by Muriel, Romero and Olver in the case of $\lambda$-symmetries for ODEs to the case of $\mu$-symmetries for PDEs, and in the case of $\mu$-symmetry of the Lagrangian, the conservation law is referred as $\mu$-conservation law.

## 3 $\mu$-prolongation And $\mu$-symmetry

In this section, the foundational results of $\mu$-prolongation and $\mu$-symmetry are briefly introduced. Let $\mu = \lambda_i dx^i$ be horizontal one-form on first order jet space $(J^{(1)}M, \pi, M)$ and compatible [7], i.e. $D_i\lambda_j - D_j\lambda_i = 0$, where $D_i$ is total derivative $x^i$ and $\lambda_i : J^{(1)}M \longrightarrow \mathbb{R}$.

Suppose $\Delta(x, u^{(k)}) = 0$ is a scalar PDEs of order $k$ for $u = u(x^1, ..., x^p)$, i.e. involving $p$ independent variables $x = (x^1, ..., x^p)$ and one dependent variable. Let $X = \xi^i \partial_{x^i} + \varphi \partial_u$ be a vector field on $M$. We define $Y = X + \sum_{J=1}^{k} \Psi_J \, \partial_{u_J}$ on $k$-th order jet space $J^k M$ as $\mu$-prolongation of $X$ if its coefficient satisfies the $\mu$-prolongation formula

$$\Psi_{J,i} = (D_i + \lambda_i)\Psi_J - u_{J,m}(D_i + \lambda_i)\xi^m\,, \tag{3.1}$$

where $\Psi_0 = \varphi$. Let $\mathcal{S} \subset J^{(k)}M$ be the solution manifold for $\Delta$. If $Y : \mathcal{S} \longrightarrow T\mathcal{S}$, we say that $X$ is a $\mu$-symmetry for $\Delta$.

if $\mu = 0$ in (3.1), then it can be assumed that ordinary prolongation is as 0-prolongation in $\mu$-prolongation and ordinary symmetry is as 0-symmetry in $\mu$-symmetry framework.

We consider an equation $\Delta$ such that $\mu = \lambda_i dx^i$ is a horizontal 1-form and compatible on $\mathcal{S}_\Delta$. Suppose $V = \exp\Big(\int \mu\Big)X$ is an exponential vector field, where $X$ is a vector field on $M$. Then $V$ is a general symmetry for $\Delta$ if and only if $X$ is a $\mu$-symmetry for $\Delta$.

In paper [7], an order reduction of PDEs under $\mu$-symmetries is shown as the following theorem.

**Theorem 3.1** *Let $\Delta$ be a scalar PDE of order $k$ for $u = u(x^1, ..., x^p)$. Let $X = \xi^i(\frac{\partial}{\partial x^i}) + \varphi(\frac{\partial}{\partial u})$ be a vector field on $M$, with characteristic $Q = \varphi - u_i\xi^i$, and let $Y$ be the $\mu$-prolong of order $k$ of $X$. If $X$ is a $\mu$-symmetry for $\Delta$, then $Y : \mathcal{S}_X \longrightarrow T\mathcal{S}_X$, where $\mathcal{S}_X \subset J^{(k)}M$ is the solution manifold for the system $\Delta_X$ made of $\Delta$ and of $E_J := D_J Q = 0$ for all $J$ with $\mid J \mid = 0, 1, ..., k-1$.*

$\mu$-symmetry of given equations (PDE): In order to determine $\mu$-symmetry of a given equation $\Delta$ of order $n$, the same way as for ordinary symmetries is considered that a generic vector field $X$ acting in $M$, and its $\mu$-prolongation $Y$ of order $n$ for a generic $\mu = \lambda_i dx^i$, acting in $J^{(n)}M$. Then applies $Y$ to $\Delta$, and restricts the obtained expression to the solution manifold $\mathcal{S}_\Delta \subset J^{(n)}M$. The equation $\Delta_*$ resulting by requiring this is zero is the determining equation for $\mu$-symmetries of $\Delta$; this is an equation for $\xi, \tau, \varphi$ and $\lambda_i$. If we require $\lambda_i$ are functions on $J^{(k)}M$, all the dependences on $u_J$ will be explicit, and one obtains a system of determining equation. This system should be complemented with the compatibility conditions between the $\lambda_i$. If we determine a priori the form $\mu$, we are left with a system of linear equation for $\xi, \tau, \varphi$; similarly, if we fix a vector field $X$ and try to find the $\mu$ for which it is a $\mu$-symmetry of the given equation $\Delta$, we have a system of quasilinear equation for the $\lambda_i$ [7].

## 4 $\mu$-symmetry for the non-isospectral KdV type equation with variable coefficients

The non-isospectral KdV type equation with variable coefficients (nvcKdV) can be shown as follows:

$$u_t + \alpha(t)(u_{xxx} + 6uu_x) + 4\beta(t)u_x - \gamma(t)(xu_x + 2u) = 0,$$

where is a scalar PDE of order 3 for $u = u(x,t)$. Let $\mu = \lambda_1 dx + \lambda_2 dt$ be a horizontal one-form and with the compatibility condition $D_t\lambda_1 = D_x\lambda_2$ when $u_t + \alpha(t)(u_{xxx} + 6uu_x) + 4\beta(t)u_x - \gamma(t)(xu_x + 2u) = 0$. Suppose $X = \xi\partial_x + \tau\partial_t + \varphi\partial_u$ is a vector field on $M$. In order to compute $\mu$-prolongation $Y$ of order 3 of $X$, we can use of (3.1); therefore, $\mu$-prolongation $Y$ of $X$ is as

$$Y = X + \Psi^x\partial_{u_x} + \Psi^t\partial_{u_t} + \Psi^{xx}\partial_{u_{xx}} + ... + \Psi^{ttt}\partial_{u_{ttt}},$$

where coefficients $Y$ are as the following

$$\begin{aligned}
\Psi^x &= (D_x + \lambda_1)\varphi - u_x(D_x + \lambda_1)\xi - u_t(D_x + \lambda_1)\tau\,,\\
\Psi^t &= (D_t + \lambda_2)\varphi - u_x(D_t + \lambda_2)\xi - u_t(D_t + \lambda_2)\tau\,,\\
\Psi^{xx} &= (D_x + \lambda_1)\Psi^x - u_{xx}(D_x + \lambda_1)\xi - u_{xt}(D_x + \lambda_1)\tau\,,\\
\Psi^{xt} &= (D_t + \lambda_2)\Psi^x - u_{xx}(D_t + \lambda_2)\xi - u_{xt}(D_t + \lambda_2)\tau\,,\\
\Psi^{tt} &= (D_t + \lambda_2)\Psi^t - u_{tx}(D_t + \lambda_2)\xi - u_{tt}(D_t + \lambda_2)\tau\,,\\
\Psi^{xxx} &= (D_x + \lambda_1)\Psi^{xx} - u_{xxx}(D_x + \lambda_1)\xi - u_{xxt}(D_x + \lambda_1)\tau,\\
\Psi^{xxt} &= (D_t + \lambda_2)\Psi^{xx} - u_{xxx}(D_t + \lambda_2)\xi - u_{xxt}(D_t + \lambda_2)\tau,\\
\Psi^{xtt} &= (D_t + \lambda_2)\Psi^{xt} - u_{xtx}(D_t + \lambda_2)\xi - u_{xtt}(D_t + \lambda_2)\tau\,,\\
\Psi^{ttt} &= (D_t + \lambda_2)\Psi^{tt} - u_{ttx}(D_t + \lambda_2)\xi - u_{ttt}(D_t + \lambda_2)\tau\,.
\end{aligned} \tag{4.1}$$

By applying $Y$ to Eq. (1.1) and substituting

$$-(1/\alpha(t))\Big(u_t + \alpha(t)6uu_x + 4\beta(t)u_x - \gamma(t)(xu_x + 2u)\Big) = 0,$$

for $u_{xxx}$ , we obtain the following system [1] :

[1] using Maple

$$
\begin{gathered}
-3\alpha(t)\tau_u = 0,\ -3\alpha(t)\tau_{uu} = 0,\ -\alpha(t)\tau_{uuu} = 0,\ -3\alpha(t)\xi_u = 0,\\
-6\alpha(t)\xi_{uu} = 0,\ \ -\alpha(t)\xi_{uuu} = 0,\ \ -3\alpha(t)(\lambda_1\tau + \tau_x) = 0,\\
\vdots\\
-3\alpha(t)(\tau_{xx} + \tau\lambda_{1x} + 2\lambda_1\tau_x + \lambda_1^2\tau) = 0.
\end{gathered} \tag{4.2}
$$

For any choice of the type

$$
\lambda_1 = D_x[f(x,t)] + g(x), \qquad \lambda_2 = D_t[f(x,t)] + h(t), \tag{4.3}
$$

where $f(x,t)$, $g(x)$ and $h(t)$ are arbitrary functions and $\lambda_1$ and $\lambda_2$ satisfy to the compatibility condition, i.e. $D_t\lambda_1 = D_x\lambda_2$ on solutions to Eq. (1.1). For instance, two cases are studied to obtain in $\mu$-symmetry of Eq. (1.1) as follows:

- When $g(x) = 0$ and $h(t) = -\gamma(t)$, also $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are arbitrary functions, then by substituting the functions

  $$
  \lambda_1 = D_x f(x,t)\ ,\ \lambda_2 = D_t f(x,t) - \gamma(t)
  $$

  into the system of (4.2) and solving them, we obtain

  $$
  \xi = F(x,t), \qquad \tau = 0, \qquad \varphi = 0\,,
  $$

  where $f(x,t) = -\ln(F(x,t))$ and $F(x,t)$ is an arbitrary positive function. Then

  $$
  X = F(x,t)\partial_x\,,
  $$

  is $\mu$-symmetry of Eq. (1.1) and corresponds to an ordinary symmetry

  $$
  V = \exp\Big(\int D_x f(x,t)dx + (D_t f(x,t) - \gamma(t))dt\Big)X\,,
  $$

  of exponential type. In this case, using Theorem 3.1, reduction of Eq. (1.1) is

  $$
  Q = \varphi - \xi u_x - \tau u_t = -F(x,t)u_x. \tag{4.4}
  $$

- When $g(x) = 0$ and $h(t) = -\beta'(t)/\beta(t)$, also $\alpha(t) = c_1\beta(t)$ and $\gamma(t) = c_2\beta(t)$ where $c_1$ and $c_2$ are arbitrary constants, then by substituting the functions

  $$
  \lambda_1 = D_x f(x,t)\ ,\ \lambda_2 = D_t f(x,t) - \beta'(t)/\beta(t)
  $$

into the system of (4.2) and solving them, we obtain

$$\xi = 0\,, \qquad \tau = F(x,t)\,, \qquad \varphi = 0\,,$$

where $f(x,t) = -\ln(F(x,t))$ and $F(x,t)$ is an arbitrary positive function. Then

$$X = F(x,t)\partial_t,$$

is $\mu$-symmetry of Eq. (1.1) and corresponds to an ordinary symmetry

$$V = \exp\Big(\int D_x f(x,t)dx + (D_t f(x,t) - \beta'(t)/\beta(t))dt\Big)X\,,$$

of exponential type. In this case, using Theorem 3.1, reduction of Eq. (1.1) is

$$Q = \varphi - \xi u_x - \tau u_t = -F(x,t)u_t. \tag{4.5}$$

## 5 Lagrangian of the non-isospectral KdV type equation with variable coefficients in potential form

In this section, we show that the nvcKdV does not admit a variational problem since it is of odd order, but the nvcKdV in potential form admitting a variational problem. In the book [16], a system admits a variational formulation if and only if its Frechet derivative is self-adjoint. In fact, we have the following theorem.

**Theorem 5.1** *Let $\Delta = 0$ be a system of differential equation. Then $\Delta$ is the Euler-Lagrange expression for some variational problem $\mathfrak{L} = \int L dx$, i.e. $\Delta = E(L)$, if and only if the Frechet derivative $D_\Delta$ is self-adjoint: $D^*_\Delta = D_\Delta$. In this case, a Lagrangian for $\Delta$ can be explicitly constructed using the homotopy formula $L[u] = \int_0^1 u.\Delta[\lambda u]d\lambda$.*

We consider the nvcKdV as

$$\Delta_{Ku} : u_t + \alpha(t)(u_{xxx} + 6uu_x) + 4\beta(t)u_x - \gamma(t)(xu_x + 2u) = 0. \tag{5.1}$$

The Frechet derivative of $\Delta_{Ku}$ is

$$\begin{aligned} D_{\Delta_{Ku}} = & D_t + \Big(6\alpha(t)u_x - 2\gamma(t)\Big) + \Big(6\alpha(t)u + 4\beta(t) - x\gamma(t)\Big)D_x \\ & + \alpha(t)D_x^3\,. \end{aligned}$$

Obviously, it does not admit a variational problem since $D^*_{\Delta_{Ku}} \neq D_{\Delta_{Ku}}$. But the well-known differential substitution $u = v_x$ yields the related transformed the nvcKdV as the following

$$\Delta_{Kv} : v_{xt} + \alpha(t)(v_{xxxx} + 6v_x v_{xx}) + 4\beta(t)v_{xx} - \gamma(t)(xv_{xx} + 2v_x) = 0. \quad (5.2)$$

This equation is called "the nvcKdV in potential form" and its Frechet derivative is

$$D_{\Delta_{Kv}} = D_x D_t + \Big(6\alpha(t)v_{xx} - 2\gamma(t)\Big)D_x + \Big(6\alpha(t)v_x + 4\beta(t) - x\gamma(t)\Big)D_x^2 + \alpha(t)D_x^4 .$$

which is self-adjoint: $D^*_{\Delta_{Kv}} = D_{\Delta_{Kv}}$. By Theorem 5.1, the nvcKdV in potential form $\Delta_{Kv}$ has a Lagrangian of the form

$$L[v] = \int_0^1 v.\Delta_{Bv}[\lambda v]d\lambda = -\frac{1}{2}\Big(v_x v_t + \alpha(t)(2v_x^3 - v_{xx}^2) + 4\beta(t)v_x^2 - \gamma(t)xv_x^2\Big) + \mathrm{Div}P.$$

Hence, Lagrangian of the nvcKdV in potential form $\Delta_{Kv}$, up to Div-equivalence is

$$\mathcal{L}_{\Delta_{Kv}}[v] = -\frac{1}{2}\Big(v_x v_t + \alpha(t)(2v_x^3 - v_{xx}^2) + 4\beta(t)v_x^2 - \gamma(t)xv_x^2\Big). \quad (5.3)$$

## 6 $\mu$-conservation laws

A *conservation law* is a relation $\mathrm{Div}\,\mathbf{P} := \sum_{i=1}^p D_i P^i = 0$, where $\mathbf{P} = (P^1, \cdots, P^p)$ is a $p-$dimensional vector. Let $\mu = \lambda_i dx^i$ be a horizontal one-form and compatibility condition, i.e, $D_i\lambda_j = D_j\lambda_i$ . A *$\mu$-conservation law* is a relation as

$$(D_i + \lambda_i)P^i = 0, \quad (6.1)$$

where $P^i$ is a vector and the $M-$vector $P^i$ is called a $\mu$-conserved vector.

The following theorem about the existence of $M-$vector $P^i$ and $\mu$-conservation law can be seen in [6]:

**Theorem 6.1** *Consider the $n-th$ order Lagrangian $\mathcal{L} = \mathcal{L}(x, u^{(n)})$, and vector field $X$, then $X$ is a $\mu$-symmetry for $\mathcal{L}$, i.e. $Y[\mathcal{L}] = 0$ if and only if there exists $M-$vector $P^i$ satisfying the $\mu$-conservation law $(D_i+\lambda_i)P^i = 0$.*

Using the other theorems in [6] and Theorem 6.1, the $M-$vector $P^i$ is obtained for first and second order Lagrangian, as the following:

- For first order Lagrangian $\mathcal{L}(x, t, u, u_x, u_t)$ and the vector field $X = \varphi\,(\partial/\partial u)$ is a $\mu$-symmetry for $\mathcal{L}$, then the $M-$vector $P^i := \varphi\,(\partial\mathcal{L}/\partial u_i)$, is a $\mu$-conserved vector.
- For second order Lagrangian $\mathcal{L}$ and the vector field $X = \varphi\,(\partial/\partial u)$ is a $\mu$-symmetry for $\mathcal{L}$, then the $M-$vector

$$P^i := \varphi\frac{\partial\mathcal{L}}{\partial u_i} + ((D_j + \lambda_j)\varphi)\frac{\partial\mathcal{L}}{\partial u_{ij}} - \varphi D_j\frac{\partial\mathcal{L}}{\partial u_{ij}}, \tag{6.2}$$

  is a $\mu$-conserved vector.

## 7 $\mu$-conservation laws of the non-isospectral KdV type equation with variable coefficients in potential form

In this section, we want to compute $\mu$-conservation law for the nvcKdV in potential form $\Delta_{Kv}$ in subsection (7.1) and using it, we compute $\mu$-conservation law for the nvcKdV $\Delta_{Ku}$ in subsection (7.2).

### *7.1 $\mu$-conservation laws of the non-isospectral KdV type equation with variable coefficients in potential forms*

We consider the second order Lagrangian (5.3), i.e.

$$\mathcal{L}_{\Delta_{Kv}}[v] = -\frac{1}{2}\Big(v_xv_t + \alpha(t)(2v_x^3 - v_{xx}^2) + 4\beta(t)v_x^2 - \gamma(t)xv_x^2\Big),$$

for the nvcKdV in potential form

$$\begin{aligned}\Delta_{Kv} &= v_{xt} + \alpha(t)(v_{xxxx} + 6v_xv_{xx}) + 4\beta(t)v_{xx} - \gamma(t)(xv_{xx} + 2v_x)\\ &= E(\mathcal{L}_{\Delta_{Kv}}).\end{aligned} \tag{7.1}$$

Suppose $X=\varphi\partial_v$ is a vector field for $\mathcal{L}_{\Delta_{Kv}}[v]$. Let $\mu=\lambda_1 dx+\lambda_2 dt$ be a horizontal one-form and with the compatibility condition $D_t\lambda_1=D_x\lambda_2$ when $\Delta_{Kv}=0$. In order to compute $\mu$-prolongation of order 2 of $X$, we can use of (3.1), we have,

$$Y=\varphi\partial_v+\Psi^x\partial_{v_x}+\Psi^t\partial_{v_t}+\Psi^{xx}\partial_{v_{xx}}+\Psi^{xt}\partial_{v_{xt}}+\Psi^{tt}\partial_{v_{tt}},$$

where coefficients $Y$ are as the following:

$$\begin{aligned}\Psi^x=(D_x+\lambda_1)\varphi,\qquad \Psi^t=(D_t+\lambda_2)\varphi,\qquad \Psi^{xx}=(D_x+\lambda_1)\Psi^x,\\ \Psi^{xt}=(D_t+\lambda_2)\Psi^x,\qquad \Psi^{tt}=(D_t+\lambda_2)\Psi^t.\end{aligned}\tag{7.2}$$

Thus, the $\mu$-prolongation $Y$ acts on the $\mathcal{L}_{\Delta_{Kv}}[v]$, and substituting $\Big(\alpha(t)(2v_x^3-v_{xx}^2)+4\beta(t)v_x^2-\gamma(t)xv_x^2\Big)/-v_x$ for $v_t$, we obtain the system as the following:

$$\begin{aligned}\alpha(t)\varphi_{vv}=0\,,\qquad -(1/2)\alpha(t)\Big(\varphi_x+\lambda_1\varphi\Big)=0,\\ -\alpha(t)\varphi_v=0\,,\qquad -2\alpha(t)\Big(\varphi_x+\lambda_1\varphi\Big)=0\,,\\ \alpha(t)\Big(\lambda_{1v}\varphi+2\lambda_1\varphi_v+2\varphi_{xv}\Big)=0\,,\\ \alpha(t)\Big(2\lambda_1\varphi_x+\lambda_{1x}\varphi+\varphi_{xx}+\lambda_1^2\varphi\Big)=0\,,\\ (1/2)\Big(x\gamma(t)\varphi_x-\varphi_t-4\beta(t)\lambda_1\varphi-\lambda_2\varphi-4\beta(t)\varphi_x+x\gamma(t)\lambda_1\varphi\Big)=0\,.\end{aligned}\tag{7.3}$$

Suppose $\varphi=F(x,t)$, where $F(x,t)$ is an arbitrary positive function satisfying $\mathcal{L}_{\Delta_{Kv}}[v]=0$, then a special solution of the system (7.3) is given by

$$\lambda_1=-\frac{F_x(x,t)}{F(x,t)}\,,\qquad \lambda_2=-\frac{F_t(x,t)}{F(x,t)},\tag{7.4}$$

where $\lambda_1$ and $\lambda_2$ are satisfying to $D_t\lambda_1=D_x\lambda_2$. Hence,

$$X=F(x,t)\partial_v$$

is a $\mu$-symmetry for $\mathcal{L}_{\Delta_{Kv}}[v]$, then, using Theorem 6.1, there exists $M-$vector $P^i$ satisfying the $\mu$-conservation law $(D_i+\lambda_i)P^i=0$. Then, by of (6.2), the $M-$vector $P^i$ is as

$$P^1 = -\Big(\frac{1}{2}v_t + \alpha(t)(v_{xxx} + 4v_x^2) + 4\beta(t)v_x - x\gamma(t)v_x\Big)F(x,t),$$
$$P^2 = -\frac{v_x}{2}F(x,t)\,, \tag{7.5}$$

and $(D_x + \lambda_1)P^1 + (D_t + \lambda_2)P^2 = 0$, or corresponds to $D_xP^1 + D_tP^2 + \lambda_1P^1 + \lambda_2P^2 = 0$ , is a $\mu$-conservation law for second order Lagrangian $\mathcal{L}_{\Delta_{Kv}}[v]$. Therefore we have obtained the following corollary:

**Corollary 7.1** *$\mu$-conservation law for the nvcKdV in potential form $\Delta_{Kv} = E(\mathcal{L}_{\Delta_{Kv}})$ is as*

$$D_xP^1 + D_tP^2 + \lambda_1P^1 + \lambda_2P^2 = 0, \tag{7.6}$$

*where $P^1$ and $P^2$ are the $M-$vector $P^i$ of (7.5).*

**Remark 7.1** *$\mu$-conservation law for the nvcKdV in potential form $\Delta_{Kv}$ , satisfying to the Noether's Theorem for $\mu$-symmetry, i.e.*

$$\begin{aligned}(D_i + \lambda_i)P^i &= (D_x + \lambda_1)P^1 + (D_t + \lambda_2)P^2\\ &= F(x,t)\Big(v_{xt} + \alpha(t)(v_{xxxx} + 6v_xv_{xx}) + 4\beta(t)v_{xx}\\ &- \gamma(t)(xv_{xx} + 2v_x)\Big)\\ &= QE(\mathcal{L}_{\Delta_{Kv}})\,.\end{aligned} \tag{7.7}$$

*7.2 $\mu$-conservation laws of the non-isospectral KdV type equation with variable coefficients*

We consider the nvcKdV in potential form

$$\Delta_{Kv} = v_{xt} + \alpha(t)(v_{xxxx} + 6v_xv_{xx}) + 4\beta(t)v_{xx} - \gamma(t)(xv_{xx} + 2v_x) = 0,$$

or equivalently

$$\begin{aligned}D_x\Big(v_t + \alpha(t)(v_{xxx} + 3v_x^2) + 4\beta(t)v_x - \gamma(t)(xv_x + 2v)\Big) &= 0,\\ v_t + \alpha(t)(v_{xxx} + 3v_x^2) + 4\beta(t)v_x - \gamma(t)(xv_x + 2v) &= F_1(t),\end{aligned}$$

where $F_1(t)$ is an arbitrary function. If we substitute

$$F_1(t) - \alpha(t)(v_{xxx} + 3v_x^2) - 4\beta(t)v_x + \gamma(t)(xv_x + 2v)$$

for $v_t$ and substitute $u$ for $v_x$ in the $M-$vector $P^i$ of (7.5), then, we obtain $M-$vectors $P^1$ and $P^2$ as the following

$$\begin{aligned} P^1 =& -\frac{1}{2}\Big(F_1(t) + \alpha(t)(u_{xx} + 5u^2) + 4\beta(t)u - \gamma(t)(xu \\ &+ 2\int udx)\Big)F(x,t), \\ P^2 =& -\frac{u}{2}F(x,t)\,. \end{aligned} \tag{7.8}$$

Therefore we have obtained the following corollary:

**Corollary 7.2** *$\mu$-conservation law for the nvcKdV is as*

$$D_xP^1 + D_tP^2 + \lambda_1P^1 + \lambda_2P^2 = 0, \tag{7.9}$$

*where $P^1$ and $P^2$ are the $M-$vector $P^i$ of (7.8).*

**Remark 7.2** *The nvcKdV satisfying to the characteristic form, i.e.*

$$\begin{aligned} (D_i + \lambda_i)P^i =& (D_x + \lambda_1)P^1 + (D_t + \lambda_2)P^2 \\ =& F(x,t)(u_t + \alpha(t)(u_{xxx} + 6uu_x) + 4\beta(t)u_x \\ &- \gamma(t)(xu_x + 2u)) \\ =& Q\Delta_{Ku}\,. \end{aligned} \tag{7.10}$$

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