Elastic-Plastic Transition of Pressurized Functionally Graded Orthotropic Cylinder using Seth's Transition Theory

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ABSTRACT

In this paper the radial deformation and the corresponding stresses in a functionally graded orthotropic hollow cylinder with the variation in thickness and density according to power law and rotating about its axis under pressure is investigated by using Seth's transition theory. The material of the cylinder is assumed to be non-homogeneous and orthotropic. This theory helps to achieve better agreement between experimental and theoretical results. Results has been mentioned analytically and numerically. From the analysis, it has been concluded that cylinder made up of orthotropic material whose thickness increases radially and density decreases radially is on the safer side of the design as circumferential stresses are high for cylinder made up of isotropic material as compared to orthotropic material. This paper is based on elastic-plastic behavior which plays important role in practical design of structures for safety factor.

Keywords: Elastic-plastic; Orthotropic; Pressure; Functionally graded material; Cylinder.

1 INTRODUCTION

ORTHOTROPIC structures are very common in present day engineering. Orthotropic cylinder has gained widespread use and acceptance, and has already earned worldwide popularity in almost all kinds of applications, housing, marine, highway bridge deck, aerospace and for strengthening of structures. In recent years, the problem of elastic-plastic deformation in composite cylinders made up of functionally graded materials (FGMs) operating at high pressure and temperature has attracted the interest of the many researchers. For an improved usage of the material, it is necessary to allow variation of the effective material properties in one direction of cylinders. The analysis of rotating functionally graded orthotropic cylinders has been reported rarely in the literature. Author A.F. Bower [1] has mentioned the behavior of orthotropic cylinders and E.J. Hearn [2] discussed the anisotropic behavior of materials. G.H. Kim et al. [3] investigated the several fracture problems using new interaction integral formulation and compared the result with analytic solutions. A.M. Zenkour [4] determined the analytic solutions for the rotating orthotropic cylinders of variable and uniform thickness and concluded that varying thickness in cylinders shows excellent result. S. Dag [5] gave a new computational technique based on the equivalent domain integral (EDI) for fracture analysis of orthotropic functionally graded materials subjected to thermal stresses and

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concluded that among the three principal thermal expansion coefficient components, the in-plane component perpendicular to the crack axis has the foremost vital influence on the stress intensity factor. M. Paschero et al. [6] analyzed the buckling of an axially-loaded orthotropic circular cylinder by defining orthotropic material properties in terms of associated geometric mean. H.M. Wang [7] has obtained closed form solutions for pressurized orthotropic cylinders using the Lame’s equations and result obtained shows good agreement with numerical simulation results using finite element analysis. G.J. Nie et al. [8] determined analytically static plane-strain deformations of functionally graded orthotropic cylinders with elliptic inner and circular outer surfaces. Authors solved the problem by employing Fourier and the Frobenius series using the assumption that four relevant elastic moduli are with same variation in the radial direction. S. Sharma et al. [9] determined thermal creep stresses and strain rates in a functionally graded stainless steel composite cylinder using finite difference method and concluded that material anisotropy may have beneficial effect on stresses. The result obtained using small strain theory is found to be on unsafe side when compared to those obtained using finite strain theory. Seth’s transition theory act as a bench mark in dealing with the problems of elastic-plastic and creep deformation i.e. applied by various authors i.e. S.K. Gupta et al. [10] determined the stresses for orthotropic rotating cylinder. B.N Borah [11] investigated the stresses in tubes and mentioned the transition points. A.K. Aggarwal et.al. [12] concluded that by introducing a suitably chosen temperature gradient, non-homogeneous compressible circular cylinder with internal and external pressure for non-linear measure is on the safer aspect of the design as compared to the cylinder without temperature. S. Sharma et al. [13] investigated stresses in transversely isotropic cylinder under pressure and concluded that transversely isotropic cylinder is on safer side as compared to isotropic cylinder. Safety analysis has been done for the torsion of a functionally graded thick-walled circular cylinder under internal and external pressure subjected to thermal loading by S. Sharma et al. [14] and concluded that in creep torsion cylinder made up of less functionally graded material under pressure is better choice for designing point of view as compared to homogeneous cylinder.

2 OBJECTIVE OF THE STUDY

In order to clarify the transition from elastic to plastic state, firstly, we need to recognize transition state as an asymptotic one and in this present study, it is our main aim to eliminate the necessity of yield condition, elastic-plastic, jump conditions and semi-empirical laws etc. The objective of this paper is to calculate stresses for thick-walled functionally graded rotating orthotropic cylinder under internal and external pressure using the concept of transition theory which will act as a benchmark and helpful in practical design of orthotropic cylinder.

3 MATHEMATICAL FORMULATION

Consider a thick-walled orthotropic cylinder made up of functionally graded material with internal and external radii \( a \) and \( b \) respectively, subjected to internal and external pressure \( p_1 \) and \( p_2 \) respectively. The non-homogeneity in the cylinder is due to variation of thickness, density and compressibility \( C \). In cylindrical polar co-ordinates, the components of displacements are given as:

\[
u = r(1-\kappa), \quad \nu = 0 \quad \text{and} \quad \nu = dz,
\]

where \( \kappa \) is a function of \( r \) only and \( d \) is a constant.

Seth has defined the generalized principal strain measure \( e_{ii} \) by taking the integral of the weighted function as:

\[
e_{ii} = \int_{0}^{\frac{1}{2}} \left[ \frac{1}{1 - 2e_{ii}^A} \right]^{(n-1)} d e_{ii}^A = \frac{1}{n} \left[ 1 - (1 - 2e_{ii}^A)^2 \right]^{\frac{n}{2}} (i = 1, 2, 3),
\]

where \( n \) is the measure and \( e_{ii}^A \) are the principal finite components of strain.

Using (2a) the generalized components of strain are,
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\[ e_{rr} = \frac{1}{n} \left[ 1 - (r \kappa' + \kappa)^n \right], e_{\theta \theta} = \frac{1}{n} \left[ 1 - \kappa^n \right], e_{zz} = \frac{1}{n} \left[ 1 - (1 - d)^n \right], e_{r \theta} = e_{r z} = e_{\theta r} = 0, \]  
\text{(2b)}

where \( n \) is the non-linear measure and \( \kappa' = \frac{d \kappa}{d r} \).

The component of stress for orthotropic material is given as:

\[
\begin{align*}
\tau_{rr} & = \frac{C_{11}}{n} \left[ 1 - (r \kappa' + \kappa)^n \right] + \frac{C_{12}}{n} \left[ 1 - \kappa^n \right] + \frac{C_{13}}{n} \left[ 1 - (1 - d)^n \right], \\
\tau_{\theta \theta} & = \frac{C_{22}}{n} \left[ 1 - (r \kappa' + \kappa)^n \right] + \frac{C_{23}}{n} \left[ 1 - \kappa^n \right] + \frac{C_{33}}{n} \left[ 1 - (1 - d)^n \right], \\
\tau_{zz} & = \frac{C_{31}}{n} \left[ 1 - (r \kappa' + \kappa)^n \right] + \frac{C_{32}}{n} \left[ 1 - \kappa^n \right] + \frac{C_{33}}{n} \left[ 1 - (1 - d)^n \right].
\end{align*}
\]
\text{(3)}

where \( \tau_{rr}, \tau_{\theta \theta} \) and \( \tau_{zz} \) are the radial, circumferential and axial stresses respectively.

Taking non-homogeneity in orthotropic material [7] as:

\[
\begin{align*}
C_{11} & = C_{011} \left( \frac{r}{b} \right)^{-a}, & C_{12} & = C_{012} \left( \frac{r}{b} \right)^{-a}, & C_{13} & = C_{013} \left( \frac{r}{b} \right)^{-a}, & C_{21} & = C_{021} \left( \frac{r}{b} \right)^{-a}, & C_{22} & = C_{022} \left( \frac{r}{b} \right)^{-a}, \\
C_{23} & = C_{023} \left( \frac{r}{b} \right)^{-a}, & C_{31} & = C_{031} \left( \frac{r}{b} \right)^{-a}, & C_{32} & = C_{032} \left( \frac{r}{b} \right)^{-a}, & C_{33} & = C_{033} \left( \frac{r}{b} \right)^{-a},
\end{align*}
\]
\text{(4)}

where \( a \leq r \leq b, k (\leq 0) \) is non-homogeneity parameter and \( C_{011}, C_{012}, C_{013}, C_{021}, C_{022}, C_{023}, C_{031}, C_{032}, C_{033} \) are material constants.

Using Eq. (4) in Eq. (3) we get,

\[
\begin{align*}
\tau_{rr} & = \frac{C_{011}}{n} \left( \frac{r}{b} \right)^{-a} \left[ 1 - (r \kappa' + \kappa)^n \right] + \frac{C_{012}}{n} \left( \frac{r}{b} \right)^{-a} \left[ 1 - \kappa^n \right] + \frac{C_{013}}{n} \left( \frac{r}{b} \right)^{-a} \left[ 1 - (1 - d)^n \right], \\
\tau_{\theta \theta} & = \frac{C_{021}}{n} \left( \frac{r}{b} \right)^{-a} \left[ 1 - (r \kappa' + \kappa)^n \right] + \frac{C_{022}}{n} \left( \frac{r}{b} \right)^{-a} \left[ 1 - \kappa^n \right] + \frac{C_{023}}{n} \left( \frac{r}{b} \right)^{-a} \left[ 1 - (1 - d)^n \right], \\
\tau_{zz} & = \frac{C_{031}}{n} \left( \frac{r}{b} \right)^{-a} \left[ 1 - (r \kappa' + \kappa)^n \right] + \frac{C_{032}}{n} \left( \frac{r}{b} \right)^{-a} \left[ 1 - \kappa^n \right] + \frac{C_{033}}{n} \left( \frac{r}{b} \right)^{-a} \left[ 1 - (1 - d)^n \right],
\end{align*}
\]
\text{(5)}

\( \tau_{r \theta} = \tau_{r z} = \tau_{r r} = 0 \), where \( \tau_{ij}, e_{ij} \) are stress and strain tensors respectively.

Equations of equilibrium [10] is given as,

\[
\frac{d}{dr} \left( h r \tau_{rr} \right) - h \tau_{r \theta} + h \rho \omega^2 r^2 = 0,
\]
\text{(6)}

where \( h = h_b \left( \frac{r}{b} \right)^{-q} \) is the wall thickness of the rotating cylinder, \( \rho = \rho_0 \left( \frac{r}{b} \right)^{-q} \) is the density of the rotating cylinder, \( \omega \) is the angular speed of the rotating cylinder.

4 IDENTIFICATION OF TRANSITION POINT

When a deformable solid is subjected to internal and external loading, it has been observed that the solid first deforms elastically. If the loading is sustained, plastic flow might set in. So, there exists an intermediate state in between elastic and plastic state that is known as transition state. Thus, differential system defining the elastic state
should reach a critical value in the transition state. The nonlinear differential equation at transition state is obtained by substituting, Eq. (5) in Eq. (6), as:

\[
\frac{dk}{dr} [hP(P + 1)^n + \frac{C_{01}(\frac{r}{b})^{-k}}{C_{02}(\frac{r}{b})^{-k}} hP - \frac{\rho q^2 r^2 h}{nC_{01}(\frac{r}{b})^{-k}} \kappa^n + \frac{h}{nC_{01}(\frac{r}{b})^{-k}} \kappa^n] \{(k + t - 1)C_{01}(\frac{r}{b})^{-k} + C_{02}(\frac{r}{b})^{-k} \} \{| - \kappa' (P + 1)^n\} \\
+ \{(k + t - 1)C_{01}(\frac{r}{b})^{-k} + C_{02}(\frac{r}{b})^{-k} \} \{| - \kappa'\} + \{(k + t - 1)C_{01}(\frac{r}{b})^{-k} + C_{02}(\frac{r}{b})^{-k} \} \{| - (1 - d')\} + C_{01}(\frac{r}{b})^{-k}} \{| - (1 - d')\} + h\kappa P(P + 1)^{n-1} = 0,
\]

where \( r\kappa' = P\kappa \).

The transition points of \( k \) in Eq. (5) are \( P \to 0, P \to -1 \) and \( P \to \pm\infty \).

The boundary conditions are

\[
\tau_r = -p_1 \text{ at } r = a \text{ and } \tau_r = -p_2 \text{ at } r = b.
\]

The resultant force normal to plane \( Z = \text{constant} \) must vanish, i.e.

\[
\int_a^b r\tau_r dr = 0.
\]

5 TRANSITIONAL AND PLASTIC STRESSES

As elastic state can go to plastic state under internal and external loading through a transition state, thus it has been shown [10-13] that the asymptotic solution through principal stress leads from elastic state to plastic state at transition point \( P \to \pm\infty \). To determine the plastic stresses at the transition point \( P \to \pm\infty \), we define the transition function \( TR \) in terms of \( \tau_{rr} \) as:

\[
TR = 1 - \frac{n\tau_r}{C_{01}(\frac{r}{b})^{-k} + C_{02}(\frac{r}{b})^{-k} + C_{03}(\frac{r}{b})^{-k} - 2(C_{01}(\frac{r}{b})^{-k} + C_{02}(\frac{r}{b})^{-k} + C_{03}(\frac{r}{b})^{-k})}. \tag{10}
\]

Using the value of \( \tau_{rr} \) from Eq. (3) in Eq. (10)

\[
TR = \frac{1}{C_{01}(\frac{r}{b})^{-k} + C_{02}(\frac{r}{b})^{-k} + C_{03}(\frac{r}{b})^{-k}} \left[ C_{01}(\frac{r}{b})^{-k} \kappa^n (P + 1)^n + C_{02}(\frac{r}{b})^{-k} \kappa^n + C_{03}(\frac{r}{b})^{-k} (1 - d')^n - \frac{n\rho q^2 r^2}{2}\right]. \tag{11}
\]

Taking the logarithmic differentiation of Eq. (11), one gets

\[
\frac{d\log TR}{dr} = \frac{k}{r} + \frac{n\beta' C_{01}(\frac{r}{b})^{-k}}{rR(C_{01} + C_{02} + C_{03})(\frac{r}{b})^{-k}} \left[ P(P + 1)^{n-1} \kappa \frac{dp}{dk} + (P + 1)^n + \frac{C_{01}(\frac{r}{b})^{-k}}{C_{01}(\frac{r}{b})^{-k}} P - \frac{k}{n} (P + 1)^{n-1} + \frac{C_{01}(\frac{r}{b})^{-k}}{C_{01}(\frac{r}{b})^{-k}} \right] \\
+ \frac{C_{01}(\frac{r}{b})^{-k}}{C_{01}(\frac{r}{b})^{-k}} \kappa', \frac{2\rho q^2 r^2 (2 - q)}{2C_{01}(\frac{r}{b})^{-k}} \kappa^n. \tag{12}
\]
Substituting the value of $\frac{dP}{d\kappa}$ in Eq. (12) from Eq. (7), we get

$$d\log TR = \frac{k}{r} + \frac{nk'C_{011}(\frac{r}{b})^{-k}}{rR(C_{011} + C_{012}, C_{013})(\frac{r}{b})^{-k}}\left[-\frac{1}{h}P(P+1)^s + \frac{C_{011}(\frac{r}{b})^{-k}}{nC_{011}(\frac{r}{b})^{-k}} - \frac{\rho\omega^2 r^2}{h} + \frac{h}{nC_{011}(\frac{r}{b})^{-k}}\right].$$

Taking asymptotic value of $\kappa$ as $P \to \pm \infty$ in Eq. (13) as,

$$d\log TR = \frac{k}{r} + \frac{C_{011}(\frac{r}{b})^{-k} - \{(1-t)C_{011}(\frac{r}{b})^{-k}\}}{rC_{011}(\frac{r}{b})^{-k}}.$$ (14)

Integrating Eq. (14) we get

$$TR = A_k r.$$ (15)

where $A_k$ is the constant of integration.

Using Eq. (10) in (15), we get

$$\tau_{rr} = B_0\left[1 - A_k r^k \right] - \frac{\rho\omega^2 r^2}{2},$$

where $B_0 = \frac{(C_{011} + C_{012}, C_{013})(\frac{r}{b})^{-k}}{n}.$ (16)

Substituting Eq. (16) in Eq. (6), we have

$$\tau_{\theta\theta} = B_0\left[1 - k - t - \frac{C_{021}(\frac{r}{b})^{-k}}{C_{011}(\frac{r}{b})^{-k}} A_k r^k \right] + \left( t + q - 1 \right) \frac{\rho\omega^2 r^2}{2}.$$ (17)

Using Eq. (2) and third equation of Eq. (5), one get

$$\tau_{zz} = \frac{C_{032}(\frac{r}{b})^{-k}}{(C_{012} + C_{022})(\frac{r}{b})^{-k}}\left[\tau_{rr} + \tau_{\theta\theta}\right] + \frac{C_{031}(\frac{r}{b})^{-k}}{(C_{012} + C_{022})(\frac{r}{b})^{-k}} e_{rr} - \frac{C_{032}(\frac{r}{b})^{-k} (C_{011}(\frac{r}{b})^{-k} + C_{021}(\frac{r}{b})^{-k})}{(C_{012} + C_{022})(\frac{r}{b})^{-k}} e_{rr}$$

$$+ \frac{C_{032}(\frac{r}{b})^{-k} (C_{031}(\frac{r}{b})^{-k} + C_{032}(\frac{r}{b})^{-k})}{(C_{012} + C_{022})(\frac{r}{b})^{-k}} e_{zz}. $$ (18)
The values of constants $A_0$ and $B_0$ are obtained by substituting boundary conditions from Eq. (8) in Eq. (16) as,

$$A_0 = \frac{\rho \omega^2 (b^2 - a^2) + p_1 - p_2}{(\rho \omega^2 b - p_2) \alpha - (\rho \omega^2 a^2 - p_1) b \alpha}$$

$$B_0 = \frac{\rho \omega^2 a^2 + p_1 - p_2}{(\rho \omega^2 b - p_2) \alpha - (\rho \omega^2 a^2 - p_1) b \alpha}$$

where $\alpha = k + t - 1 + \frac{C_{021}}{C_{011}}$.

Tresca specifies that yielding in any material occurs i.e. material will flow plastically when maximum shear stress is equal to yield stress of the material. This maximum shear stress is equal to half the difference of maximum principle stress and minimum principle stress. In the classical theory, assumptions are used by the authors for this yield criterion to join the two spectrums i.e. elastic region and plastic region while in the case of transition theory this yield criterion has been calculated from the constitutive equations in transition state. Thus, from Eqs. (16) and (17),

$$|\tau_n - \tau_{\theta\theta}| = \left| B_0 [k + t + \left( \frac{C_{021} - C_{011}}{C_{011}} \right) A_0 \alpha] - (p + q) \frac{\rho \omega^2 \alpha^2}{2} \right|.$$  

(20)

It can be seen from Eq. (20) that $|\tau_{rr} - \tau_{\theta\theta}|$ is maximum at $r = a$ which means yielding of the cylinder will takes place at the internal surface. Thus, Eq. (20) can be rewritten as:

$$|\tau_n - \tau_{\theta\theta}|_{_{\nu, \nu}} = |A_1 \omega^2 + A_2 p_1 - A_3 p_2| = Y,$$  

(21)

where $A_1 = \frac{\rho}{2A} \left[ \frac{\alpha b^2}{a} - (t + q) \left( \frac{b}{a} \right) \frac{\alpha^2}{a^2} \right]$, $A_2 = \frac{[(k + t) \left( \frac{b}{a} \right) \frac{\alpha^2}{a^2} + \frac{C_{021} - C_{011}}{C_{011}}]}{A}$, and $A_3 = \frac{\alpha^2}{A}, A = 1 - \left( \frac{b}{a} \right)^a$.

For the material to become fully plastic, the change in volume must be zero under the set of applied forces i.e. volumetric strain = 0. For full plasticity [10], $C_{11} = C_{13} = C_{12}, C_{21} = C_{23} = C_{22}, C_{31} = C_{32} = C_{33}$, Eq. (20) becomes

$$|\tau_n - \tau_{\theta\theta}|_{_{\nu, b}} = \left| B_1 \omega^2 + B_2 p_1 - B_3 p_2 \right| = Y^*,$$  

(22)

where $B_1 = \frac{\rho}{2B} \left[ \left( k + t \right) \left( \frac{a}{b} \right) \frac{\alpha^2}{b^2} - \left( k + t \right) \left( \frac{b}{a} \right) \frac{\alpha^2}{a^2} + \left( \frac{C_{022} - C_{011}}{C_{011}} \right) (1 - \frac{a^2}{b^2}) - (t + q)B \right], B_2 = \frac{\xi}{B}$, and $B_3 = \frac{\xi}{B}$.

Now we introduce the following non-dimensional component as:

$$R = \left( \frac{r}{b} \right), R_0 = \left( \frac{a}{b} \right), \sigma_{rr} = \left[ \frac{\tau_n}{Y} \right], \sigma_{\theta\theta} = \left[ \frac{\tau_{\theta\theta}}{Y} \right].$$

Angular speed required for initial yielding can be rewritten from Eq. (21) in non-dimensional form as:

$$\left| \frac{\rho \omega^2 b^2}{Y} \right| = \Omega_1^2 = \frac{1}{A_1} \left| 1 - A_2 p_1 + A_3 p_2 \right|,$$  

(23)

where $p_1 = \frac{P_1}{Y}$ and $p_2 = \frac{P_2}{Y}$.

Also, angular speed required for fully plasticity can be rewritten from Eq. (22) in non-dimensional form as:
\[
\left| \frac{\rho \omega^2 b^2}{Y'} \right| = \Omega_p^2 = \frac{1}{B_i} \left| \frac{1}{P_i} \right| \left| I - B^2 P_i + B_2 P_2 \right|, \text{ where } P_1 = \frac{P_i}{Y'} \text{ and } P_2 = \frac{P_2}{Y'}.
\]

(24)

Transitional stresses are given as:

\[
\sigma_n = \left| \frac{\tau_n}{Y'} \right| = \left( \frac{\Omega_p^2 - P_2 R_0^a - (\Omega_p^2 R_0^a - P_1) - (\Omega_p^2 - \Omega_2^2 R_0^a) R^a - (P_1 - P_2) R^a}{R_0^a - 1} \right) - \Omega_p^2 R^a,
\]

(25)

\[
\sigma_0 = \left| \frac{\tau_0}{Y'} \right| = \left( \frac{(1-k-t)(\Omega_p^2 - P_2 R_0^a - (\Omega_p^2 R_0^a - P_1) - \frac{C_{021}}{C_{011}} (\Omega_p^2 - \Omega_2^2 R_0^a) R^a)}{R_0^a - 1} \right) + (t+q-1)\Omega_p^2 R^a.
\]

(26)

Stresses for full plasticity [10] \( C_{11} = C_{13} = C_{12} \), \( C_{21} = C_{23} = C_{22} \), \( C_{31} = C_{32} = C_{33} \) are given as:

\[
\sigma_n^* = \left| \frac{\tau_n^*}{Y'} \right| = \left( \frac{\Omega_p^2 R_0^a - (\Omega_p^2 R_0^a - P_1) - (\Omega_p^2 - \Omega_2^2 R_0^a) R^a - (P_1 - P_2) R^a}{R_0^a - 1} \right) - \Omega_p^2 R^a,
\]

(27)

\[
\sigma_0^* = \left| \frac{\tau_0^*}{Y'} \right| = \left( \frac{(1-k-t)(\Omega_p^2 - P_2 R_0^a - (\Omega_p^2 R_0^a - P_1) - \frac{C_{021}}{C_{011}} (\Omega_p^2 - \Omega_2^2 R_0^a) R^a)}{R_0^a - 1} \right) + (t+q-1)\Omega_p^2 R^a.
\]

(28)

The Eqs. (27) and (28) are fully plastic stresses for orthotropic cylinder made up of functionally graded material under internal and external pressure.

If we substitute \( k = 0, t = 0, q = 0, P_1 = P_2 = 0 \) in Eq. (4), Eq. (6), we have

\[
C_{ij} = C_{ij}, \quad h = h_0, \quad \rho = \rho_0.
\]

(29)

Using Eq. (29) in Eqs. (27-28), the radial, circumferential stresses of orthotropic cylinder becomes

\[
\sigma_n^* = \left| \frac{\tau_n^*}{Y'} \right| = \left( \frac{\Omega_p^2 R_0^a - (\Omega_p^2 R_0^a - P_1) - (\Omega_p^2 - \Omega_2^2 R_0^a) R^a}{R_0^a - 1} \right) - \Omega_p^2 R^a,
\]

(30)

\[
\sigma_0^* = \left| \frac{\tau_0^*}{Y'} \right| = \left( \frac{(1-k-t)(\Omega_p^2 R_0^a - (\Omega_p^2 R_0^a - P_1) - \frac{C_{021}}{C_{011}} (\Omega_p^2 - \Omega_2^2 R_0^a) R^a)}{R_0^a - 1} \right) + (t+q-1)\Omega_p^2 R^a, \text{ where } \xi = \frac{(C_{22} - C_{11})}{C_{11}}
\]

(31)

The Eqs. (30) and (31) are same as obtained by Gupta [10] for orthotropic cylinder made up of homogeneous material.

6 RESULTS AND NUMERICAL DISCUSSION

The material properties of the cylinder made up of functionally graded orthotropic material (Barite and Topaz) and isotropic material (Mild Steel) are defined as:
The inner and outer radii of the cylinder are taken as \( a = 1 \) and \( b = 2 \) respectively. To calculate the transitional and fully plastic stresses based on the above analysis Eq. (23) to Eq. (28) have been evaluated by the use of Mathematica. Curves have been made for angular speed required for initial yielding and fully plastic state with respect to radii ratio \( R_0 \) as shown in Figs. 1-4 for \( k = -5, -4, -3, -2 \) respectively under various internal and external pressure.

It has been observed from Fig. 1 and Tables 2-4, that angular speed required for initial yielding in a rotating cylinder under internal and external pressure is maximum at the external surface. It has also been observed that high angular speed is required for initial yielding for the rotating cylinder made up of less non-homogeneous material as compared to rotating cylinder made up of highly non-homogeneous material. It has been noticed from Table 2, Table 3 and Table 4 that angular speed required for initial yielding is maximum for orthotropic material i.e. topaz as compared to orthotropic material i.e. barite and isotropic material i.e. mild steel. It has also been observed that with the decrease in non-homogeneity (\( k = -2 \) to \( k = -5 \)) angular speed required for initial yielding increases significantly. It has been noticed from Fig. 2 that with the increase in internal and external pressure angular speed required for initial yielding increases significantly for rotating cylinder made up of orthotropic and isotropic materials.

It has been observed from Fig. 3, Table 2, Table 3, and Table 4 that angular speed required for fully plastic state is maximum at the internal surface for rotating cylinder made up of orthotropic and isotropic materials. It has also been observed that angular speed required for full plasticity is less for highly non-homogeneous rotating cylinder. It has also been observed that angular speed required for fully plasticity decreases with the increase in non-homogeneity of rotating cylinder under internal and external pressure.

It has been noticed from Tables 2-4, that percentage increase in angular velocity required for initial yielding to become fully plastic is high for isotropic material as compared to orthotropic materials at the internal surface. Also this percentage increase in angular velocity is maximum for less non-homogeneous rotating cylinder as compared to highly non-homogeneous rotating cylinder. Out of two orthotropic materials i.e. barite and topaz, angular speed required for initial yielding to become fully plastic is high for barite as compared to topaz. With the increase in pressure, this percentage increases significantly as can be seen from Tables 5-7. It has also been observed that angular speed required for full plasticity is high for orthotropic material barite as compared to orthotropic material topaz and isotropic material mild steel. It has been noticed from Fig. 2 that with the increase in internal and external pressure angular speed required for initial yielding increases significantly for rotating cylinder made up of orthotropic and isotropic materials. It can be seen from Fig. 4 that with the increase in pressure, angular speed required for full plasticity increases significantly for both orthotropic and isotropic materials.

### Table 1

Elastic constants \( C_{ij} \) used (in units of \( 10^{11} \ N/m^2 \)).

<table>
<thead>
<tr>
<th>Materials</th>
<th>( C_{11} )</th>
<th>( C_{12} )</th>
<th>( C_{13} )</th>
<th>( C_{21} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel (Isotropic Material)</td>
<td>2.908</td>
<td>1.27</td>
<td>1.27</td>
<td>2.908</td>
</tr>
<tr>
<td>Barite (Orthotropic Material)</td>
<td>0.907</td>
<td>0.273</td>
<td>0.275</td>
<td>0.273</td>
</tr>
<tr>
<td>Topaz (Orthotropic Material)</td>
<td>2.813</td>
<td>1.258</td>
<td>0.846</td>
<td>1.258</td>
</tr>
</tbody>
</table>

### Table 2

Percentage in angular speed required for initial yielding to become fully plastic state for the orthotropic cylinder made up of barite material under internal pressure \( = 2 \) and external pressure \( = 0.5 \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \Omega^2 / R_0 )</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-5)</td>
<td>Initial Yielding</td>
<td>0.0196756</td>
<td>0.0782957</td>
<td>0.162032</td>
<td>96.064</td>
<td>84.132</td>
<td>61.844</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.499902</td>
<td>0.493412</td>
<td>0.424652</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-4)</td>
<td>Initial Yielding</td>
<td>0.0196068</td>
<td>0.0773809</td>
<td>0.149209</td>
<td>96.075</td>
<td>84.004</td>
<td>60.221</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.499517</td>
<td>0.483754</td>
<td>0.375098</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-3)</td>
<td>Initial Yielding</td>
<td>0.0194884</td>
<td>0.0752184</td>
<td>0.123576</td>
<td>96.084</td>
<td>83.655</td>
<td>57.863</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.497633</td>
<td>0.460204</td>
<td>0.293275</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-2)</td>
<td>Initial Yielding</td>
<td>0.0192104</td>
<td>0.0694246</td>
<td>0.065533</td>
<td>96.068</td>
<td>82.808</td>
<td>59.029</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.488589</td>
<td>0.403819</td>
<td>0.15995</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Table 3
Percentage in angular speed required for initial yielding to become fully plastic state for the orthotropic cylinder made up of topaz material under internal pressure = 2 and external pressure = 0.5.

<table>
<thead>
<tr>
<th>k</th>
<th>$\Omega^2 / R_0$</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>Initial Yielding</td>
<td>0.0211374</td>
<td>0.0841131</td>
<td>0.174897</td>
<td>95.771</td>
<td>82.855</td>
<td>57.466</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.499804</td>
<td>0.490602</td>
<td>0.411193</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-4</td>
<td>Initial Yielding</td>
<td>0.0213603</td>
<td>0.0843304</td>
<td>0.164596</td>
<td>95.72</td>
<td>82.331</td>
<td>53.6349</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.499054</td>
<td>0.477277</td>
<td>0.355</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>Initial Yielding</td>
<td>0.0216771</td>
<td>0.0838446</td>
<td>0.090883</td>
<td>95.389</td>
<td>78.499</td>
<td>29.0805</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.479644</td>
<td>0.375913</td>
<td>0.12815</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4
Percentage in angular speed required for initial yielding to become fully plastic state for the cylinder made up of isotropic material under internal pressure = 2 and external pressure = 0.5.

<table>
<thead>
<tr>
<th>k</th>
<th>$\Omega^2 / R_0$</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>Initial Yielding</td>
<td>0.0210355</td>
<td>0.0837079</td>
<td>0.173999</td>
<td>95.7919</td>
<td>83.0143</td>
<td>58.7234</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.499884</td>
<td>0.492814</td>
<td>0.421544</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-4</td>
<td>Initial Yielding</td>
<td>0.0212381</td>
<td>0.0838462</td>
<td>0.163521</td>
<td>95.7475</td>
<td>82.6174</td>
<td>55.8530</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.49943</td>
<td>0.482358</td>
<td>0.370401</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>Initial Yielding</td>
<td>0.0215246</td>
<td>0.0832431</td>
<td>0.141413</td>
<td>95.6711</td>
<td>81.7877</td>
<td>50.6532</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.486822</td>
<td>0.397403</td>
<td>0.265246</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5
Percentage in angular speed required for initial yielding to become fully plastic state for the orthotropic cylinder made up of barite material under internal pressure = 3 and external pressure = 1.

<table>
<thead>
<tr>
<th>k</th>
<th>$\Omega^2 / R_0$</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>Initial Yielding</td>
<td>0.0373455</td>
<td>0.148935</td>
<td>0.319417</td>
<td>96.265</td>
<td>84.955</td>
<td>64.003</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.999849</td>
<td>0.989961</td>
<td>0.887354</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-4</td>
<td>Initial Yielding</td>
<td>0.0368102</td>
<td>0.146073</td>
<td>0.30047</td>
<td>96.316</td>
<td>85.02</td>
<td>62.999</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.999251</td>
<td>0.975088</td>
<td>0.812056</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>Initial Yielding</td>
<td>0.0359897</td>
<td>0.140838</td>
<td>0.263744</td>
<td>96.388</td>
<td>84.991</td>
<td>61.542</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.996298</td>
<td>0.938373</td>
<td>0.685791</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>Initial Yielding</td>
<td>0.0345259</td>
<td>0.129363</td>
<td>0.181622</td>
<td>96.484</td>
<td>84.754</td>
<td>61.68</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.98185</td>
<td>0.84849</td>
<td>0.473966</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6
Percentage in angular speed required for initial yielding to become fully plastic state for the orthotropic cylinder made up of topaz material under internal pressure = 3 and external pressure = 1.

<table>
<thead>
<tr>
<th>k</th>
<th>$\Omega^2 / R_0$</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>Initial Yielding</td>
<td>0.0392945</td>
<td>0.156705</td>
<td>0.336841</td>
<td>96.069</td>
<td>84.101</td>
<td>61.1422</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.999698</td>
<td>0.985639</td>
<td>0.866855</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-4</td>
<td>Initial Yielding</td>
<td>0.0391484</td>
<td>0.155372</td>
<td>0.32144</td>
<td>96.079</td>
<td>83.899</td>
<td>58.836</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.998529</td>
<td>0.965009</td>
<td>0.780877</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>Initial Yielding</td>
<td>0.0389094</td>
<td>0.152419</td>
<td>0.290089</td>
<td>96.081</td>
<td>83.356</td>
<td>54.7177</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.992944</td>
<td>0.915752</td>
<td>0.640624</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>Initial Yielding</td>
<td>0.0384054</td>
<td>0.144748</td>
<td>0.216741</td>
<td>96.029</td>
<td>81.947</td>
<td>47.9497</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.967176</td>
<td>0.801784</td>
<td>0.416407</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 7
Percentage in angular speed required for initial yielding to become fully plastic state for the cylinder made up of isotropic material under internal pressure = 3 and external pressure = 1.

<table>
<thead>
<tr>
<th>K</th>
<th>$\Omega^2 / R_0$</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>Initial Yielding</td>
<td>0.0391587</td>
<td>0.156164</td>
<td>0.335625</td>
<td>96.0834</td>
<td>84.2106</td>
<td>61.9746</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.999821</td>
<td>0.989043</td>
<td>0.88263</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-4</td>
<td>Initial Yielding</td>
<td>0.0389855</td>
<td>0.154724</td>
<td>0.319974</td>
<td>96.0980</td>
<td>84.0970</td>
<td>60.2418</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.999117</td>
<td>0.972922</td>
<td>0.804801</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>Initial Yielding</td>
<td>0.038706</td>
<td>0.151612</td>
<td>0.288241</td>
<td>96.1125</td>
<td>83.7574</td>
<td>57.3038</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.995664</td>
<td>0.933423</td>
<td>0.675097</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>Initial Yielding</td>
<td>0.0381351</td>
<td>0.143672</td>
<td>0.214268</td>
<td>96.1046</td>
<td>82.8536</td>
<td>53.3947</td>
</tr>
<tr>
<td></td>
<td>Full Plasticity</td>
<td>0.97898</td>
<td>0.837914</td>
<td>0.45975</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 1
Angular speed required for initial yielding for Barite, Topaz and Mild Steel respectively with ($P_1 = 2$ and $P_2 = 0.5$).

Fig. 2
Angular speed required for initial yielding for Barite, Topaz and Mild Steel respectively with ($P_1 = 3$ and $P_2 = 1$).
It has been observed from Table 8, that circumferential transitional stresses are maximum at the internal surface for rotating cylinder under internal and external pressure with angular speed $\Omega = 5$. From Fig. 5, it has also been noticed that circumferential transitional stresses are less for highly non-homogeneous rotating cylinder and these stresses increases with the decrease in non-homogeneity. Also, these circumferential transitional stresses are high for barite as compared to topaz and mild steel. From Fig. 6, it can be seen that with the increase in pressure, circumferential stresses decrease significantly which further decrease with the increase in pressure. It has been observed from Table 9 that fully plastic circumferential stresses are maximum at the internal surface for isotropic rotating cylinder i.e. mild steel while these stresses are maximum at the centre of the cylinder for orthotropic rotating cylinder made up of barite and topaz. Also, from Fig. 7, it can be seen that these circumferential stresses are high for less non-homogeneous rotating cylinder as compared to highly non-homogeneous rotating cylinder. Also circumferential stresses are high for cylinder made up of isotropic material as compared to orthotropic material.
has also been noticed from Table 9, Fig. 8 that with the increase in pressure these circumferential stresses decrease significantly.

**Table 8**
Transitional circumferential stresses under internal and external pressure with angular speed (Ω^2 = 5) for barite, topaz and mild steel materials respectively.

<table>
<thead>
<tr>
<th>Internal Pressure=2</th>
<th>Barite</th>
<th>Topaz</th>
<th>Mild Steel</th>
</tr>
</thead>
<tbody>
<tr>
<td>k/R</td>
<td>0.5</td>
<td>0.75</td>
<td>1</td>
</tr>
<tr>
<td>-5</td>
<td>26.0228</td>
<td>27.9035</td>
<td>38.5812</td>
</tr>
<tr>
<td>-3</td>
<td>19.7203</td>
<td>21.5012</td>
<td>32.7619</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Internal Pressure=3</th>
<th>Barite</th>
<th>Topaz</th>
<th>Mild Steel</th>
</tr>
</thead>
<tbody>
<tr>
<td>k/R</td>
<td>0.5</td>
<td>0.75</td>
<td>1</td>
</tr>
<tr>
<td>-5</td>
<td>22.0388</td>
<td>23.2306</td>
<td>34.9815</td>
</tr>
<tr>
<td>-3</td>
<td>15.0756</td>
<td>16.4056</td>
<td>28.7586</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Internal Pressure=4</th>
<th>Barite</th>
<th>Topaz</th>
<th>Mild Steel</th>
</tr>
</thead>
<tbody>
<tr>
<td>k/R</td>
<td>0.5</td>
<td>0.75</td>
<td>1</td>
</tr>
<tr>
<td>-5</td>
<td>18.0499</td>
<td>20.4856</td>
<td>32.1489</td>
</tr>
<tr>
<td>-3</td>
<td>11.638</td>
<td>14.086</td>
<td>26.5924</td>
</tr>
</tbody>
</table>

**Table 9**
Fully plastic circumferential stresses under internal and external pressure with angular speed (Ω^2 = 5) for barite, topaz and mild steel materials respectively.

<table>
<thead>
<tr>
<th>Internal Pressure=2</th>
<th>Barite</th>
<th>Topaz</th>
<th>Mild Steel</th>
</tr>
</thead>
<tbody>
<tr>
<td>k/R</td>
<td>0.5</td>
<td>0.75</td>
<td>1</td>
</tr>
<tr>
<td>-5</td>
<td>26.0228</td>
<td>27.9035</td>
<td>38.5812</td>
</tr>
<tr>
<td>-3</td>
<td>19.7203</td>
<td>21.5012</td>
<td>32.7619</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Internal Pressure=3</th>
<th>Barite</th>
<th>Topaz</th>
<th>Mild Steel</th>
</tr>
</thead>
<tbody>
<tr>
<td>k/R</td>
<td>0.5</td>
<td>0.75</td>
<td>1</td>
</tr>
<tr>
<td>-5</td>
<td>22.0388</td>
<td>23.2306</td>
<td>34.9815</td>
</tr>
<tr>
<td>-3</td>
<td>15.0756</td>
<td>16.4056</td>
<td>28.7586</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Internal Pressure=4</th>
<th>Barite</th>
<th>Topaz</th>
<th>Mild Steel</th>
</tr>
</thead>
<tbody>
<tr>
<td>k/R</td>
<td>0.5</td>
<td>0.75</td>
<td>1</td>
</tr>
<tr>
<td>-5</td>
<td>18.0499</td>
<td>20.4856</td>
<td>32.1489</td>
</tr>
<tr>
<td>-3</td>
<td>11.638</td>
<td>14.086</td>
<td>26.5924</td>
</tr>
</tbody>
</table>

**Fig. 5**
Transitional circumferential stresses for Barite, Topaz and Mild Steel respectively with (P_1 = 2 and P_2 = 0.5).
Fig. 6
Transitional circumferential stresses for Barite, Topaz and Mild Steel respectively with \( P_1 = 3 \text{ and } P_2 = 1 \).

Fig. 7
Fully plastic circumferential stresses for Barite, Topaz and Mild Steel respectively with \( P_1 = 2 \text{ and } P_2 = 0.5 \).
7 CONCLUSIONS

On the basis of above discussion, it has been concluded that circular cylinder made up of highly functionally graded orthotropic material (Topaz) under internal and external pressure is better choice for designing as compared to cylinder made up of functionally graded orthotropic material (Barite) and isotropic material (Mild Steel). It is because of the reason that circumferential stresses are less for Topaz as compared to Steel and Barite. Also, the cylinder whose thickness increases radially and density decreases radially is on the safer side of design. This leads to the idea of stress savings that minimizes the possibility of fracture of cylinder due to internal and external pressure.

REFERENCES


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