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A Third Order Iterative Method for Finding Zeros of Nonlinear Equations

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Abstract. In this paper, we present a new modification of Newton's method for finding a simple root of a nonlinear equation. It has been proved that the new method converges cubically.

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1. Introduction

Solving nonlinear equations is one of the most important and very basic and an old problem in numerical analysis and has many applications in engineering and other applied sciences. In this paper, we consider iterative methods to find a simple root of a non-linear equation $f(x) = 0$.

The best known and the most widely used example of the types of the one–point $f : \mathbb{R} \to \mathbb{R}$, where $x_{n+1} = g(x_n),\ n = 0, 1, ...$ is the classical Newton's method given by iteration method

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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots.$$

Time to time the method has been derived and modified in a variety of ways. Weerakoon and Fernando have suggested an one-step method for finding simple roots of non-linear equations (WF) [15] that requires having the $(n+1)^{th}$ iterative step to calculate $(n+1)^{th}$ derivative of the function at the iterative itself. Frontiny and Sormani also generalized the approach of Weerakoon and Fernando (MP) [3,4,5]. Also one of the most useful literatures in this field is [17]. A recent work has been done by Khattri and Argyros [9]

Here, we will apply these ideas to obtain a new modification of Newton's method (AM). Analysis of convergence show the new method is cubically convergent. Per iteration the method requires one evaluation of the function and two evaluations of its first derivative.

In Section 2, we briefly introduce some definitions and concepts. In Section 3 we introduce the most important and main ideas of the modified Newton methods with cubic convergence and in Section 4 we will suggest a new type of Newton's method to find simple roots of non-linear equations with third-order convergence. Finally in the last Section we compare our method with some well-known methods for non-linear equations. The numerical result are provided in Table 1.

2. Preliminaries

Definition 2.1 *If the sequence $\{x_n \mid n \geq 0\}$ tends to limit α in such a way that*

$$\lim_{x_n \to \alpha} \left| \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} \right| = C > 0 \tag{1}$$

for some $C \neq 0$ and $p \geq 1$, then the order of convergence of the sequence is said to be p, and C is known as the asymptotic error constant.

When $p = 1$ the convergence is linear, while for $p = 2$ and $p = 3$ the sequence is said to converge quadratically and cubically, respectively. The value of p is called the order of convergence of the method which produces the sequence $\{x_n \mid n \geq 0\}$. Let $e_n = x_n - \alpha$. Then the relation $e_{n+1} = Ce_n^p + O(e_n^{p+1})$ is called the error equation for the method, p being the order of convergence[12].

Definition 2.2 *Let α be a zero of the function f and suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the zero α. Then the computational order of convergence ρ can be approximated using the formula[2,12]*

$$\rho \approx \frac{\ln |(x_{n+1} - \alpha)/(x_n - \alpha)|}{\ln |(x_n - \alpha)/(x_{n-1} - \alpha)|} \tag{2}$$

Definition 2.3 *The efficiency index method (EI) is defined as $p^{\frac{1}{m}}$, in which p is the order of the method and m is the number of functional evaluations per each iteration by the method.*

3. The modified Newton's methods

The Weerakoon and Fernando's method for obtaining a simple root of equation $f(x) = 0$ uses the iteration formula [15]

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n - \frac{f(x_n)}{f'(x_n)})} \tag{3}$$

The method defined by (3) satisfies the following error equation:

$$e_{n+1} = \frac{1}{6}(\frac{3}{2}c_2^2 - \frac{1}{4}c_3)e_n^3 + O(e_n^4), \tag{4}$$

where $c_2\, and\, c_3$ are some constants. The method derived from midpoint rule defined by[16],

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - \frac{f(x_n)}{2f'(x_n)})}\ , \tag{5}$$

that satisfies the following error equation:

$$e_{n+1} = (c_2^2 - \frac{1}{4}c_3)e_n^3 + O(e_n^4), \tag{6}$$

where c_2 and c_3 are some constants.

The Homeier's method [HM] that may be rewritten as the iterative schema [6,7,8]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - \frac{f(x_n)}{2f'(x)})}, \quad n = 0, 1, 2, \dots\ . \tag{7}$$

The method defined by (7) satisfies the following error equation .

$$e_{n+1} = \frac{1}{12}c_3e_n^3 + O(e_n^4),$$

that c_3 is a constant.

4. The new method and its convergence

We consider and analyze one new iterative method as follows

$$x_{n+1} = x_n - \frac{h(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots\ , \tag{8}$$

where $h(x) = f(x + a(x)f(x))$ and $a(x)$ is some function of x to be determined.

Let

$$x_{n+1} = g(x_n), \quad n = 0, 1, \dots \tag{9}$$

where

$$g(x) = x - \frac{h(x)}{f'(x)}. \tag{10}$$

Thus

$$g'(x) = 1 - \frac{(1 + a'(x)f(x) + a(x)f'(x))\,(f'(x + a(x)f(x)))}{f'(x)} + \frac{f(x + a(x)f(x))f''(x)}{f'^2(x)}$$

We want to have $g'(\alpha) = 0$, So we have $g'(\alpha) = a(\alpha)f'(\alpha)$. Thus

$$a(\alpha) = 0. \tag{11}$$

For function $g(x)$ we have

$$\begin{aligned} g''(x) = &\frac{1}{f'^2(x)}\{2(1 + a'(x)f(x) + a(x)f'(x))f'(x + a(x)f''(x))\} \\ &- \frac{1}{f'^3(x)}\{2f(x + a(x))f''^2(x)\} \\ &- \frac{1}{f'(x)}\{f'(x + a(x)f(x))(2a'(x)f'(x) + a''(x)f(x) + a(x)f''(x))\} \\ &- \frac{1}{f'(x)}\{(1 + a'(x)f(x) + a(x)f'(x))^2 f''(x + a(x)f(x))\} \\ &+ \frac{1}{f'^2(x)}\{f(x + a(x))f'''(x)\} \end{aligned} \tag{12}$$

Since $f(\alpha) = 0$, we have

$$g''(\alpha) = -2a'(\alpha)f'(\alpha) + \frac{f''(\alpha)}{f'(\alpha)}. \tag{13}$$

We want to have at least three-order convergence so

$$a'(\alpha) = \frac{f''(\alpha)}{2f'(\alpha)^2}. \tag{14}$$

Therefore we have

$$a(\alpha) = -\frac{1}{2f'(\alpha)} + c. \tag{15}$$

From (11) and (15) we have

$$a(x) = -\frac{1}{2}\left(\frac{1}{f'(x)} - \frac{1}{f'(\alpha)}\right). \tag{16}$$

In this formula instead of unknown α, one can use the first iteration of a simpler method such as bisection method (with efficiency index 1.24573) or one can use the classical Newton method iteration (with efficiency index 1.16993). Thus the final formula for root finding is:

$$x_{n+1} = x_n - \frac{1}{f'(x_n)} f\left(x_n - \frac{1}{2} y_n f(x_n)\right) \tag{17}$$

in which

$$y_n = \left(\frac{1}{f'(x_n)} - \frac{1}{f'(x_n - \frac{f(x_n)}{f'(x_n)})} \right) \tag{18}$$

5. Convergence analysis

In this section, we discuss the convergence of formula (8) under some conditions.

THEOREM 5.1 *Let $\alpha \in D$ (D is an open interval), be the root of a sufficiently differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ with assumption $c_k = \frac{f^{(k)}(\alpha)}{k! f'(\alpha)}$, $k = 2, 3, \cdots$. Let $e_n = x_n - \alpha$. Then under the conditions:*

$$a(\alpha) = 0, \quad a'(\alpha) = \frac{f''(\alpha)}{2(f'(\alpha))^2}, \quad a''(\alpha) = \frac{f'''(\alpha) f'(\alpha) - 2f''^2(\alpha)}{2(f'(\alpha))^3}. \tag{19}$$

the sequence obtained from (8), satisfies the following error equation:

$$e_{n+1} = \left(\frac{1}{2} c_3 - 3c_2^2 \right) e_n^3 + O(e_n^4). \tag{20}$$

Proof

Using the definition of error in the Formula (8), we have

$$e_{n+1} + \alpha = e_n + \alpha - \frac{f(e_n + \alpha + a(e_n + \alpha) f(e_n + \alpha))}{f'(e_n + \alpha)}. \tag{21}$$

Using Taylor expansion and taking into account $f(\alpha) = 0$, we have

$$f(e_n + \alpha) = e_n f'(\alpha) + \frac{1}{2} e_n^2 f''(\alpha) + \frac{1}{6} e_n^3 f'''(\alpha) + O(e_n^4), \tag{22}$$

and

$$a(e_n + \alpha) = a(\alpha) + e_n a'(\alpha) + \frac{1}{2} e_n^2 a''(\alpha) + \frac{1}{6} e_n^3 a'''(\alpha) + O(e_n^4). \tag{23}$$

Thus

$$\begin{aligned} a(e_n + \alpha) f(e_n + \alpha) &= e_n a(\alpha) f'(\alpha) + \tfrac{1}{2} e_n^2 a(\alpha) f''(\alpha) \\ &+ \tfrac{1}{6} e_n^3 a(\alpha) f'''(\alpha) + e_n^2 a'(\alpha) f'(\alpha) \\ &+ \tfrac{1}{2} e_n^3 a'(\alpha) f''(\alpha) + \tfrac{1}{2} e_n^3 f'(\alpha) a''(\alpha) + O(e_n^4). \end{aligned} \tag{24}$$

Also we have

$$f'(e_n+\alpha)=f'(\alpha)+e_nf''(\alpha)+\frac{e_n^2}{2}f'''(\alpha)+\frac{e_n^3}{6}f^4(\alpha)+O(e_n^4), \tag{25}$$

and

$$\begin{aligned}
f(e_n+\alpha+a(e_n+\alpha)f(e_n+\alpha)) =&(e_n+a(e_n+\alpha)f(e_n+\alpha))f'(\alpha)\\
&+\frac{1}{2}(e_n+a(e_n+\alpha)f(e_n+\alpha))^2f''(\alpha)\\
&+\frac{1}{6}(e_n+a(e_n+\alpha)f(e_n+\alpha))^3f'''(\alpha)\\
=&e_nf'(\alpha)+e_na(\alpha)f'^2(\alpha)+\frac{1}{2}e_n^2a(\alpha)f'(\alpha)f''(\alpha)\\
&+\frac{1}{6}e_n^3a(\alpha)f'(\alpha)f'''(\alpha)+e_n^2a'(\alpha)f'^2(\alpha)\\
&+\frac{1}{2}e_n^3a'(\alpha)f'(\alpha)f''(\alpha)+\frac{1}{2}e_n^3a''(\alpha)f'^2(\alpha)\\
&+\frac{1}{2}e_n^2f''(\alpha)+\frac{1}{2}e_n^2a^2(\alpha)f'^2(\alpha)f''(\alpha)\\
&+\frac{1}{2}e_n^3a^2(\alpha)f'(\alpha)f''^2(\alpha)+e_n^2a(\alpha)f'(\alpha)f''(\alpha)\\
&+\frac{1}{2}e_n^3a(\alpha)f''^2(\alpha)+e_n^3a'(\alpha)f'(\alpha)f''(\alpha)\\
&+\frac{1}{6}e_n^3f'''(\alpha)+\frac{1}{6}e_n^3a^3(\alpha)f'^3(\alpha)f'''(\alpha)\\
&+\frac{1}{2}e_n^3a(\alpha)f'(\alpha)f'''(\alpha)\\
&+\frac{1}{2}e_n^3a^2(\alpha)f'^2(\alpha)f'''(\alpha)+O(e_n^4).
\end{aligned} \tag{26}$$

Substituting (25) and (26) into (28) we have

$$\begin{aligned}
e_{n+1}=&-e_na(\alpha)f'(\alpha)-\frac{1}{2}e_n^2a(\alpha)f''(\alpha)-\frac{1}{6}e_n^3a(\alpha)f'''(\alpha)\\
&-e_n^2a'(\alpha)f'(\alpha)-\frac{1}{2}e_n^3a'(\alpha)f''(\alpha)-\frac{1}{2}e_n^3a''(\alpha)f'(\alpha)-2e_n^2\frac{f''(\alpha)}{f'(\alpha)}\\
&-\frac{1}{2}e_n^2a^2(\alpha)f'(\alpha)f''(\alpha)\ -\frac{1}{2}e_n^3a^2(\alpha)f''^2(\alpha)\ -e_n^2a(\alpha)f''(\alpha)\\
&-\frac{1}{2}e_n^3a(\alpha)\frac{f''^2(\alpha)}{f'(\alpha)}-e_n^3a'(\alpha)f''(\alpha)\ -\frac{1}{6}e_n^3\frac{f'''(\alpha)}{f'(\alpha)}\\
&-\frac{1}{6}e_n^3a^3(\alpha)f'^2(\alpha)f'''(\alpha)-\frac{1}{2}e_n^3a(\alpha)f'''(\alpha)-\frac{1}{2}e_n^3a^2(\alpha)f'(\alpha)f'''(\alpha)\\
&+e_n^2\frac{f''(\alpha)}{f'(\alpha)}+e_n^2a(\alpha)f''(\alpha)+\frac{1}{2}e_n^3a(\alpha)\frac{f''^2(\alpha)}{f'(\alpha)}+e_n^3a'(\alpha)f''(\alpha)\\
&+\frac{1}{2}e_n^3a(\alpha)\frac{f''^2(\alpha)}{f'^2(\alpha)}\ +\frac{1}{2}e_n^3a^2(\alpha)f''^2(\alpha)\ +e_n^3a(\alpha)\frac{f''^2(\alpha)}{f'(\alpha)}
\end{aligned} \tag{27}$$

$$+\frac{1}{2}e_n^3\frac{f'''(\alpha)}{f'(\alpha)}+\frac{1}{2}e_n^3a(\alpha)f'''(\alpha)-e_n^3\frac{f''^2(\alpha)}{f'^2(\alpha)}$$
$$-e_n^3a(\alpha)\frac{f''^2(\alpha)}{f'(\alpha)}+O(e_n^4).$$

For function $a(x)$ we have

$$e_{n+1}+\alpha=e_n+\alpha-\frac{f(e_n+\alpha+a(e_n+\alpha)f(e_n+\alpha))}{f'(e_n+\alpha)}. \tag{28}$$

By (27) and (19) we have

$$e_{n+1}=-\frac{3}{4}e_n^3\frac{f''^2(\alpha)}{f'^2(\alpha)}+\frac{1}{12}e_n^3\frac{f'''(\alpha)}{f'(\alpha)}+O(e_n^4). \tag{29}$$

By the assumption $c_k=\frac{f^{(k)}(\alpha)}{k!f'(\alpha)},\ \ k=2,3,\cdots$, We obtain

$$e_{n+1}=\left(\frac{1}{2}c_3-3c_2^2\right)e_n^3+O(e_n^4).$$

■

6. Numerical results

In this section, we present the results of some numerical examples to compare the efficiencies of the method. Numerical computations reported here have been carried out in a Mathematica 6 environment. The stopping criterion has been taken as $|x_{n+1}-x_n|\leq 10^{-15}$. In Table 1, we give the number of iterations (i) and the computational order of convergence (coc) ρ.

The computational results show that for some functions the computational order of convergence is even more than three. In this table, NM is the famous Newton method, MP is the method introduced in [3-5], WF is Weerakoon and Fernando method [15], HM is Homeier's method [6-8] and AM is the method introduced in this work. In the Table 1, for function $f_1(x)=(x-1)^3-2$, AM in less number of iterations we yeilds more accurate approximation with better ρ than others. For function $f_2(x)=\sin^2 x-x^2+1$, AM in less number of iterations we yeilds more accurate approximation but ρ is not better than MP's method. For function $f_2(x)=x^3-10$, AM in less number of iterations we yeilds more accurate approximation. Finally for function $f_2(x)=e^{x^2+7x-30}-1$, AM in less number of iterations we yeilds more accurate approximation.

7. Conclusion

In this work we presented a new method for finding the root of an equation by third order of convergence. We compared this method by some other methods and showed the advantages of this method.

Table 1. Comparison between the results.

$f_1(x) = (x-1)^3 - 2,\ x_0 = 1.85$	i	x_i	$f(x_i)$	ρ
NM	4	2.25992	2.8×10^{-6}	1.90409
MP	4	2.25999	-3.9×10^{-11}	3.28075
WF	3	2.25992	-1.5×10^{-9}	3.03351
HM	3	2.25992	-3.9×10^{-11}	3.01927
AM	3	2.25992	1.3×10^{-15}	3.78046
$f_2(x) = \sin^2 x - x^2 + 1,\ x_0 = 1$				
NM	6	1.40450	-1.8×10^{-25}	1.99983
MP	4	1.40449	0.0×10^{-30}	3.02739
WF	5	1.40449	0.0×10^{-30}	3.04104
HM	4	1.40449	0.0×10^{-29}	3.02739
AM	4	1.40450	0.0×10^{-30}	3.02960
$f_3(x) = x^3 - 10,\ x_0 = 1.5$				
NM	4	2.15443	3.8×10^{-6}	1.92243
MP	4	2.15443	-2.1×10^{-11}	3.24993
WF	3	2.15443	-8.2×10^{-10}	3.02399
HM	3	2.15443	-2.1×10^{-11}	3.01394
AM	3	2.15443	-0.8×10^{-11}	3.02123
$f_4(x) = e^{x^2+7x-30} - 1,\ x_0 = 3.25$				
NM	7	3.00000	1.7×10^{-8}	1.97623
MP	5	3.00000	2.5×10^{-6}	2.53315
WF	5	3.00000	2.2×10^{-12}	2.80573
HM	4	3.00000	2.5×10^{-6}	2.53315
AM	4	3.00000	3.0×10^{-12}	2.80248

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