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On the Convergence Analysis of Gravitational Search Algorithm

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Abstract

Gravitational search algorithm (GSA) is one of the newest swarm based optimization algorithms, which has been inspired by the Newtonian laws of gravity and motion. GSA has empirically shown to be an efficient and robust stochastic search algorithm. Since introducing GSA a convergence analysis of this algorithm has not yet been developed. This paper introduces the first attempt to a formal convergence analysis of the standard gravitational search algorithm which involves with randomness and time varying parameters. In this analysis the behavior of GSA on the facet of mass interaction is considered. The paper provides a formal proof that each object converges to a stable point.

Keywords: *Analysis of algorithms, Heuristic optimization, Swarm intelligence, Gravitational Search Algorithm, Convergence*

1. Introduction

Over the last decades, there has been a growing interest in algorithms inspired by the behaviors of natural phenomena. It has been shown by many researches that these algorithms are good replacements as tools to solve complex computational problems. Various heuristic approaches are adopted by researches so far, for example genetic algorithms (GA) [1], simulated annealing (SA) [2], ant colony optimization (ACO) [3] particle swarm optimization (PSO) [4], gravitational search algorithm (GSA) [5-7], and etc. These algorithms are progressively analyzed or powered by researchers in many different areas.

Various researchers addressed heuristic search algorithms both empirically and theoretically. However, convergence proof of these algorithms is a challenging topic which is an interesting area for researchers. Although heuristic algorithms are widely applied in many research fields nowadays, the theoretical analysis on them is still quite limited. There are only few theoretical convergence analyses of ACO variants [8-10],or to the best of our knowledge there exist only a few attempts to develop a proof of convergence of PSO such as [11][26].

Gravitational Search Algorithm (GSA) is one of the modern met heuristic optimization algorithms introduced by Rashedi et al. [5]. It is a new stochastic population-based optimization tool which works based on the metaphor of gravitational interaction between masses. This algorithm provides an iterative optimization method that simulates the interactions between objects based on the laws of gravity and motion. In the GSA the objects (searcher agents) moves through a multi-dimensional search space in the influence of gravitation. The effectiveness of GSA and its binary version (BGSA) [6] in solving a set of canonical benchmark functions has been verified [5, 6].Furthermore, the conclusions achieved in [7][27-30] confirm that GSA is a suitable tool for linear and nonlinear filter modeling [7], parameter identification of hydraulic turbine governing system [27], feature selection and classifier design [28][29], and data clustering [30].

Although some experimental and empirical research has already been done on GSA, a convergence analysis of this algorithm has not been developed yet. This paper is devoted to the presentation of an attempt to develop such an analysis. Therefore, in this paper a formal convergence analysis of the GSA is presented while the randomness characteristics of the algorithm, time varying parameters and interactions between masses are considered.

The reminder of the paper is organized as follows: Section 2 reviews in brief the GSA. A convergence analysis of GSA is presented in Section 3, which is followed by conclusion in the last Section.

2. An brief introduction to GSA

In this section, we introduce a brief review of GSA. In GSA, searcher agents are considered as objects and their performance is measured by their masses. All these objects attract each other by a gravity force, and this force causes a movement of all objects globally towards the objects with heavier masses. The heavy objects correspond to good solutions of the problem. The position of the object (agent) corresponds to a solution of the problem, and its mass is determined using a fitness function. By lapse of time, objects are attracted by the object with heaviest mass. Hopefully, this object would present the optimum solution of the problem at hand. The GSA could be considered as an isolated system of objects obeying the Newtonian laws of gravitation and motion [5].

To mathematically describe the GSA consider a system with Sobjects in which the position of the i^{th} object is defined as follows:

$$X_i = (x_i^1, \ldots, x_i^d, \ldots, x_i^n) \quad , i = 1,2,\ldots, S \tag{1}$$

where x_i^d is the position of i^{th} object in the d^{th} dimension and n is dimension of the search space. It is noted that the positions of objects correspond to the solutions of the problem. Based on [5], the mass of each object is calculated after computing current population's fitness as follows:

$$q_i(t) = \frac{f_i(t) - worst(t)}{best(t) - worst(t)} \tag{2}$$

$$M_i(t) = \frac{q_i(t)}{\sum_{j=1}^{S} q_j(t)} \tag{3}$$

where $f_i(t)$ represents thefitness value of the object i at t, and, $best(t)$ and $worst(t)$ are defined as follows (for a maximization problem):

$$best(t) = \max_{j \in \{1,..,S\}} f_j(t) \tag{4}$$

$$worst(t) = \min_{j \in \{1,..,S\}} f_j(t) \tag{5}$$

According to Eqs.(2) and(3) $M_i \geq 0,\ \text{i} = 1,2,...,\text{S}$ and the summation of all masses is equal to 1. At iteration "t", the force acting on object "i" from object "j" is:

$$F_{ij}^d(t) = G(t)\frac{M_i(t) \times M_j(t)}{R_{ij}(t) + \varepsilon}(x_j^d(t) - x_i^d(t)) \tag{6}$$

To update the acceleration of anobject, total forces from all other objects that apply to it should be considered based on law of gravity (Equation (7)). This term is tracked by calculation of agent acceleration using law of motion (Equation (8)).

$$F_i^d(t) = \sum_{j=1,\, j \neq i}^{S} r_j F_{ij}^d(t) = G(t)M_i(t)\sum_{j=1,\, j \neq i}^{S} r_j \frac{M_j(t)}{R_{ij}(t) + \varepsilon}(x_j^d(t) - x_i^d(t)) \tag{7}$$

$$a_i^d(t) = \frac{F_i^d(t)}{M_i(t)} = G(t)\sum_{j=1,\, j \neq i}^{S} r_j \frac{M_j(t)}{R_{ij}(t) + \varepsilon}(x_j^d(t) - x_i^d(t)) \tag{8}$$

Later, the next velocity of an object is determined as a fraction of its current velocity added to its acceleration (Equation (9)). Then, its next position could be calculated using Equation (10).

$$v_i^d(t+1) = r_i \times v_i^d(t) + a_i^d(t) \tag{9}$$

$$x_i^d(t+1) = x_i^d(t) + v_i^d(t+1) \tag{10}$$

where r_i and r_j are two uniformly distributed random numbers in the interval [0,1] , and ε is a small value. $G(t) = G(G_0, t)$ isthe gravitational constant at time t whichis a decreasing function of time where is set to $G_0 > 0$ at the beginning ($t = 0$) and will be decreased exponentially [5] or linearly [6] towards zero by lapse oftime (iterations). $R_{ij}(t)$ is the Euclidean distance between two agents i and j defined as:

$$R_{ij}(t) = \left\| X_i(t), X_j(t) \right\|_2 \tag{11}$$

It should be noted that to have a good balance between exploration and exploitation, Rashedi et al. [5] proposed a modified version of Equation (8) by limiting objects which can apply the force to attract the other objects to a set of $kbest$ objects (Equation (12)) . The set of $kbest$ includes the objects with heavier masses. $kbest$ is a function of time, with the initial value S(number of objects) at the beginning and is decreased with time.The pseudo code of the GSA is given by Figure1.

$$a_i^d(t) = G(t)\sum_{\substack{j \in kbest \\ j \neq i}} r_j \frac{M_j(t)}{R_{ij}(t) + \varepsilon}(x_j^d(t) - x_i^d(t)) \tag{12}$$

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Initialize a n-dimensional swarm, S;
repeat
for each object i = 1,2,..., S
evaluate fitness function;
calculate mass using Eqs. (2) and (3);
calculate the acceleration using Eq. (12);
update the velocity using Eq. (9);
update the position using Eq. (10);
end
until stopping condition is met;
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Figure 1.The pseudo code of GSA [29].

3. Convergence analysis of GSA

Before introducing convergence analysis, it is mentioned that the term of convergence is used for the sequence $\{A(t)\}, t = 0,1,...,$ to the property that $\lim_{t\to\infty} A(t) = A$ exists, where A is the convergent value (a constant), and both $A(t)$ and A are scalars or vectors [22].

It should be noted that to proof the convergence of GSA, the position of object i is analytically traced considering its interactions with all other objects of the artificial system, its randomness characteristics and time varying parameters. The convergence proof of GSA is obtained if one could show that $lim_{t\to\infty} x_i^d(t) = C$ $(a\ constant)$, where it means $\lim_{t\to\infty}\left(x_i^d(t+1) - x_i^d(t)\right) = 0$.

To proof convergence analysis of GSA,two theorems are used as follow:

Theorem1: *Let $B(t)$ and $E(t)$ be two real value functions and for some $N > 0, |B(t)| < N$ for all t, and $lim_{t\to\infty} E(t) = 0$,* then it can be concluded*:*

$$\lim_{t\to\infty} B(t)E(t) = 0$$

Proof: *let $\varepsilon > 0$, then there exist $T > 0$ such that for all $t > T$, $|E(t)| < \frac{\varepsilon}{N}$.* Therefore, for all $t > T$:

$$B(t)E(t) < N \times \frac{\varepsilon}{N} = \varepsilon$$

Theorem2: Let $lim_{t\to\infty} G(t) = 0$, then

$$\lim_{t\to\infty}\left(x_i^d(t+1) - x_i^d(t)\right) = 0$$

Proof:

From (11), itcan be seen that

$$R_{ij}(t) = \left\|X_i(t), X_j(t)\right\|_2 \ge \left|x_j^d(t) - x_i^d(t)\right| \Rightarrow$$
$$R_{ij} + \varepsilon > \left|x_j^d(t) - x_i^d(t)\right| \Rightarrow \left|\frac{x_j^d(t) - x_i^d(t)}{R_{ij} + \varepsilon}\right| < 1 \tag{13}$$

Based on (2) and (3) one can conclude that $M_j(t) \ge 0$ and $\sum_{j=1}^{S} M_j = 1$, here a more general case is supposed and it is assumed that $\sum_{j=1}^{S} M_j = K > 0$, which K can also get a finite value bigger than 1. On the other hand, $0 \le r_j(t) \le 1$. Therefore, considering (13),it can be discovered that:

$$\left|\sum_{j=1,j\neq i}^{S}\left(r_j(t)M_j(t)\frac{(x_j^d(t)-x_i^d(t))}{R_{ij}(t)+\varepsilon}\right)\right| \le \left|\sum_{j=1,j\neq i}^{S}\left(M_j(t)\frac{(x_j^d(t)-x_i^d(t))}{R_{ij}(t)+\varepsilon}\right)\right| < \left|\sum_{j=1,j\neq i}^{S}M_j(t)\right| < \left|\sum_{j=1}^{S}M_j(t)\right| < K \quad (14)$$

On the other hand, for a special case that we limit the agents which attract other agents by applying the force to the set of$kbest$, we can also write:

$$\left|\sum_{j\in kbest,j\neq i}^{S}\left(r_j(t)M_j(t)\frac{(x_j^d(t)-x_i^d(t))}{R_{ij}(t)+\varepsilon}\right)\right| \le \left|\sum_{j\in kbest,j\neq i}^{S}\left(M_j(t)\frac{(x_j^d(t)-x_i^d(t))}{R_{ij}(t)+\varepsilon}\right)\right| < \left|\sum_{j\in kbest,j\neq i}^{S}M_j(t)\right| \le \left|\sum_{j=1,j\neq i}^{S}M_j(t)\right| < \left|\sum_{j=1}^{S}M_j(t)\right| < K \quad (15)$$

According to (8):

$$a_i^d(t) = G(t)\sum_{j=1,j\neq i}^{S}\left(r_j(t)M_j(t)\frac{(x_j^d(t)-x_i^d(t))}{R_{ij}(t)+\varepsilon}\right)$$

From (14), and based on theorem1, this conclusion can made that $lim_{t\to\infty} a_i^d(t) = 0$. Once more, by replacing (8) by (12), based on (15) and theorm1 yet again it is achieved that $lim_{t\to\infty} a_i^d(t) = 0$. In other word, in both cases in use of (8) and (12) we will have

$$lim_{t\to\infty} a_i^d(t) = 0 \quad (16)$$

From (9), it can be seen that

$$a_i^d(t) = v_i^d(t+1) - r_i(t)v_i^d(t) \quad (17)$$

According to (16):

$$\lim_{t\to\infty} a_i^d(t) = 0 \overset{(17)}{\Longrightarrow} \lim_{t\to\infty}\left(v_i^d(t+1) - r_i(t)v_i^d(t)\right) = 0 \Rightarrow \lim_{t\to\infty}(v_i^d(t) - r_i(t)v_i^d(t)) = 0 \Rightarrow \lim_{t\to\infty}(1-r_i(t))\,v_i^d(t) = 0 \quad (18)$$

It is known that $r_i(t)$ is a uniformly distributed random number in the interval [0, 1] and is independent of time, t. Therefore the term $(1 - r_i(t))$ cannot converge to zero by lapse of time. Hence, it is concludedthat $\lim_{t\to\infty} v_i^d(t) = 0$. Formally,it can be demonstrated by mathematics as follows.

Let $r'_i(t) = 1 - r_i(t)$, nowbased on (18) it can be seen that

$$lim_{t\to\infty} r'_i(t)\,v_i^d(t) = 0 \quad (19)$$

where$r_i(t)$ and$r'_i(t)$ are two random numbersin the interval [0, 1] and act similarly, therefore the term$r_i(t)$is used instead of $r'_i(t)$. This means that

$$lim_{t\to\infty} r_i(t)v_i^d(t) = 0 \quad (20)$$

On the other hand,

$$v_i^d(t) = v_i^d(t)\left(1 - r_i(t) + r_i(t)\right) = (1-r_i(t))v_i^d(t) + r_i(t)v_i^d(t) \Rightarrow$$

$$\lim_{t\to\infty} v_i^d(t) = \lim_{t\to\infty}(1-r_i(t))v_i^d(t) + \lim_{t\to\infty} r_i(t)v_i^d(t) \overset{(19),(20)}{\Longrightarrow}$$

$$\lim_{t\to\infty} v_i^d(t) = 0 \quad (21)$$

From (10), it can be seen that

$$x_i^d(t+1) - x_i^d(t) = v_i^d(t) \overset{(21)}{\Longrightarrow} \lim_{t\to\infty}(x_i^d(t+1) - x_i^d(t)) = \lim_{t\to\infty} v_i^d(t) = 0 \quad (22)$$

And this means that $x_i^d(t)$ is a convergent sequence and proof is complete. In other words, each object of the GSA may converge to a stable point.

Arevision to the theorems 1 and 2 shows that the convergence proof of GSA is obtained by considering the following conditions, which can be used in setting parameters.

a) $M_j \geq 0,\ j = 1,2,\dots,S$

b) $\sum_{j=1}^{S} M_j = K > 0$

c) $lim_{t\to\infty} G(t) = 0$

4. Conclusion

This paper is devoted to address the first attempt to the convergence properties of standard gravitational search algorithm. In this study, the convergence analysis of the GSA is done in presence of time-varying parameter $G(t)$ (gravitational constant), considering the randomness characteristics of the algorithm and interactions of objects in the swarm. The result derived in this paper reveals that each object of the standard GSA converges to a stable point. Moreover, the analysis results confirm the convergent condition of GSA according to the existing parameters ranges of GSA.

While the paper offers a proof of convergence to a stable point for each object, work remains about where this point is and the accuracy of the stable point with respect to the global optima. Another possibility would be to determine the expected number of iterations for the algorithm to hit a ball around the global optimum. In future work, we try to answer these questions.

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