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A Note on Solving Prandtl's Integro-Differential Equation

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Abstract. A simple method for solving Prandtl's integro-differential equation is proposed based on a new reproducing kernel space. Using a transformation and modifying the traditional reproducing kernel method, the singular term is removed and the analytical representation of the exact solution is obtained in the form of series in the new reproducing kernel space. Compared with known investigations, its advantages are that the representation of exact solution is obtained in a reproducing kernel Hilbert space and accuracy in numerical computation is higher. On the other hand, the approximate solution and its derivatives converge uniformly to the exact solution and its derivatives. The final numerical experiments illustrate the method is efficient.

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1. Introduction

Prandtl's singular integro-differential equation and related equations appear in problems of aerofoil and propeller theory in fluid mechanics. Singular integral and integro-differential equations are usually difficult to solve analytically so it is required to obtain the approximate solution. There are a few numerical methods on singular integral and integro-differential equations with Cauchy kernel including Galerkin and collocation methods, Chebyshev polynomials and so on (see [1-11]).

In recent years, the reproducing kernel method is applied for singular integral, integro-differential equations, partial differential equation and fractional differential

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equations [12-27]. In the traditional reproducing kernel method, the representation of reproducing kernel function is complicated and requirement for image space of operator is high.

The purpose of this work is to introduce a reproducing kernel method for solving a singular integro-differential equation of Prandtl's type:

$$v(x)u(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{u'(t)}{t-x}\,\mathrm{d}t + \int_{-1}^{1} h(x,t)u(t)\,\mathrm{d}t = f(x), \quad (|x|<1) \tag{1}$$

where $u(x)$ is the unknown function, and $v(x)$ and $f(x)$ are known functions, is usually supplemented by the conditions [19]

$$u(-1) = u(1) = 0. \tag{2}$$

Now, in ordere to get rid of the derivative of $u(x)$ in (1), we apply the rule of integration by part in the integral containing $u'(x)$. Thus by taking into account (2), we obtain the following equivalent form of (1):

$$v(x)u(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{u(t)}{(t-x)^2}\,\mathrm{d}t + \int_{-1}^{1} h(x,t)u(t)\,\mathrm{d}t = f(x), \quad |x|<1. \tag{3}$$

Following Kalandiya [16] we set

$$u(x) = \omega(x)\phi(x), \quad |x|<1, \tag{4}$$

where $\phi(x)$ is a well-behaved function on the interval $-1<x<1$ and

$$w(x) = \sqrt{1-x^2}.$$

Substituting (4) into (3), we have

$$v(x)\omega(x)\phi(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{\phi(t)}{(t-x)^2}\omega(t)\,\mathrm{d}t + \int_{-1}^{1} h(x,t)\phi(t)\omega(t)\,\mathrm{d}t = f(x). \tag{5}$$

On the other hand, the hypersingular integral term of (5) is equal to (see [5, 8]),

$$\int_{-1}^{1} \omega(t)\frac{\phi(t)}{(t-x)^2}\,\mathrm{d}t = -\pi x\phi'(x) - \pi\phi(x) + \int_{-1}^{1} \frac{-\phi'(t)(t-x)+\phi(t)-\phi(x)}{(t-x)^2}\omega(t)\,\mathrm{d}t. \tag{6}$$

Then (1) can be converted into

$$\begin{aligned} -x\phi'(x) + (v(x)\omega(x)-1)\phi(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{-\phi'(t)(t-x)+\phi(t)-\phi(x)}{(t-x)^2}\omega(t)\,\mathrm{d}t \\ + \int_{-1}^{1} h(x,t)u(t)\,\mathrm{d}t = f(x), \end{aligned} \tag{7}$$

where $\frac{-\phi'(t)(t-x)+\phi(t)-\phi(x)}{(t-x)^2} = \frac{\phi''(x)}{2}$ while $t=x$. It means singular term have been removed.

The remainder of the paper is organized as follows: A reproducing kernel space $W_2^3[-1,1]$ is introduced in Section 2. In Section 3, the analytical representation of the exact solution is obtained in the form of series. We also illustrate the numerical experiment in secation 4. Section 5 ends this paper with a brief conclusion.

2. The reproducing kernel space $W_2^3[-1,1]$

The reproducing kernel space $W_2^3[-1,1]$ is defined by
$W_2^3 \equiv W_2^3[-1,1] = \{\phi(x)|\phi''(x)$ is an absolutely continuous real-valued function on $[-1,1], \phi'''(x) \in L^2[-1,1]\}\}$.

The inner product and norm in W_2^3 are defined respectively by

$$\langle \phi, \varphi \rangle = \sum_{i=0}^{2} \phi^{(i)}(-1)\varphi^{(i)}(-1) + \int_{-1}^{1} \phi'''(t)\varphi'''(t)\,\mathrm{d}t, \quad \forall \phi, \varphi \in W_2^3,$$

$$\|\phi\| = \sqrt{\langle \phi, \phi \rangle}, \quad \forall \phi \in W_2^3.$$

Theorem 2.1 W_2^3 is a reproducing kernel space with reproducing kernel

$$R_y(x) = \begin{cases} \frac{1}{120}\left(276 + x^5 + 195y - 5x^4y + 40y^2 + 10x^3y^2 \right. \\ \qquad \left. +10x^2\left(4+9y+6y^2\right) + 5x\left(39+56y+18y^2\right)\right), & x \le y, \\ \frac{1}{120}\left(276 + 195y + 40y^2 + y^5 + 10x^2(1+y)^2(4+y) \right. \\ \qquad \left. -5x\left(-39-56y-18y^2+y^4\right)\right), & y < x, \end{cases} \tag{8}$$

that is, for every $x \in [-1,1]$ and $\phi \in W_2^3$, $\langle \phi, R_y \rangle = \phi(y)$ is hold.

Theorem 2.2 Let $R_y(x)$ be the reproducing kernel of the space W_2^3. Then,

$$\frac{\partial^{i+j} R_y(x)}{\partial x^i \partial y^j} \in W_2^3[-1,1], \quad i+j=2$$

with respect to x or y.

The proof of Theorems 2.1 and 2.2 can be found in [6].

3. The solution of (7)

Define $\mathbb{L} : W_2^3 \to L^2[-1,1]$ as follows

$$(\mathbb{L}\phi)(x) = -x\phi'(x) + (v(x)\omega(x) - 1)\phi(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{-\phi'(t)(t-x) + \phi(t) - \phi(x)}{(t-x)^2}\omega(t)\,\mathrm{d}t + \int_{-1}^{1} h(x,t)\phi(t)\omega(t)\,\mathrm{d}t. \tag{9}$$

Then (1) turns into

$$(\mathbb{L}\phi) = f \quad |x| < 1, \tag{10}$$

where if $t = x$, then $\frac{-\phi'(t)(t-x)+\phi(t)-\phi(x)}{(t-x)^2} = \frac{\phi''(x)}{2}$. In order to obtain the exact solutions of (10), define

$$\psi_i(x) = [\mathbb{L}_y R_x(y)]\,(x_i) \quad i = 1, 2, \ldots, \tag{11}$$

where $\{x_i\}_{i=1}^{\infty}$ in the domain $[-1, 1]$. Hence, one gets

$$\begin{aligned}\psi_i(x) &= -x_i \frac{\partial R_x(x_i)}{\partial x} + (v(x_i)\omega(x_i) - 1)R_x(x_i)\\ &+ \frac{1}{\pi}\int_{-1}^{1} \frac{\frac{\partial R_x(x_i)}{\partial x}(t - x_i) + R_x(t) - R_x(x_i)}{(t - x_i)^2}\omega(t)\,\mathrm{d}t\\ &+ \int_{-1}^{1} h(x_i, t)R_x(t)\omega(t)\,\mathrm{d}t. \end{aligned}\tag{12}$$

According to [12, 14], we have the following theorem:

Theorem 3.1 Linear operator $\mathbb{L}$ maps W_2^3 into $L_2[-1, 1]$.

Lemma 3.0.1 $\{\psi_i(x)\}_{i=1}^{\infty}$ is complete in W_2^3 if $\{x_i\}_{i=1}^{\infty}$ is dense in $[-1, 1]$.

Proof. For each $\phi(x) \in W_2^3$, let $\langle\phi, \psi_i\rangle = 0, (i = 1, 2, \ldots)$, which means

$$\langle\phi, \psi_i\rangle = \langle\phi, [\mathbb{L}_y R_x]\,(x_i)\rangle \tag{13}$$
$$= \mathbb{L}_y\langle\phi, R_x\rangle(x_i) \tag{14}$$
$$= [\mathbb{L}_y\phi(y)]\,(x_i) = 0. \tag{15}$$

From the density of $\{x_i\}_{i=1}^{\infty}$ and uniqueness of solution on (10), It follows that $\phi(x) \equiv 0$. So $\{\psi_i(x)\}_{i=1}^{\infty}$ is complete in W_2^3.

The orthonormal system $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ of W_2^3 can be derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$ as

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \qquad \beta_{ii} > 0, \quad i = 1, 2, \ldots. \tag{16}$$

Theorem 3.2 Let $\{x_i\}_{i=1}^{\infty}$ be dense on $[-1, 1]$, then the exact solution of (10) could be represented by

$$\phi(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\bar{\psi}_i(x). \tag{17}$$

Proof. Let $\phi(x)$ be solution of (10) in W_2^3. From $\{\bar{\psi}\}_{i=1}^{\infty}$ is an orthonormal system,

$\phi(x)$ could be expressed Fourier series

$$\begin{aligned}\phi(x) &= \sum_{i=1}^{\infty}\langle\phi,\bar{\psi}_i\rangle\bar{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle\phi,\psi_k\rangle\bar{\psi}_i(x)\\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left[\mathbb{L}_y\phi(y)\right](x_k)\bar{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\mathbb{L}\phi(x_k)\bar{\psi}_i(x)\\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k)\bar{\psi}_i(x).\end{aligned}$$

Now the approximate solution $\phi_n(x)$ can be obtained by the n-term intercept of the analytical solution $\phi(x)$, that is,

$$\phi_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}f(x_k)\bar{\psi}_i(x). \tag{18}$$

From (4), we know

$$u(x) = \omega(x)\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k)\bar{\psi}_i(x) \tag{19}$$

and

$$u_n(x) = \omega(x)\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}f(x_k)\bar{\psi}_i(x) \tag{20}$$

are the exact solution and the approximate solution of (1), respectively.

Theorem 3.3 The approximate solution $u_n(x)$ and its derivatives $u_n^{(k)}(x)$, $0 \le k \le 2$, are all uniformly convergent.

Proof. Let $R_y(x)$ be the reproducing kernel of the space W_2^3. Because

$$\begin{aligned}u_n(x) - u(x) &= \langle u_n - u, R_x\rangle_{W_2^3},\\ u_n^{(k)}(x) - u^{(k)}(x) &= (u_n(x) - u(x))^{(k)}\\ = \frac{\partial^k}{\partial x^k}\left[\langle u_n - u, R_x\rangle_{W_2^3}\right] &= \left\langle u_n - u, \frac{\partial^k}{\partial x^k}R_x\right\rangle_{W_2^3}\end{aligned}$$

Since $\frac{\partial^k}{\partial x^k}R_x(y) \in W_2^3$, by Theorem (2.2), one obtains

$$|u_n^{(k)}(x) - u^{(k)}(x)| \le \|u_n - u\|_{W_2^3}\|\frac{\partial^k}{\partial x^k}R_x\|_{W_2^3}.$$

Also $\|\frac{\partial^k}{\partial x^k}R_x\|_{W_2^3}$ is continuous with respect to x in $[-1, 1]$, then

$$|u_n^{(k)}(x) - u^{(k)}(x)| \le M\|u_n - u\|_{W_2^3},$$

where M is a positive number. So that,

$$\lim_{n\to\infty} u_n(x) = u(x) \Longrightarrow \lim_{n\to\infty} u_n^{(k)}(x) \overset{unif}{=} u^{(k)}(x).$$

Theorem 3.4 Let that $u(x)$ and $u_n(x)$ are the exact solution and the approximate solution of (1) in space W_2^3, respectively. and $e_n(x)$ is the error between the exact solution $u(x)$ and the approximate solution $u_n(x)$.
Then,

(1) $e_n(x)$ is monotonically decreasing in the sense of norm $\|.\|_{W_2^3}$.
(2) $\|e_n^{(k)}\| = O(h)$, where $h = max_{1\leqslant i\leqslant n-1}|h_{i+1} - h_i|$ and $k = 0, 1, 2$.

Proof.

(1) Note that

$$\|e_n\|^2 = \|u - u_n\|^2 = \|\omega \sum_{i=n+1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\bar{\psi}_i\|^2 = \|\omega\| \sum_{i=n+1}^{\infty}(\sum_{k=1}^{i} \beta_{ik} f(x_k))^2.$$

Similarly,

$$\|e_{n-1}\|^2 = \|u-u_{n-1}\|^2 = \|\omega \sum_{i=n}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\bar{\psi}_i\|^2 = \|\omega\| \sum_{i=n}^{\infty}(\sum_{k=1}^{i} \beta_{ik} f(x_k))^2.$$

This means that the error $\{e_n\}$ is monotonically decreasing in the sense of $\|.\|_{W_2^3}$. From Theorem 3.2, it is noted that (20) is convergent in the norm of W_2^3.

(2) Note here that

$$\mathbb{L}u(x) = \omega(x) \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\mathbb{L}\bar{\psi}_i(x)$$

then

$$\begin{aligned}
(\mathbb{L}u)(x_n) &= \omega(x_n) \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\mathbb{L}\bar{\psi}_i(x_n) \\
&= \omega(x_n) \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)(\varphi_n, \mathbb{L}\bar{\psi}_i) \\
&= \omega(x_n) \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)(\mathbb{L}^*\varphi_n, \bar{\psi}_i) \\
&= \omega(x_n) \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)(\psi_n, \bar{\psi}_i).
\end{aligned}$$

Therefore,

$$\sum_{j=1}^{n}\beta_{nj}(\mathbb{L}u)(x_j) = \omega(x_j)\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k)(\sum_{j=1}^{n}\beta_{nj}\psi_j,\bar{\psi}_i)$$
$$= \omega(x_j)\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k)(\bar{\psi}_n,\bar{\psi}_i) = \omega(x_j)\sum_{k=1}^{n}\beta_{nk}f(x_k).$$

Moreover, it is easy to see by induction that

$$(\mathbb{L}u)(x_j) = \omega(x_j)f(x_j), \quad j = 1,2,...,n. \tag{21}$$

Similarly, one can show that

$$(\mathbb{L}u_n)(x_j) = \omega(x_j)f(x_j), \quad j = 1,2,...,n. \tag{22}$$

Now, from (21), (22), we have

$$(\mathbb{L}u)(x_j) = (\mathbb{L}u_n)(x_j) = \omega(x_j)f(x_j), \quad j = 1,2,\ldots,n.$$

Therefore, $(\mathbb{L}u_n)(x_j)$ is the interpolating function of $(\mathbb{L}u)(x_j)$, where $x_j(j = 1,2,\ldots,n)$ are the interpolation nodes in $[-1,1]$.

By (21), (22) and by means of value theorem for differentials, we have

$$\begin{aligned}\mathbb{L}(u(x) - u_n(x)) &= \mathbb{L}(u(x) - u(x_j) + u_n(x_j) - u_n(x))\\ &= \mathbb{L}(u'(\zeta)(x - x_j) + u_n'(\varsigma)(x_j - x))\\ &= (x_{j+1} - x_j)(\mathbb{L}u'(\zeta)\frac{x - x_j}{x_{j+1} - x_j}\\ &+ \mathbb{L}u_n'(\varsigma)\frac{x_j - x}{x_{j+1} - x_j}) = h\rho(x),\end{aligned}$$

where $\zeta,\varsigma \in (x,x_j), \rho(x) = \mathbb{L}u'(\zeta)\frac{x-x_j}{x_{j+1}-x_j} + \mathbb{L}u_n'(\varsigma)\frac{x_j-x}{x_{j+1}-x_j}$. It is clear that $\rho(x) \in C[-1,1]$ and since $\mathbb{L}^{-1}$ exists, then

$$\mathbb{L}^{-1}f(x) = u(x) = \omega(x)\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k)\bar{\psi}_i(x). \tag{23}$$

Therefore, $\mathbb{L}^{-1}$ can be obtained by (23). Note here that $\mathbb{L}^{-1}\rho(x) \in W_2^3$, thus, $\|\mathbb{L}^{-1}\rho\|$ is bounded. it follows that

$$\|e_n\| = \|u - u_n\| = \|\mathbb{L}^{-1}(h\rho)\| = h\|\mathbb{L}^{-1}\rho\| = O(h).$$

The proof is complete.

4. Numerical examples

In this section, two numerical examples are provided to show the accuracy of the present method. In our numerical calculations, we have chosen

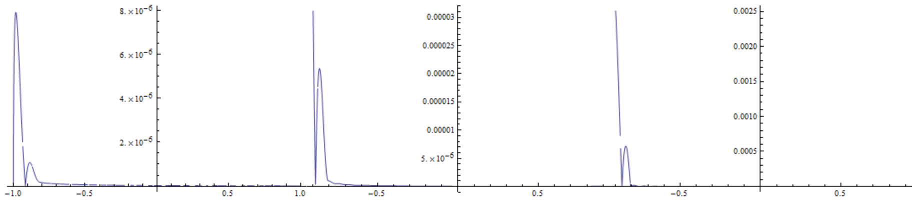

Figure 1. The absolute errors of $u(x)$ (left), $u'(x)$ (center) and $u''(x)$ (right) for case (a).

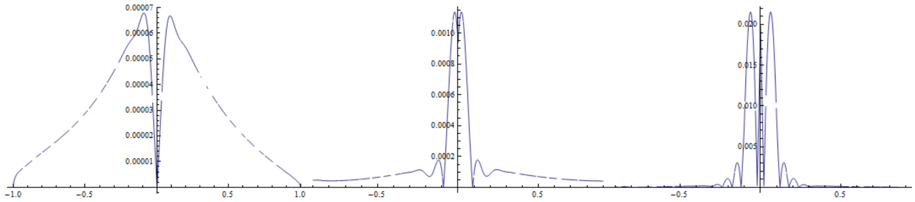

Figure 2. The absolute errors of $u(x)$ (left), $u'(x)$ (center) and $u''(x)$ (right) for case (b).

the four-point Gauss-Chebyshev quadrature rule of the second kind. All computations are performed by Mathematica 8.

The original (1) or, which is the same, (10) in the case of $v(x) = \vartheta_0$ =const and $h(x,t) = h_1(x,t) - \vartheta_1 \ln|x-t|$ which means an equation of the form

$$\vartheta_0 u(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{u'(t)}{(t-x)}\,\mathrm{d}t + \int_{-1}^{1} \left[h_1(x,t) - \vartheta_1 \ln|x-t|\right] u(t)\,\mathrm{d}t = f(x). \quad (24)$$

We apply the present method in Section 3 to (24) with

a) $\vartheta_0 = 0, \vartheta_1 = 0, h_1(x,t) = x + t, f(x) = \frac{\pi}{2}\left(1 - 6x^2\right) + \frac{\pi}{8}x,$

b) $\vartheta_0 = 1, \vartheta_1 = 0, h_1(x,t) = t(x^2|x| + t|t|),$
$f(x) = x\left(\left(1 + \frac{4x}{15\pi}\right)\mathrm{Abs}[x] + \frac{6}{\pi} + \left(\frac{(3x^2-2)}{\pi\sqrt{1-x^2}}\right)\mathrm{Log}\left[\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right]\right).$

In case (a) (24) possesses the solution $u(x) = \sqrt{1-x^2}x^2$, and in case (b) the solution $u(x) = \sqrt{1-x^2}x|x|$.

Using the method presented in Section 3, taking $n = 30$, $x_i = -1 + \frac{2i}{\mathrm{n}+1}, i = 1, 2, ..., n$, the numerical results of $u_n(x)$ are shown in Figs. $1-2$. It can be seen that the approximate solution and its derivatives converge uniformly to the exact solution and its derivatives, respectively.

5. Conclusion

In this paper, a new method is proposed for solving Prandtl's integro-differential equation. Here we obtained a series of solutions for Prandtl's integro-differential equation in reproducing kernel space. The weak singularity of (1) was successfully removed by applying smooth transformations. A representation of the solutions was obtained by applying a linear operator and a reproducing kernel function. It is worthy to note that, in our work, the approximate solution and its derivatives converge uniformly to the exact solution and its derivatives, respectively. The method used in this paper can be generalized to solve a two-dimensional thermoelastic contact problem involving frictional heating [3], the other appropriate integral and integro-differential equations with singular kernel.

References

[1] O. A. Arqub, M. Al-Smadi and S. Momani, Application of reproducing kernel method for solving nonlinear Fredholm–Volterra integro-differential equations, J. Abstr. Appl. Anal, (2012) 1-16.

[2] H. Beyrami, T. Lotfi and K. Mahdiani, Stability and error analysis of the reproducing kernel Hilbert space method for the solution of weakly singular Volterra integral equation on graded mesh, J. Appl. Numer. Math, **120** (2017) 197-214.

[3] J. R. Barber, Some thermoelastic contact problems involving frictional heating, Quart. J. Mech. Appl. Math, **19** (1976) 1-13.

[4] Z. Chen and Y. F. Zhou, A new method for solving Hilbert type singular integral equations, J. Appl. Math. Comput, **218** (2011) 406-412.

[5] Z. Chen and Y. F. Zhou, A new method for solving hypersingular integral equations of the first kind, J. Appl. Math. Lett, **24** (2011) 636-641.

[6] M. Cui and Y. Lin, Nonlinear Numerical Analysis in the Reproducing Kernel Space, Nova Science Publisher, New York, (2009).

[7] A. Dezhbord, T. Lotfi and K. Mahdiani, A numerical approach for solving the high-order nonlinear singular Emden-Fowler type equations, Journal of Advances in Difference Equations, **2018** (2018) 1-17.

[8] A. Dezhbord, T. Lotfi and K. Mahdiani, A new efficient method for cases of the singular integral equation of the first kind, J. Comput. Appl. Math, **296** (2016) 156-169.

[9] H. Du and J. H. Shen, Reproducing kernel method of solving singular integral equation with cosecant kernel, J. Math. Anal. Appl, **348** (2008) 308-314.

[10] H. Du, G. Zhao and C. Zhao, Reproducing kernel method for solving Fredholm integro-differential equations with weakly singularity, J. Comput. Appl. Math, **255** (2014) 122-132.

[11] V. J. Ervin and E. P. Stephan, Collocation with Chebyshev polynomials for a hypersingular integral equation on an interval, J. Comput. Appl. Math, **43** (1992) 221-229.

[12] F. Z. Geng and M. G. Cui, Solving a nonlinear system of second order boundary value problems, J. Math. Anal. Appl, **327** (2007) 1167-1181.

[13] F. Z. Geng and Z. Q. Tang, Piecewise shooting reproducing kernel method for linear singularly perturbed boundary value problems, Appl. Math. Lett, **52** (2016) 156-161.

[14] M. A. Golberg, The convergence of several algorithms for solving integral equations with finite-part integrals, J. Int. Eqs, **5** (1983) 329-340.

[15] P. Junghanns and B. Silbermann, The numerical treatment of singular integral equations by means of polynomial approximations. I, preprint P-Math-35/86, AdW der DDR, Karl-WeierstraB-Institut ffir Mathematik, Berlin, (1986).

[16] A. I. Kalandiya, Mathematical Methods of Two-Dimensional Elasticity, Mir, Moscow, (1975).

[17] A. C. Kaya and F. Erdogan, On the solution of integral equations with strongly singular kernels, Quart. Appl. Math, **XLV** (1987) 105-122.

[18] Y. Z. Lin, M. G. Cui and L. H. Yang, Representation of the exact solution for a kind of nonlinear partial differential equation, J. Appl. Math. Lett, **19** (2006) 808-813.

[19] N. I. Loakimidits, A natural interpolation formula for Prandtl's singular integroctifferential equation, Internat. J. Numer. Methods Fluids, **4** (1984) 283-290.

[20] X. Lv and Y. Gao, Reproducing Kernel Space Method for the Solution of Linear Fredholm Integro-Differential Equations and Analysis of Stability, J. Abs. Appl. Anal, (2012).

[21] K. S. Parihar and M. P. Ramachandran, On the numerical solution of Prandtel's integral equaion, j. Appl. Num. Math, **5** (1989) 223-235.

[22] S. Prössdorf and B. Silbermann, Numerical Analysis for Integral and Related Operator Equations, Akademie Verlag, Berlin, (1991).

[23] M. Rosaria, G. Criscuolo and P. Junghanns, A fast algorithm for Prandtl's integro-differential equation, J. Comput. Appl. Math, **77** (1997) 103-128.

[24] V. V. Sil'vestrov and A. V. Smirnov, The Prandtl integrodifferential equation and the contact problem for a piecewise homogeneous plate, J. Appl. Math. Mech, **74** (2010) 679-691.

[25] B. Wu and X. Li, Iterative reproducing kernel method for nonlinear oscillator with discontinuity, J. Appl. Math. Lett, **23** (2010) 1301-1304.

[26] M. Q. Xu and Y. Z. Lin, Simplified reproducing kernel method for fractional differential equations with delay, Appl. Math. Lett, **52** (2016) 156-161.

[27] Y. Zhou and Y. Lin, Solving integro-differential equations with Cauchy kernel, J. Appl. Math. Comput, **215** (2009) 2438-2444.