Journal of Advances in Computer Research
Quarterly ISSN: 2008-6148
Sari Branch, Islamic Azad University, Sari, I.R.Iran
(Vol. 3, No. 1, February 2012), Pages: 33-49
www.jacr.iausari.ac.ir

Degree Reduction of Disk Wang-Bézier Type Generalized Ball Curves

Davood Bakhshesh[1]✉, Seyyed Abdol Hamid Esfahani[2]

(1) Department of Computer Science, University of Bojnord, Bojnord, Iran
(2) MA in Engineering and Faculty Member of Islamic Azad University, Qaenat Branch, Iran

dbakhshesh@gmail.com; hamidcom86@gmail.com

Received: 2011/11/09; Accepted: 2011/12/20

Abstract

A disk Wang-Bézier type generalized Ball curve is a Wang-Bézier type generalized Ball curve whose control points are disks in a plane. It can be viewed as a parametric curve with error tolerances. In this paper, we discuss the problem of degree reduction of disk Wang-Bézier type generalized Ball curve, that is, bounding disk Wang-Bézier type generalized Ball curves with lower degree disk Wang-Bézier type generalized Ball curves. We propose an algorithm to solve this problem. The algorithm starts by finding an optimal approximation to the center curve of the original disk Wang-Bézier type generalized Ball curve, which is served as the center curve of the degree reduced disk Wang-Bézier type generalized Ball curve. Then the radius of the degree reduced disk Wang-Bézier type generalized Ball curve is computed by solving some linear programming problems, and for which analytic solutions are obtained.

Keywords: *Wang-Bézier Type Generalized Ball Curve, Degree Reduction, Optimal, Approximation, Degree Elevation*

1. Introduction

In recent years, many investigations have focused on the problem which lacks robustness in geometric design and computing. In order to provide an efficient and reliable method, interval arithmetic is brought into many domains [3-4], [9-12], [16-18]. Interval arithmetic offers an essential and faultless way to monitor error propagation in numerical algorithms that utilize the floating-point arithmetic. Using interval arithmetic will substantially increase the numerical stability in the geometric computations. In addition, it improves the robustness of current CAD/CAM systems. The geometric transform based on rectangle (the sides of which are parallel to the coordinate axes) in a plane is easy to compute in terms of intervals. However it also has some drawbacks [4]. In order to overcome these shortcomings, the symmetrical disk is used to replace asymmetrical rectangle in interval arithmetic. For example [20], a disk with center (7, 0) and the radius 1 is transformed to the disk whose center is $(7/2, 7\sqrt{3}/2)$ and the radius is 2 after a rotation of 60 degrees. Thereby, the area of the disk after transformation is doubled. The cubic Ball curves are first introduced by Ball [1-2]. Since then several researchers have studied the generalized Ball curves [5-8], [13-15]. These have aided in a theoretically conclusion that generalized Ball curves are more efficient than Bézier curves in calculation also with the degree elevation and reduction.

In this paper, the degree reduction problem of disk Wang-Bézier type generalized Ball curves (disk WBGB curves) is discussed. Degree reduction of parametric curves has been widely studied and a wealthy of literature has focused on the problem. This problem is necessary because of the practical need to communicate product data between diverse CAD/CAM systems that impose fundamentally incompatible constraints on their representation schemes. Communicating product data between diverse CAD/CAM systems is a major problem. For example, difference systems have their own limits on the degrees of polynomial curves/surfaces. However, all these degree reduction schemes guarantee only that an approximation satisfies the prescribed tolerance, and none has ever proposed to carry the detailed information on the approximation errors to subsequent applications. Such information can be of crucial importance in geometrical operations in other systems. To solve the major problem we solve the degree reduction problem of disk WBGB curves that we discuss two cases involving non-interpolating endpoints and interpolating endpoints of disk WBGB curves, and an algorithm based on Linear Programming is presented (algorithm based on Optimal Approximation can be obtained in the same way as the Linear Programming). The algorithm is divided into two steps. First, the center curve optimal approximation of the original disk WBGB curves with lower degree is obtained. Then the radius of the degree reduced by WBGB curves is presented. We organize the paper as follows. In Section 2, we introduce the definition of disk WBGB curves. In Section 3, we solve the problem of degree elevation of disk WBGB curves. In Section 4, we propose an efficient algorithm to solve the problem of degree reduction of disk WBGB curves. Two cases-constrained degree reduction and non-constrained degree reduction are considered. In section 5, an example is presented. Finally, we conclude the paper in Section 6.

2. Disk WBGB Curve

In this section, we present some preliminary results for disk arithmetic and disk WBGB curves.

2.1 Disk arithmetic

Let us use $\mathbb{R}$ to denote the set of all the real numbers and $\mathbb{R}^+$ the set of all nonnegative real numbers. A real *disk* in a plane is the point set

$$(\boldsymbol{P}) \coloneqq (\boldsymbol{x_0}, \boldsymbol{y_0})_{\boldsymbol{r_0}} \coloneqq \left\{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^2 \middle| (\boldsymbol{x} - \boldsymbol{x_0})^2 + (\boldsymbol{y} - \boldsymbol{y_0})^2 \le \boldsymbol{r_0^2}\right\}, \qquad (\mathbf{1})$$

where $(x_0, y_0) \in \mathbb{R}^2$ is the center and $r_0 \in \mathbb{R}^+$ is the radius. For any two disks $(x_i, y_i)_{r_i}, i = 1,2,$ the following operations are defined:

$$\boldsymbol{a}(\boldsymbol{x_1}, \boldsymbol{y_1})_{\boldsymbol{r_1}} = (\boldsymbol{a}\boldsymbol{x_1}, \boldsymbol{a}\boldsymbol{y_1})_{|\boldsymbol{a}|\boldsymbol{r_1}}, \ \ \boldsymbol{for}\ \boldsymbol{a} \in \mathbb{R},$$

$$(x_1, y_1)_{r_1} + (x_2, y_2)_{r_2} = (x_1 + x_2, y_1 + y_2)_{r_1 + r_2}. \qquad (2)$$

For the linear combination of n disks, one has

$$\sum_{i=1}^{n} a_i (x_i, y_i)_{r_i} = \left(\sum_{i=1}^{n} a_i x_i\,, \sum_{i=1}^{n} a_i y_i\right)_{\sum_{i=1}^{n} |a_i| r_i}. \qquad (3)$$

2.2 Disk WBGB curves

A disk Wang-Bézier type generalized Ball curve (disk WBGB curve) is a WBGB curve whose control points are disks. Given $n + 1$ disks $(p_i) = (x_i, y_i)_{r_i}, i = 0, 1, \dots, n$, the corresponding disk Wang-Bézier type generalized Ball curve is defined:

$$(\boldsymbol{P})(\boldsymbol{t}, \boldsymbol{L}) = \sum_{i=0}^{n} (\boldsymbol{p_i}) \boldsymbol{\omega_i^n}(\boldsymbol{t}, \boldsymbol{L}), \qquad \mathbf{0} \le \boldsymbol{t} \le \mathbf{1}; \quad \mathbf{0} \le \boldsymbol{L} \le \lfloor (\boldsymbol{n} - \mathbf{1})/\mathbf{2} \rfloor. \tag{4}$$

where L is the given position parameter, $(p_i), i = 0, 1, \dots, n$, are called control disks and $\omega_i^n(t, L), i = 0, 1, \dots, n$ are Wang-Bézier type generalized Ball bases and are defined as follows.

$$\boldsymbol{\omega_i^n}(\boldsymbol{t}, \boldsymbol{L}) = \boldsymbol{\omega_{n-i}^n}(\mathbf{1} - \boldsymbol{t}, \boldsymbol{L}) = \begin{cases} \binom{n-2L}{i} t^i (1-t)^{n-2L-i}, & 0 \le i \le m - L - 1; \\ 2^{i-m+L} \binom{n-2L}{m-L} t^i (1-t)^{i+2}, & m - L \le i \le m - 1; \\ 2^L \binom{n-2L}{m-L} t^{\lfloor \frac{n}{2} \rfloor} (1-t)^{\lceil \frac{n}{2} \rceil}, & i = m; \end{cases} \tag{5}$$

where m is a natural number and $n = 2m + 1$ or $n = 2m$.
The disk WBGB curve $(P)(t, L)$ can also be written as

$$(P)(t, L) = \big(x(t), y(t)\big)_{r(t)}, \tag{6}$$

where

$$x(t) = \sum_{i=0}^{n} x_i \omega_i^n(t, L), \quad y(t) = \sum_{i=0}^{n} y_i \omega_i^n(t, L),$$
$$r(t) = \sum_{i=0}^{n} r_i \omega_i^n(t, L). \tag{7}$$

$C(t) = \big(x(t), y(t)\big) = \sum_{i=0}^{n} c_i \omega_i^n(t, L)$ and $r(t)$ are called the center curve and the radius of the disk WBGB curve $(P)(t, L)$ respectively and $c_i = (x_i, y_i)$. The width of the disk WBGB curve $(P)(t, L)$ is defined to be $\|r(t)\|$. Denote by

$$lec\big(r(t)\big) = \begin{cases} 2^{L-1}(-1)^m \binom{2m-2L}{m-L} (2r_m - r_{m+1} - r_{m-1}) = \\ \qquad 2^{L-1}(-1)^{m+1} \binom{2m-2L}{m-L} \Delta^2 r_{m-1}, \quad n = 2m, \\ 2^L(-1)^m \binom{2m+1-2L}{m-L} (r_{m+1} - r_m) = \\ \qquad 2^L(-1)^m \binom{2m+1-2L}{m-L} \Delta r_m, \quad n = 2m+1, \end{cases} \tag{8}$$

The leading coefficient of polynomial $r(t)$ of degree n.

3. Degree Elevation of Disk WBGB Curve

The degree elevation problem is defined as follows.

Definition 1. Given a degree m disk WBGB curve $(Q)(t,L)=\sum_{j=0}^{m}(q_j)\omega_j^m(t,L)$, The degree elevation problem is to find a degree n $(n > m)$ disk WBGB curve $(\tilde{Q})(t,L)=\sum_{j=0}^{n}(\tilde{q}_j)\omega_j^n(t,L)$ such that $(Q)(t,L) = (\tilde{Q})(t,L)$ for $0 \le t \le 1$.

In the following Theorem, we present $\{(\tilde{q}_j)\}_{j=0}^{n}$ in terms of $\{(q_j)\}_{j=0}^{m}$.

Lemma 1(see [22]). The degree-n Bernstein basis $\{B_i^n(t)\}_{i=0}^n$ can be expressed by the degree-n WBGB basis $\{\omega_i^n(t)\}_{i=0}^n$ as follows:

$$\begin{cases} B_i^n(t)=\sum_{j=0}^{m} a_{i,j}\,\omega_j^n(t,L)+\sum_{j=0}^{m} a_{i,n-j}\,\omega_{n-j}^n(t,L), \\ \qquad i=0,1,\cdots,m, \quad n=2m+1, \\ B_{n-i}^n(t)=\sum_{j=0}^{m} a_{n-i,n-jj}\,\omega_{n-j}^n(t,L)+\sum_{j=0}^{m} a_{n-i,j}\,\omega_j^n(t,L), \end{cases} \tag{9}$$

where

a)

$\mathbf{0 \le j \le m-L}$,

$$a_{i,j}=\binom{n-2L}{j}^{-1}\binom{n}{i}\binom{-2L}{j-i}, \quad a_{i,n-j}=0;$$

b)

$\boldsymbol{m-L+1 \le j \le m}$,

$$a_{i,j}=2^{m-j-L}\binom{n-2L}{m-L}^{-1}\binom{n}{i}\left[\binom{2j-n}{j-i}-\sum_{p=0}^{j-m+l-2}2^p\binom{2j-2-2p-n}{j-i-2-p}\right];$$

$$a_{i,n-j}=\begin{cases} 0, & j=m-L+1, \\ 2^{m-j+L}\binom{n-2L}{m-L}^{-1}\binom{n}{i}\times \\ \left\{-\binom{2j-2-n}{j-i-2}-\sum_{p=1}^{j-m+L-2}2^p\binom{2j-2-2p-n}{j-i-2-p}\right\}, \\ & m-L+2 \le j \le m. \end{cases}$$

c)

$\boldsymbol{a_{i,j} = a_{n-i,j}}, \;\; \boldsymbol{for\;\; m+1 \le i \le n}.$

$$\begin{cases} B_i^n(t) = \sum_{j=0}^{m} a_{i,j}\,\omega_j^n(t,L) + \sum_{j=0}^{m} a_{i,n-j}\,\omega_{n-j}^n(t,L), \\ \qquad\qquad i = 0,1,\cdots,m, \quad n = 2m, \\ B_{n-i}^n(t) = \sum_{j=0}^{m} a_{n-i,n-jj}\,\omega_{n-j}^n(t,L) + \sum_{j=0}^{m} a_{n-i,j}\,\omega_j^n(t,L), \\ \qquad\qquad i = 0,1,\cdots,m-1, \quad n = 2m, \end{cases} \tag{10}$$

where:

a)

$$\boldsymbol{a_{i,j} = \binom{n-2L}{j}^{-1}\binom{n}{i}\binom{-2L}{j-i}}, \qquad \boldsymbol{0 \le j \le m-L},$$

$$a_{i,n-j} = 0, \qquad 0 \le j \le m-L-1;$$

b)

$$\boldsymbol{a_{i,j} = 2^{m-j-L}\binom{n-2L}{m-L}^{-1}\binom{n}{i}\left[\binom{2j-n}{j-i} - \sum_{p=0}^{j-m+L-1} 2^p\binom{2j-2-2p-n}{j-i-2-p}\right]},$$

$$\boldsymbol{m-L+1 \le j \le m}$$

$$a_{i,n-j} = 2^{m-j+L}\binom{n-2L}{m-L}^{-1}\binom{n}{i}\left\{-\binom{2j-2-n}{j-i-2} - \sum_{p=1}^{j-m+L-1} 2^p\binom{2j-2-2p-n}{j-i-2-p}\right\},$$

$$m-L+1 \le j \le m-1;$$

c)

$\boldsymbol{a_{i,j} = a_{n-i,n-j}}, \quad \boldsymbol{for\;\; m+1 \le i \le n}.$

Suppose

$$W_n(t) = \left(w_0^n(t,L), w_1^n(t,L), \cdots, w_n^n(t,L)\right)^T$$

and

$$B_n(t) = \left(B_0^n(t), B_1^n(t), \cdots, B_n^n(t)\right)^T$$

are column vectors of WBGB and Bernstein bases. Then the following formulas hold true:

$$B_n(t) = A_n W_n(t), \quad W_n(t) = A_n^{-1} B_n(t). \tag{11}$$

Here $A_n\left(a_{i,j}\right)_{(n+1)\times(n+1)}$ is defined by (9) and (10) respectively.

Lemma 2 (see [23]). Bernstein polynomial $B_i^m(t)$ of degree m, $0 \le t \le 1$, can be represented by

$$B_i^m(t) = \sum_{j=i}^{n+i-m} b_{i,j}^{(m,n)} B_j^n(t), \quad n > m, \qquad i = 0,1 \cdots, m, \tag{12}$$

where

$$b_{i,j}^{(m,n)} = \binom{m}{i}\binom{n-m}{j-i}\Big/\binom{n}{i}. \tag{13}$$

Suppose

$$K = \begin{pmatrix} b_{0,0}^{(m,n)} & b_{0,1}^{(m,n)} & \cdots & b_{0,n-m}^{(m,n)} & & & \\ & b_{i,j}^{(m,n)} & \cdots & b_{1,n-m}^{(m,n)} & b_{1,n-m+1}^{(m,n)} & & \\ & & \cdots & & & & \\ & & b_{m,m}^{(m,n)} & b_{m,m+1}^{(m,n)} & & \cdots & b_{m,n}^{(m,n)} \end{pmatrix}, \tag{14}$$

then the following formula hold true:

$$B_m(t) = KB_n(t). \tag{15}$$

Lemma 3 (see [3]). Given the vector $v = (v_0, v_1, \cdots, v_n)^T$, the vector $u = (u_0, u_1, \cdots, u_n)^T$ $(n > m)$ in the formula

$$u^T B_m(t) = v^T B_n(t), \tag{16}$$

can be obtained by the recursive formula

$$u_0 = v_0,$$

$$u_k = \left[\binom{n}{m} v_k - \sum_{i=max\,(0,k+m-n)}^{k-1} v_i \binom{k}{i}\binom{n-k}{m-i}\right]\Big/\binom{n-k}{m-k}, \quad k = 1,2,\cdots,m \tag{17}$$

Theorem 1. Given a disk WBGB curve$(Q)(t,L) = q^T W_m(t)$, the degree elevated disk WBGB curve $(\tilde{Q})(t,L) = \tilde{q}^T W_n(t), (n > m)$ is specified as follows

$$\tilde{q}^T = q^T A_m^{-1} K A_n, \tag{18}$$

where $q = \left((q_0), (q_1), \cdots, (q_m)\right)^T$, $\tilde{q} = \left((\tilde{q}_0), (\tilde{q}_1), \cdots, (\tilde{q}_m)\right)^T$.

Conversely, if we know the degree-elevated expression of $(Q)(t,L)$, then the entries of vector q^T can be obtained by the formula

$$q^T = u^T A_m, \tag{19}$$

where

$$u_0 = v_0,$$

$$u_k = \left[\binom{n}{m} v_k - \sum_{i=max\,(0,k+m-n)}^{k-1} v_i \binom{k}{i}\binom{n-k}{m-i}\right] \Big/ \binom{n-k}{m-k}, \tag{20}$$

and $v^T = \tilde{q}^T A_n^{-1}$.

Proof. By the Definition 1

$q^T W_m(t) = \tilde{q}^T W_n(t),$
then using Lemma 1 and Lemma 2, we have the following formulas:

$$q^T A_m^{-1} B_m(t) = q^T A_m^{-1} K B_n(t) = \tilde{q}^T W_n(t) = \tilde{q}^T A_n^{-1} B_n(t),$$

then

$$(q^T A_m^{-1} K - \tilde{q}^T A_n^{-1}) B_n(t) = 0,$$

then

$$\tilde{q}^T = q^T A_m^{-1} K A_n.$$

Now, assume we know the degree-elevated expression of $Q(t, L)$, then by using the Lemma 3 into formula

$$q^T A_m^{-1} B_m(t) = \tilde{q}^T A_n^{-1} B_n(t),$$

we have the following formula:

$$u^T B_n(t) = v^T B_n(t),$$

where $u^T = q^T A_m^{-1}$ and $v^T = \tilde{q}^T A_n^{-1}$, then

$q^T = u^T A_m,$
and

$$u_0 = v_0,$$
$$u_k = \left[\binom{n}{m} v_k - \sum_{i=max\,(0,k+m-n)}^{k-1} v_i \binom{k}{i}\binom{n-k}{m-i}\right] \Big/ \binom{n-k}{m-k}. \square$$

According to Theorem 1, it is easy to get the following algorithm for degree elevation of a disk WBGB curve.

Algorithm1. Degree elevation of disk WBGB curve.

1. Compute A_n by Eqs. (9) and (10).
2. Compute A_m^{-1} by Eqs. (9) and (10).

3. Compute K by Eq. (14).
4. Compute $\tilde{q}$ by Eq. (18).

4. Degree Reduction of Disk WBGB Curve

Given a disk WBGB curve $(P)(t,L)$ of degree n, the problem of the degree reduction of disk WBGB curve is to find a disk WBGB curve of degree $m < n$ (in what follows we only discuss the case $m = n - 1$. If$m < n - 1$, we can apply this algorithm recursively)

$$(Q)(t,L) = \sum_{i=0}^{m} (q_i)\omega_i^m(t,L), \quad 0 \le t \le 1, \tag{21}$$

such that

$$\{(P)(t,L)|0 \le t \le 1\} \subset \{(Q)(t,L)|0 \le t \le 1\} \tag{22}$$

and the difference between the widths of $(Q)(t,L)$ and $(P)(t,L)$ is as small as possible (which will be specified later).

The general bounding condition (22) is not easy to handle. In practice, we will use the following condition instead

$$(P)(t,L) \subset (Q)(t,L) \qquad for\ any\ 0 \le t \le 1. \tag{23}$$

We also need to write the disk WBGB curve $(Q)(t,L)$ in degree elevated form for the convenience of derivations:

$$(Q)(t,L) = \left(\tilde{x}(t), \tilde{y}(t)\right)_{\tilde{r}(t)} = \sum_{i=0}^{n} (\tilde{q}_i)\omega_i^n(t,L), \qquad 0 \le t \le 1, \tag{24}$$

where $(\tilde{q}_i), i = 0,1,\cdots,n,$ can be easily obtained by the formula (18). Also $(q_i), i = 0,1,\ldots,m$ can be obtained by (19), (20).

Suppose $(\tilde{q}_i) = (\tilde{x}_i, \tilde{y}_i)_{\tilde{r}_i}$ $(i = 0,1,\cdots,n), (\tilde{c}_i) = (\tilde{x}_i, \tilde{y}_i)$.
Let

$$\tilde{r}(t) = \sum_{i=0}^{n} \tilde{r}_i\, \omega_i^n(t,L),$$

$$\tilde{C}(t) = \left(\tilde{x}(t), \tilde{y}(t)\right) = \sum_{i=0}^{n} \tilde{c}_i \omega_i^n(t,L), \tag{25}$$

be the radius and center of $(Q)(t,L)$ respectively and $(Q)(t,L)$ must satisfy the degeneracy condition:

$$lec\left(\tilde{C}(t)\right) = 0, \;\; lec\left(\tilde{r}(t)\right) = 0. \tag{26}$$

That is,

$$\Delta^2\tilde{c}_{m-1} = 0, \;\; \Delta^2\tilde{r}_{m-1} = 0, \;\; for\ \ n = 2m; \tag{27}$$

$$\Delta\tilde{c}_m = 0, \qquad \Delta\tilde{r}_m = 0, \ for\, n = 2m+1; \tag{28}$$

Then the condition of (23) is written as

$$\tilde{r}(t) \geq r(t) + dist\left(C(t) - \tilde{C}(t)\right) = r(t) + \sqrt{\left(x(t) - \tilde{x}(t)\right)^2 + \left(y(t) - \tilde{y}(t)\right)^2}. \tag{29}$$

The condition that the difference between the width of $(P)(t, L)$ and $(Q)(t, L)$ is small expressed as

$$Min\|r(t) - \tilde{r}(t)\|. \tag{30}$$

Let $T_n(t),\ t \in [0,1]$ be the Chebyshev polynomial of degree$n$,

$$\begin{cases} T_n(t) = cos\left(n\, arccos(t)\right), \\ T_0(t) = 1,\ \ T_1(t) = t, \\ T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t),\ \ n = 1,2,\cdots \end{cases} \tag{31}$$

The following lemma is derived by referring [18].

Lemma 4. The transformation between WBGB basis $\omega_i^n(t, L)$ and Chebyshev polynomial $T_n(t)$ of degree n can be expressed as follows:

$$T_n(2t-1) = t^T W_n(t), \tag{32}$$

where $t^T = a^T A_n$ and entries of vector a^T can be specified as follows.

$$a_j = (-1)^{n+j} \frac{\binom{2n}{2j}}{\binom{n}{j}}. \tag{33}$$

The following lemma is derived by referring [3], [18].

Lemma 5. Suppose

$$\hat{T}_n(t) = \frac{1}{2^{2n-1}cos^n\left(\frac{\pi}{2n}\right)} T_n\left[(2t-1)cos\left(\frac{\pi}{2n}\right)\right] = f^T B_n(t) = \hat{t}^T W_n(t), \tag{34}$$

then $\hat{t}^T = f^T A_n,\ \hat{T}_n(0) = \hat{T}_n(1) = 0,$ and entries of vector f^T can be specified as follows.

$$f_i = \frac{1}{2^{2n-1}cos^n\left(\frac{\pi}{2n}\right)} \sum_{k=0}^{n} \frac{a(k)}{2^k}, \sum_{j=0}^{i} \binom{i}{j} \left\{ \binom{n-i}{k-j} \left(1 + cos\left(\frac{\pi}{2n}\right)\right)^i \left(1cos\left(\frac{\pi}{2n}\right)\right)^{k-j} \right\}, \tag{35}$$

where

$$a(k) = (-1)^{n+k} \sum_{i=0}^{k} \binom{k}{i} \binom{2n}{2i} \binom{n}{i}^{-1}.$$

4.1 Optimal approximation of the center curve

Similar to the research [18], the following propositions about the degree reduction of the center curve $C(t)$ is defined in section 2.

Proposition 1. The optimal approximation of degree n center curve $C(t)$ in all the curves of degree $n-1$ is

$$\tilde{C}(t) = C(t) - \frac{lec(C(t))}{2^{2n-1}} T_n(2t-1), \tag{36}$$

the control points satisfy

$$\tilde{c}_i = c_i - \frac{lec(C(t))}{2^{2n-1}} t_i, \qquad i = 0,1,\cdots,n, \tag{37}$$

where t_j is jth entry of t^T satisfies Lemma 4.

Proposition 2. The optimal approximation interpolating the two endpoints of degree n center $C(t)$ in all the curves of degree $n-1$ is

$$\tilde{C}(t) = C(t) - lec(C(t))\tilde{T}_n(t), \tag{38}$$

the control points satisfy

$$\tilde{c}_i = c_i - lec(C(t))\tilde{t}_i, \; i = 0,1,\cdots,n, \tag{39}$$

where $\tilde{t}_j$ is jth entry of $\tilde{t}^T$ satisfies Lemma 5.

4.2 Approximating the radius

A In this section, the solution for the approximation of the radius in two cases respectively is presented. First case is for the degree reduction of disk WBGB curve without interpolating the two endpoints. The other is for the disk curve interpolating with the two endpoints. The former case is discussed first.

a) Without interpolating the two endpoints

Since $\|T_n(t)\|_\infty \leq 1$, and $\left\|\tilde{C}(t) - C(t)\right\|_\infty \geq dist\left(C(t) - \tilde{C}(t)\right)$ then a sufficient condition for (29) to hold is

$$\tilde{r}_i \geq r_i + d, \qquad i = 0,1,\cdots,n; \;\; lec(\tilde{r}(t)) = 0, \tag{40}$$

where

$$d=\begin{cases}\frac{1}{2^{2n-L}}\binom{2m-2L}{m-L}\|\Delta^2 c_{m-1}\|_2, & n=2m;\\ \frac{1}{2^{2n-L-1}}\binom{2m+1-2L}{m-L}\|\Delta c_m\|_2, & n=2m+1.\end{cases} \tag{41}$$

In order to meet the need that the difference between the width of $(P)(t,L)$ and $(Q)(t,L)$is small, let $\tilde{r}_i-r_i-d=\varepsilon_i^2$ $(i=0,1,\cdots,n)$, then ε_i is describe based on $\tilde{r}(t)$ which is approximated to $r(t)$. So the problem of degree reduction is reduced to the following problem

$$\begin{cases}Min\|r(t)-\tilde{r}(t)-d\|=Min\left\|\sum_{i=0}^{n}\varepsilon_i^2\omega_i^n(t,L)\right\|,\\ s.t\ \Delta^2 r_{m-1}+\Delta^2\varepsilon_{m-1}^2=0,\quad n=2m;\\ \quad\Delta r_m+\Delta\varepsilon_m^2=0,\quad n=2m+1.\end{cases} \tag{42}$$

Now one obtains the solution by means of the form $\left\|\sum_{i=0}^{n}\varepsilon_i^2\omega_i^n(t,L)\right\|_1$ to compute the optimal problem.

Theorem 2. The radius of $\tilde{r}_i$ the degree reduced disk WBGB $(Q)(t,L)$ is determined as follows

1) for $\boldsymbol{n=2m}$,

[1] *if* $\Delta^2 r_{m-1}\le 0$, *then*

$$\tilde{r}_i=\begin{cases}r_i+d;\ \ i=0,1,\cdots,n,\ \ i\ne m\pm 1;\\ \frac{1}{2}(2r_m+r_{m-1}-r_{m+1})+d;\ \ i=m-1;\\ \frac{1}{2}(2r_m+r_{m+1}-r_{m-1})+d;\ \ i=m+1,\end{cases} \tag{43}$$

[2] *if* $\Delta^2 r_{m-1}>0$, *then*

$$\tilde{r}_i=\begin{cases}r_i+d;\ \ i=0,1,\cdots,n,i\ne m;\\ \frac{1}{2}(r_{m-1}+r_{m+1})+d;i=m;\end{cases} \tag{44}$$

2) for n=2m+1,

[3] *if* $\Delta r_m\le 0$, *then*

$$\tilde{r}_i=\begin{cases}r_i+d;\ \ i=0,1,\cdots,n,i\ne m+1;\\ r_m+d;\ \ i=m+1;\end{cases} \tag{45}$$

[4] *if* $\Delta r_m > 0$, *then*

$$\tilde{r}_i = \begin{cases} r_i + d; & i = 0,1,\cdots,n, i \neq m; \\ r_{m+1} + d; & i = m. \end{cases} \tag{46}$$

Proof. At first, we consider the case $n = 2m$, then

$$\left\|\sum_{i=0}^{n} \varepsilon_i^2 \omega_i^n(t,L)\right\|_1 = \sum_{i=0}^{2m} \varepsilon_i^2 \int_0^1 \omega_i^{2m}(t,L)dt = \sum_{i=0}^{m-L-1} \left(\frac{\varepsilon_i^2 + \varepsilon_{2m-i}^2}{2m-2L+1}\right)$$

$$+ \sum_{i=m-L}^{m-1} 2^{i-m+L} \binom{2m-2L}{m-L} B(i,i+2)(\varepsilon_i^2 + \varepsilon_{2m-i}^2) + 2^L \binom{2m-2L}{m-L} B(m,m)\varepsilon_m^2,$$

such that

$$B(\alpha,\beta) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, \Gamma(\alpha+1) = \alpha! \ (\alpha \epsilon \mathbb{N}),$$

Now, we define Lagrange function

$$L(\varepsilon_0, \epsilon_1, \cdots, \epsilon_{2m}, \lambda) = \left\|\sum_{i=0}^{n} \varepsilon_i^2 \omega_i^n(t,L)\right\|_1 + \lambda(\Delta^2 r_{m-1} + \Delta^2 \varepsilon_{m-1}^2)$$

$$= \left\|\sum_{i=0}^{n} \varepsilon_i^2 \omega_i^n(t,L)\right\|_1$$

$$+\lambda(r_{m+1} + r_{m-1} - 2r_m + \varepsilon_{m+1}^2 + \varepsilon_{m-1}^2 - 2\varepsilon_m^2).$$

From

$$\frac{\partial L}{\partial \lambda} = 0, \frac{\partial L}{\partial \varepsilon_k} = 0, \qquad k = 0,1,\cdots,2m,$$

we have the following equations:

$$\begin{cases} \varepsilon_i = 0, i = 0,1,\cdots,n, i \neq m \pm 1, m, \\ 2^{L+1} \binom{2m-2L}{m-L} B(m,m)\varepsilon_m - 4\lambda\varepsilon_m = 0, \\ 2^L \binom{2m-2L}{m-L} B(m-1,m+1)\varepsilon_{m\pm1} + 2\lambda\varepsilon_{m\pm1} = 0, \end{cases}$$

since $\Delta^2 r_{m-1} + \Delta^2 \varepsilon_{m-1}^2 = 0$, then

[1] *if* $\Delta^2 r_{m-1} \leq 0$, *then,*

$$\begin{cases} \varepsilon_i = 0, i = 0,1,\cdots,n, i \neq m \pm 1, m, \\ \varepsilon_{m-1}^2 = \varepsilon_{m+1}^2 = \dfrac{2r_m - r_{m+1} - r_{m-1}}{2}, \\ \varepsilon_{m+1}^2 = 0, \end{cases} \quad (47)$$

with substituting $\varepsilon_i^2 = \tilde{r}_i - r_i - d$ into (47), we have the following equations.

$$\tilde{r}_i = \begin{cases} r_i + d; \ \ i = 0,1,\cdots,n, \ \ i \neq m \pm 1; \\ \dfrac{1}{2}(2r_m + r_{m-1} - r_{m+1}) + d; \ \ i = m-1; \\ \dfrac{1}{2}(2r_m + r_{m+1} - r_{m-1}) + d; \ \ i = m+1, \end{cases} \quad (48)$$

[2] *if* $\Delta^2 r_{m-1} > 0$, *then,*

$$\begin{cases} \varepsilon_i = 0, i = 0,1,\cdots,n, i \neq m \pm 1, m, \\ \varepsilon_{m-1}^2 = \varepsilon_{m+1}^2 = 0, \\ \varepsilon_m^2 = \dfrac{2r_m - r_{m+1} - r_{m-1}}{2}, \end{cases} \quad (49)$$

with substituting $\varepsilon_i^2 = \tilde{r}_i - r_i - d$ into (49), we have the following equations.

$$\tilde{r}_i = \begin{cases} r_i + d; \ \ i = 0,1,\cdots,n, i \neq m; \\ \dfrac{1}{2}(r_{m-1} + r_{m+1}) + d; i = m; \end{cases} \quad (50)$$

Now, we consider the case $n = 2m + 1$, then

$$\begin{aligned} \left\| \sum_{i=0}^{n} \varepsilon_i^2 \omega_i^n(t,L) \right\|_1 &= \sum_{i=0}^{2m+1} \varepsilon_i^2 \int_0^1 \omega_i^{2m+1}(t,L)dt \\ &= \sum_{i=0}^{m-L-1} \left(\frac{\varepsilon_i^2 + \varepsilon_{2m+1-i}^2}{2m - 2L + 2} \right) \\ &+ \sum_{i=m-L}^{m-1} 2^{i-m+L} \binom{2m+1-2L}{m-L} B(i, i+2)(\varepsilon_i^2 + \varepsilon_{2m+1-i}^2) \\ &+ 2^L \binom{2m+1-2L}{m-L} B(m, m+1)(\varepsilon_m^2 + \varepsilon_{m+1}^2). \end{aligned}$$

we define Lagrange function

$$R(\varepsilon_0,\epsilon_1,\cdots,\epsilon_{2m+1},\lambda)=\left\|\sum_{i=0}^{n}\varepsilon_i^2\omega_i^n(t,L)\right\|_1+\lambda(\Delta r_m+\Delta\varepsilon_m^2)$$
$$=\left\|\sum_{i=0}^{n}\varepsilon_i^2\omega_i^n(t,L)\right\|_1+\lambda(r_{m+1}-r_m+\varepsilon_{m+1}^2-\varepsilon_m^2).$$

From

$$\frac{\partial R}{\partial\lambda}=0,\frac{\partial R}{\partial\varepsilon_k}=0,\qquad k=0,1,\cdots,2m+1,$$

we have the following equations:

$$\begin{cases}\varepsilon_i=0, i=0,1,\cdots,n, i\neq m\pm1, m,\\ 2^{L+1}\binom{2m+1-2L}{m-L}B(m,m+1)\varepsilon_m-2\lambda\varepsilon_m=0,\\ 2^{L+1}\binom{2m+1-2L}{m-L}B(m,m+1)\varepsilon_{m+1}+2\lambda\varepsilon_{m+1}=0,\end{cases}$$

since $\Delta r_m+\Delta\varepsilon_m^2=0$, then

[3] *if* $\Delta r_m\leq 0$, *then*

$$\begin{cases}\varepsilon_i=0, i=0,1,\cdots,n, i\neq m\pm1,\\ \varepsilon_{m+1}^2=-\Delta r_m,\end{cases}\tag{51}$$

with substituting $\varepsilon_i^2=\tilde{r}_i-r_i-d$ into (51), we have the following equations

$$\tilde{r}_i=\begin{cases}r_i+d;\ \ i=0,1,\cdots,n, i\neq m+1;\\ r_m+d;\ \ i=m+1;\end{cases}\tag{52}$$

[4] *if* $\Delta r_m>0$, *then*

$$\begin{cases}\varepsilon_i=0, i=0,1,\cdots,n, i\neq m,\\ \varepsilon_m^2=\Delta r_m,\end{cases}\tag{53}$$

with substituting $\varepsilon_i^2=\tilde{r}_i-r_i-d$ into (53), we have the following equations

$$\tilde{r}_i=\begin{cases}r_i+d;\ \ i=0,1,\cdots,n, i\neq m;\\ r_{m+1}+d;\ \ i=m.\square\end{cases}\tag{54}$$

According to Theorem 2, it is easy to get the following algorithm for degree reduction of a disk WBGB curve without interpolating the two endpoints.

Algorithm2. Degree Reduction of disk WBGB curve without interpolating the two endpoints.

1. Compute $\tilde{r}_0, \tilde{r}_1, \cdots, \tilde{r}_n$ by Eqs. (43), (44), (45) and (46).
2. Compute A_m^{-1} by Eqs. (9) and (10).
3. Compute A_n by Eqs. (9) and (10).
4. Compute $r_0, r_1, \cdots, r_m$, by Eqs. (19) and (20).

b) Interpolating the two endpoints

Comparing *Proposition 1* and *Proposition 2*, and modified the d in *Theorem 2* as

$$\tilde{d} = \begin{cases} \dfrac{1}{2^{2n-L} cos^n\left(\frac{\pi}{2n}\right)} \dbinom{2m-2L}{m-L} \|\Delta^2 c_{m-1}\|_2, n = 2m; \\ \dfrac{1}{2^{2n-L-1} cos^n\left(\frac{\pi}{2n}\right)} \dbinom{2m+1-2L}{m-L} \|\Delta c_m\|_2, n = 2m+1. \end{cases} \quad (55)$$

It is easy to obtain the radius approximation of the degree reduced disk WBGB curve which interpolates the original two endpoints.

5. Example

In the example, we use the proposed algorithms for reducing the degree of a disk WBGB curve of degree 7 (Figure 1). Using the Algorithm 2 and Eq. (55), we can compute the corresponding degree reduction of the disk WBGB curve. Given a disk WBGB curve $P(t)$ of degree 7 (Figure 1). Two methods are illustrated here to reduce the degree of. Now IDR and WIDR are utilized to represent the solutions for degree reduction problems with interpolation and without interpolation of endpoints, respectively. Figure 2 and Figure 3 demonstrate the corresponding degree reduction of disk WBGB curves. As shown in Figure 1 – Figure 3, the original disk WBGB curves are the degree reduction curve by WIDR and the degree reduction curve by IDR respectively. The difference between Figure 2 and Figure 3 is that in Figure 3 the endpoints interpolated but in Figure 2 the endpoints didn't interpolate.

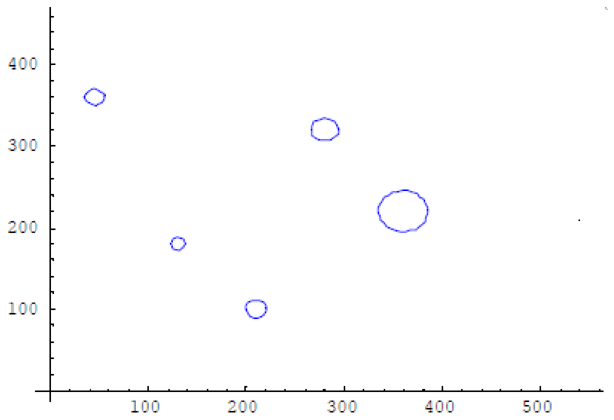

Figure 1. Original Disk WBGB curve of degree 7

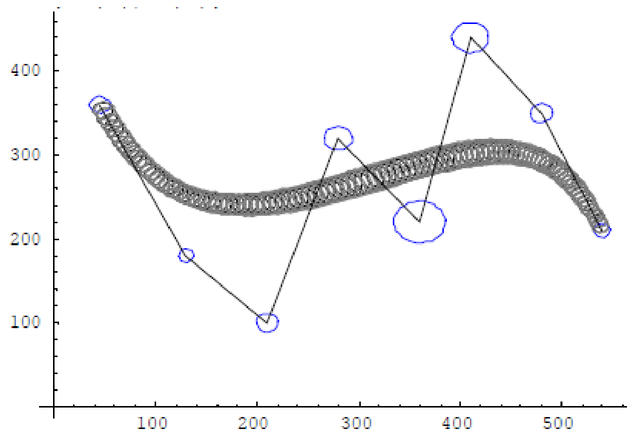

Figure 2. Degree reduction of the original curve without interpolating endpoints

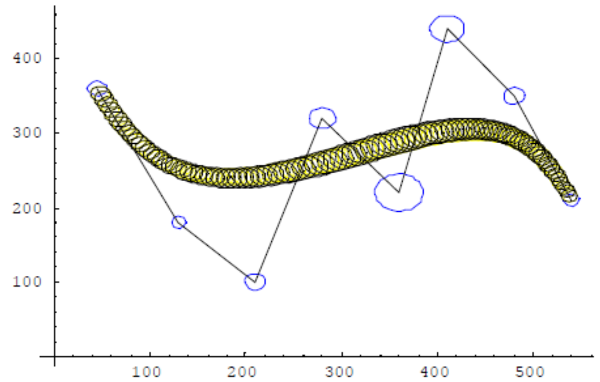

Figure 3. Degree reduction of the original curve with interpolating endpoints

6. Conclusion

In this paper, an algorithm of the degree reduction problem of disk Wang-Bézier type generalized Ball curves is presented. The advantage of the algorithms lies in its one-into two policy, that is, it first aims at seeking for the center curve optimal approximation of the original disk Wang-Bézier type generalized Ball curve with lower degree and then approximating the radius by the degree reduced Wang-Bézier type generalized Ball curve, and both approaches are adopted to find the approximation solution to the radius, which brings about the flexibility in the degree reduction of disk Wang-Bézier type generalized Ball curves. The idea presented in this paper can be easily generalized to solve the other degree reduction problems of surfaces.

7. References

[1] A.A. Ball, Consurf Part I: Introduction of conic lofting title, Computer Aided Design, 1974, 6(4): 243-249.

[2] A.A. Ball, Consurf Part II: Description of the algorithms, Computer Aided Design, 1975, 7(4): 237-242.

[3] F.L. Chen, W.P. Lou, Degree reduction of interval Bezier curves, Computer-Aided Design, 2000, 32 (8), 571-582.

[4] F.L. Chen, W. Yang, Degree reduction of disk Bezier curves, Computer Aided Geometric Design, 2004, 21(3): 263-280.

[5] J. Delgado and J.M. Pena, A shape preserving representation with an evaluation algorithm of linear complexity, Computer Aided Geometric Design, 2003, 14(6): 1-10.

[6] T.N.T. Goodman and H.B. Said, Shape-preserving properties of the generalized Ball basis, Computer Aided Geometric Design, 1991, 8(2): 115-121.

[7] T.N.T. Goodman and H.B. Said, Properties of the generalized Ball basis and surfaces, Computer Aided Design, 1991, 23(8): 554-560.

[8] S.M. Hu, G.Z. Wang and T.G. Jin, Properties of two types of generalized Ball curves, Computer Aided Design, 1996, 28(2): 125-133.

[9] C-Y Hu, T. Maekawa, E.C. Sherbrooke, N.M. Patrikalakis, Robust interval algorithm for intersections, Computer Aided Design, 1996, 28(6/7): 495-506.

[10] C-Y Hu, N.M. Patrikalakis, X. Ye, Robust interval solid modeling: Part I: representations, Computer Aided Design, 1996, 28(10): 807-817.

[11] C-Y Hu, N.M. Patrikalakis, X. Ye, Robust interval solid modeling: Part II: boundary evaluation, Computer Aided Design, 1996, 28(10): 819-830.

[12] C-Y Hu, T. Maekawa, N.M. Patrikalakis, X. Ye, Robust interval algorithm for surface intersections, Computer Aided Design, 1997, 29(9): 617-627.

[13] P. Jiang, H.Y. Wu, J.Q. Tan, The dual functionals for the generalized Ball basis of Wang-Said type and basis transformation formulas, Numer. Math.-A J. Chinese Univ., 2006 (to appear).

[14] W. Othman and R.N. Goldman, The dual basis functions for the generalized Ball basis of odd degree, Computer Aided Geometric Design, 1997, 14(6): 571-582.

[15] H.B. Said, Generalized Ball curve and its recursive algorithm, ACM Trans. Graph. 1989, 8(4): 360-371.

[16] T.W. Sederberg, R.T. Farouki, Approximation by interval Bezier curves, IEEE Comput. Graph. Appl., 1992, 15 (2), 87-95.

[17] R.E. Moore, Interval analysis, Englewood Cliffs, NJ: Prentice-Hall, 1996.

[18] J.Q. Tan, P. Jiang, Boundary and Degree Reduction of Interval Ball Curves, Journal of Computer- Aided Design & Computer Graphics, 2006 (to appear).

[19] G.J. Wang, G.Z. Wang, J.M. Zheng, Computer Aided Geometric Design, Beijing: Higher Education Press; Berlin Heidelberg: Springer-Verlag, 2001.

[20] W. Yang, F.L. Chen, Representation of geometric transform errors, Journal of Computer-Aided Design & Computer Graphics, 2003, 15(12): 1506-1512.

[21] P. Jiang, J.Q. Tan, Degree Reduction of Disk Said-Ball Curves, Journal of Computational Information Systems, 2005, 1(3): 389-398.

[22] L. Zhang, J.Q. Tan, H.Y. Wu, Z. Liu, The weighted dual functions for Wang-Bézier type generalized Ball bases and their applications, 2009, 22-36.

[23] Chen, G.-D., Wang, G.-J., Optimal multi-degree reduction of Bézier curves with constraints of endpoints continuity. Computer Aided Geometric Design, 2002, 19, 365-377.