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Somewhat-connectedness and *somewhat*-continuity in the product space

M. S. Bilao [a], M. A. Labendia[a,*]

[a] *Department of Mathematics and Statistics, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City, Philippines.*

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Abstract. In this paper, the concept of *somewhat*-connected space will be introduced and characterized. Its connection with the other well-known concepts such as the classical connectedness, the ω_θ-connectedness, and the ω-connectedness will be determined. Moreover, the concept of *somewhat*-continuous function from an arbitrary topological space into the product space will be characterized.

Keywords: *Somewhat*-open, *somewhat*-closed, *somewhat*-connected, *somewhat*-continuous.

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1. Introduction and preliminaries

In literature, substituting several concepts in topology with concepts acquiring either of weaker or stronger properties is often studied. The first attempt was done by Levine [24] when he introduced the concepts of semi-open set, semi-closed set, and semi-continuity of a function. Several mathematicians then became interested in introducing other topological concepts which can replace the concept of open set, closed set, and continuity of a function.

In 1968, Velicko [27] introduced the concepts of θ-continuity between topological spaces and subsequently defined the concepts of θ-closure and θ-interior of a subset of topological space. The concept of θ-open sets and its related topological concepts had been deeply studied and investigated by numerous authors, see [1, 7, 8, 15, 16, 20–22, 25, 26].

*Corresponding author.
E-mail address: mohamidin.bilao@g.msuiit.edu.ph (M. S. Bilao); mhelmar.labendia@g.msuiit.edu.ph (M. A. Labendia).

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Let (X, T) be a topological space and $A \subseteq X$. The *θ-closure* and *θ-interior* of A are, respectively, denoted and defined by $Cl_\theta(A) = \{x \in X : Cl(U) \cap A \neq \varnothing$ for every open set U containing $x\}$ and $Int_\theta(A) = \{x \in X : Cl(U) \subseteq A$ for some open set U containing $x\}$, where $Cl(U)$ is the closure of U in X. A subset A of X is *θ-closed* if $Cl_\theta(A) = A$ and *θ-open* if $Int_\theta(A) = A$. Equivalently, A is θ-open if and only if $X \setminus A$ is θ-closed.

In 1971, Hoyle and Gentry [19] introduced the class of *somewhat*-continuous functions and *somewhat*-open functions. The *somewhat*-continuous functions, which are generalization of continuity requiring nonempty inverse images of open sets to have nonempty interiors instead of being open, have proved to be very useful in topology. Since then, the concepts of *somewhat*-interior and *somewhat*-closure of a subset of a topological space have been subsequently defined and the concept of *somewhat*-open and *somewhat*-closed sets have been used to characterize *somewhat*-continuity, see [5, 6].

A subset U of a space X is said to be *somewhat*-open if $U = \varnothing$ or if there exists $x \in U$ and an open set V such that $x \in V \subseteq U$. A set is called *somewhat*-closed if its complement is somewhat open. Denote by T_{sw}, the collection of all *somewhat*-open sets in X. Let A be a subset of a space X. The *somewhat*-closure and *somewhat*-interior of A are, respectively, denoted and defined by $swCl(A) = \cap\{F : F$ is *somewhat*-closed and $A \subseteq F\}$ and $swInt(A) = \cup\{U \subseteq A : U$ is *somewhat*-open$\}$.

Let X and Y be topological spaces. A function $f : X \to Y$ is said to be

(*i*) *somewhat*-open provided that for every open set U of X such that $U \neq \varnothing$, there exists an open set V of Y such that $V \neq \varnothing$ and $V \subseteq f(U)$;

(*ii*) *somewhat*-closed if for every closed set F of X such that $F \neq \varnothing$, there exists a closed set G of Y such that $\varnothing \neq G \subseteq f(F)$; and

(*iii*) *somewhat*-continuous if for every open set V of Y such that $f^{-1}(V) \neq \varnothing$ there exists an open set U of X such that $U \neq \varnothing$ and $U \subseteq f^{-1}(V)$.

In 1982, Hdeib [18] introduced the concepts of ω-open and ω-closed sets and ω-closed mappings on a topological space. He showed that ω-closed mappings are strictly weaker than closed mappings and also showed that the Lindelöf property is preserved by counter images of ω-closed mappings with Lindelöf counter image of points. The concepts of ω-open sets and its corresponding topological concepts had been studied in several papers, see [3, 4, 9–14, 23].

In 2010, Ekici et al. [17] introduced the concepts of ω_θ-open and ω_θ-closed sets on a topological space. They showed that the family of all ω_θ-open sets in a topological space X forms a topology on X. They also introduced the notions of ω_θ-interior and ω_θ-closure of a subset of a topological space.

A point x of a topological space X is called a condensation point of $A \subseteq X$ if for each open set G containing x, $G \cap A$ is uncountable. A subset B of X is ω-closed if it contains all of its condensation points. The complement of B is ω-open. Equivalently, a subset U of X is ω-open (resp., ω_θ-open) if and only if for each $x \in U$, there exists an open set O containing x such that $O \setminus U$ (resp., $O \setminus Int_\theta(U)$) is countable. A subset B of X is ω_θ-closed if its complement $X \setminus B$ is ω_θ-open. The ω-closure (resp., ω_θ-closure) and ω-interior (resp., ω_θ-interior) of $A \subseteq X$ are, respectively, denoted and defined by $Cl_\omega(A) = \cap\{F : F$ is an ω-closed set containing $A\}$ (resp., $Cl_{\omega_\theta}(A) = \cap\{F : F$ is an ω_θ-closed set containing $A\}$) and $Int_\omega(A) = \cup\{G : G$ is an ω-open set contained in $A\}$ (resp., $Int_{\omega_\theta}(A) = \cup\{G : G$ is an ω_θ-open set contained in $A\}$). It is worth noting that $A \subseteq Cl_{\omega_\theta}(A)$ (resp., $A \subseteq Cl_\theta(A)$, $A \subseteq Cl_\omega(A)$) and $Int_{\omega_\theta}(A) \subseteq A$ (resp., $Int_\theta(A) \subseteq A$, $Int_\omega(A) \subseteq A$). Let T_{ω_θ} (resp., T_θ, T_ω) be the family of all ω_θ-open (resp., θ-open, ω-open) subsets of a topological space X. Since T_{ω_θ} (resp., T_θ, T_ω)

is a topology on X for any set $A \subseteq X$, $Int_{\omega_\theta}(A)$ (resp., $Int_\theta(A)$, $Int_\omega(A)$) is ω_θ-open (resp., θ-open, ω-open) and the largest ω_θ-open (resp., θ-open, ω-open) set contained in A. Moreover, for any set $A \subseteq X$, $Cl_{\omega_\theta}(A)$ (resp., $Cl_\theta(A)$, $Cl_\omega(A)$) is ω_θ-closed (resp., θ-closed, ω-closed) and the smallest ω_θ-closed (resp., θ-closed, ω-closed) set containing A.

A topological space X is said to be *somewhat*-connected (resp., θ-connected, ω-connected, ω_θ-connected) if X cannot be written as the union of two nonempty disjoint *somewhat*-open (resp., θ-open, ω-open, ω_θ-open) sets. Otherwise, X is *somewhat*-disconnected (resp., θ-disconnected, ω-disconnected, ω_θ-disconnected). A subset B of X is *somewhat*-connected (resp., θ-connected, ω-connected, ω_θ-connected) if it is *somewhat*-connected (resp., θ-connected, ω-connected, ω_θ-connected) as a subspace of X.

A function f from a topological space X to another topological space Y is said to be

- (i) ω_θ-open (resp., θ-open, ω-open) if $f(G)$ is ω_θ-open (resp., θ-open, ω-open) in Y for every open set G in X; and
- (ii) ω_θ-closed (resp., θ-closed, ω-closed) if $f(G)$ is ω_θ-closed (resp., θ-closed, ω-closed) in Y for every closed set G in X;

Let A be an indexing set and $\{Y_\alpha : \alpha \in A\}$ be a family of topological spaces. For each $\alpha \in A$, let T_α be the topology on Y_α. The Tychonoff topology on $\Pi\{Y_\alpha : \alpha \in A\}$ is the topology generated by a subbase consisting of all sets $p_\alpha^{-1}(U_\alpha)$, where the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in A\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$, U_α ranges over all members of T_α, and α ranges over all elements of A. Corresponding to $U_\alpha \subseteq Y_\alpha$, denote $p_\alpha^{-1}(U_\alpha)$ by $\langle U_\alpha \rangle$. Similarly, for finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$, and sets $U_{\alpha_1} \subseteq Y_{\alpha_1}$, $U_{\alpha_2} \subseteq Y_{\alpha_2}, \ldots, U_{\alpha_n} \subseteq Y_{\alpha_n}$, the subset

$$\langle U_{\alpha_1} \rangle \cap \langle U_{\alpha_2} \rangle \cap \cdots \cap \langle U_{\alpha_n} \rangle = p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap p_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \cdots \cap p_{\alpha_n}^{-1}(U_{\alpha_n})$$

is denoted by $\langle U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n} \rangle$. We note that for each open set U_α subset of Y_α, $\langle U_\alpha \rangle = p_\alpha^{-1}(U_\alpha) = U_\alpha \times \Pi_{\beta \neq \alpha} Y_\beta$. Hence, a basis for the Tychonoff topology consists of sets of the form $\langle B_{\alpha_1}, B_{\alpha_2}, ..., B_{\alpha_k} \rangle$, where B_{α_i} is open in Y_{α_i} for every $i \in K = \{1, 2, ..., k\}$.

Now, the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in A\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$ for each $\alpha \in A$. It is known that every projection map is a continuous open surjection. Also, it is well known that a function f from an arbitrary space X into the Cartesian product Y of the family of spaces $\{Y_\alpha : \alpha \in A\}$ with the Tychonoff topology is continuous if and only if each coordinate function $p_\alpha \circ f$ is continuous, where p_α is the α-th coordinate projection map.

In this paper, we revisit the concept of *somewhat*-open sets and characterize its related topological concepts such as *somewhat*-connected space and *somewhat*-continuous functions from an arbitrary topological space into the product space.

2. *Somewhat*-open and *somewhat*-closed functions

In this section, we investigate the connection of *somewhat*-open (resp., *somewhat*-closed) function to the other well-known functions such as the classical open, θ-open, ω-open, and ω_θ-open (resp., closed, θ-closed, ω-closed, ω_θ-closed) functions. We also give some characterizations of a *somewhat*-open function. Throughout, if no confusion arises, let X and Y be topological spaces.

Remark 1 *The arbitrary union of somewhat-open sets is somewhat-open, but the finite intersection of somewhat-open sets is not necessarily somewhat-open. This means that*

T_{sw} may not be a topology on a set X.

Consider topological space (X, T) where $X = \{1, 2, 3\}$ and $T = \{\varnothing, X, \{1\}, \{2\}, \{1, 2\}\}$. Then $\{1, 3\}$ and $\{2, 3\}$ are both *somewhat*-open sets but their intersection, which is $\{3\}$, is not *somewhat*-open.

Remark 2 *Let X be a topological space and $A, B \subseteq X$.*

- (i) *$swInt(A)$ is somewhat-open and $swInt(A) \subseteq A$.*
- (ii) *$swCl(A)$ is somewhat-closed and $A \subseteq swCl(A)$.*
- (iii) *$swInt(A)$ is the largest somewhat-open set contained in A;*
- (iv) *If $A \subseteq B$, then $swInt(A) \subseteq swInt(B)$;*
- (v) *$x \in swInt(A)$ if and only if there exists a somewhat-open set U containing x such that $U \subseteq A$;*
- (vi) *A is somewhat-open if and only if $A = swInt(A)$;*
- (vii) *$swInt(swInt(A)) = swInt(A)$;*
- $(viii)$ *$swInt(A \cap B) \subseteq swInt(A) \cap swInt(B)$;*
- (ix) *$swCl(A)$ is the smallest somewhat-closed set containing A;*
- (x) *$A \subseteq B$ implies that $swCl(A) \subseteq swCl(B)$;*
- (xi) *$swCl(swCl(A)) = swCl(A)$;*
- (xii) *$swCl(A) \cup swCl(B) \subseteq swCl(A \cup B)$;*

The following three results can proved by modifying the construction given in [1], [2], and [17], respectively.

Theorem 2.1 Let $f : X \to Y$ be a function. If f is θ-open (resp., θ-closed), then f is open (resp., closed), but not conversely.

Theorem 2.2 Let $f : X \to Y$ be a function. If f is open (resp., closed), then f is ω-open (resp., ω-closed), but not conversely.

Theorem 2.3 Let $f : X \to Y$ be a function.

- (i) If f is θ-open (resp., θ-closed), then f is ω_θ-open (resp., ω_θ-closed), but not conversely.
- (ii) If f is ω_θ-open (resp., ω_θ-closed), then f is ω-open (resp., ω-closed), but not conversely.

Theorem 2.4 Let X be topological space and $A \subseteq X$. If A is open (resp., closed) then A is *somewhat*-open (resp., *somewhat*-closed), but not conversely.

Proof. Suppose that A is open. It is immediate when $A = \varnothing$. Now, assume that $A \neq \varnothing$ and let $x \in A$. Then $x \in A \subseteq A$, implying that A is *somewhat*-open. If A is closed then $X \setminus A$ is *somewhat*-open. By the definition of a *somewhat*-closed set, $X \setminus (X \setminus A) = A$ is *somewhat*-closed.

To show that the converses do necessarily hold, consider the topological space (X, T), where $X = \{a, b, c\}$ and $T = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. Let $U = \{a, c\}$. Then U is not an open set. Note that $a \in U$ and $\{a\} \in T$. Hence, U is *somewhat*-open. Moreover, $\{b\}$ is *somewhat*-closed but not closed. ■

Theorem 2.5 Let $f : X \to Y$ be a function. If f is open (resp., closed), then f is *somewhat*-open (resp., *somewhat*-closed), but not conversely.

Remark 3 *The following diagrams hold for a function $f : X \to Y$.*

Remark 4 *Somewhat-open (resp., closed) set and ω-open (resp., closed) set are two independent notions.*

$$\begin{array}{ccccc} \omega_\theta - open & \longrightarrow & \omega - open & & \\ \uparrow & & \uparrow & & \\ \theta - open & \longrightarrow & open & \longrightarrow & somewhat - open \end{array}$$

$$\begin{array}{ccccc} \omega_\theta - closed & \longrightarrow & \omega - closed & & \\ \uparrow & & \uparrow & & \\ \theta - closed & \longrightarrow & closed & \longrightarrow & somewhat - closed \end{array}$$

To see this, first we will construct an ω-open (resp., ω-closed) set which is not *somewhat*-open (resp., *somewhat*-closed). Second, we will construct a *somewhat*-open (resp., *somewhat*-closed) set which is not an ω-open (resp., ω-closed) set. Now, consider the topological space $X = \{a, b, c\}$ with the topology $T = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}\}$. We will show that $A = \{c\}$ is ω-open but not *somewhat*-open in (X, T). Since X is countable, A is ω-open, but not *somewhat*-open since we cannot find open subset V containing c such that $V \subseteq A$. Moreover, $X \setminus A = \{a, b\}$ is ω-closed but not *somewhat*-closed. For the second part, consider $\mathbb{R}$ as the real line with the topology $T = \{\mathbb{R}, \varnothing, \mathbb{N} \cup \{0\}\}$. We will show that $\mathbb{Z}$ is *somewhat*-open but not ω-open set in $(\mathbb{R}, T)$. Note that, $0 \in \mathbb{Z}$, $\mathbb{N} \cup \{0\}$ is an open set containing 0 and $\mathbb{N} \cup \{0\} \subseteq \mathbb{Z}$. Hence, $\mathbb{Z}$ is *somewhat*-open. Next, we will show that $\mathbb{Z}$ is not ω-open. Proceeding by contradiction, suppose that $\mathbb{Z}$ is ω-open. Since $-1 \in \mathbb{Z}$ but $\mathbb{N} \cup \{0\}$ does not contain -1. Thus, $\mathbb{R}$ is the only open set containing -1 which implies that $\mathbb{R} \setminus \mathbb{Z}$ is countable which is a contradiction. Thus, $\mathbb{Z}$ is not ω-open. Furthermore $\mathbb{R} \setminus \mathbb{Z}$ is *somewhat*-closed but not ω-closed.

Remark 5 *Somewhat-open (resp., closed) set and ω_θ-open (resp., closed) set are two independent notions.*

To see this, consider the topological space $X = \{x, y, z\}$ with the topology $T = \{\varnothing, X, \{x\}, \{z\}, \{x, z\}\}$. We will show that $A = \{y\}$ is ω_θ-open but not *somewhat*-open in (X, T). Since X is countable, A is ω_θ-open. But A is not *somewhat*-open since we cannot find open subset V containing y such that $V \subseteq A$. Moreover, $\{x, z\}$ is ω_θ-closed but not somewhat closed.

On the other hand, suppose that *somewhat*-open (resp., *somewhat*-closed) implies ω_θ-open (resp., ω_θ-closed). Then *somewhat*-open (resp., *somewhat*-closed) implies ω-open (resp., ω-closed), which is a contradiction.

The next result presents a characterization of *somewhat*-open functions.

Theorem 2.6 Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:

(i) f is *somewhat*-open on X.
(ii) $f(O)$ is *somewhat*-open in Y for every open set O in X.
(iii) $f(Int(A)) \subseteq swInt(f(A))$ for every $A \subseteq X$.
(iv) f sends each member of a basis for X to a *somewhat*-open set in Y.
(v) For each $p \in X$ and every open set O in X containing p, there exists a *somewhat*-open set W in Y such that $f(p) \in swInt(W) \subseteq f(O)$.

Proof. $(i)\Rightarrow(ii)$: Suppose that f is *somewhat*-open and let O be an open set in X. If $O = \varnothing$ then $f(O) = \varnothing$ and so $f(O)$ is *somewhat*-open in Y. Now, assume that $O \neq \varnothing$. Then, by the definition of a *somewhat*-open function, there exists an open set V in Y such that $V \neq \varnothing$ and $V \subseteq f(O)$. Thus $f(O)$ is *somewhat*-open in Y.

$(ii)\Rightarrow(iii)$: Suppose that $f(O)$ is *somewhat*-open in Y for every open set O in X and let $A \subseteq X$. Then $f(Int(A)) \subseteq f(A)$. Since $Int(A)$ is open in X, $f(Int(A))$ is *somewhat*-open in Y. Thus, $f(Int(A)) \subseteq swInt(f(A))$ since $swInt(f(A))$ is the largest *somewhat*-open set contained in $f(A)$.

$(iii)\Rightarrow (iv)$: Let B be a basic open set in X. Then $f(B) = f(Int(B))$. By assumption, $f(B) = f(Int(B)) \subseteq swInt(f(B)) \subseteq f(B)$. Hence, $f(B) = swInt(f(B))$. By Remark *2* (vi), $f(B)$ is *somewhat*-open in Y.

$(iv)\Rightarrow (v)$: Let O be open in X and let $p \in O \subseteq X$. Since O is open, there exists a basic open set B containing p such that $B \subseteq O$. This implies that $f(p) \in f(B) \subseteq f(O)$. By assumption, there exists a *somewhat*-open set W in Y such that $f(p) \in W \subseteq f(B)$. By Remark *2* (v), $f(p) \in swInt(W) \subseteq f(O)$.

$(v)\Rightarrow (i)$: Let O be open in X and $y \in f(O)$. Then there exists $x \in O$ such that $f(x) = y$. By assumption, there exists a *somewhat*-open set W in Y containing y such that $W = swInt(W) \subseteq f(O)$. Thus, f is *somewhat*-open on X. ■

Theorem 2.7 Let $f : X \to Y$ be a bijective function. Then f is *somewhat*-open on X if and only if $f(G)$ is *somewhat*-closed for every closed set G in X.

3. *Somewhat*-continuity in the product space

In this section, a characterization of a *somewhat*-continuous function from an arbitrary topological space into the product space will be presented.

We shall give first a characterization of a *somewhat*-continuous function from a topological space to another topological space.

Theorem 3.1 Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:

(i) f is *somewhat*-continuous on X.
(ii) $f^{-1}(G)$ is *somewhat*-open in X for each open subset G of Y.
(iii) $f^{-1}(F)$ is *somewhat*-closed in X for each closed subset F of Y.
(iv) $f^{-1}(B)$ is *somewhat*-open in X for each (subbasic) basic open set B in Y.
(v) For every $p \in X$ and every open set V of Y containing $f(p)$, there exists a *somewhat*-open set U containing p such that $f(U) \subseteq V$.
(vi) $f(swCl(A)) \subseteq Cl(f(A))$ for each $A \subseteq X$.
(vii) $swCl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$ for each $B \subseteq Y$.

Proof. Statements (i), (ii), (iii), and (v) are equivalent by [6, Theorem 3.2].

$(ii) \Rightarrow (iv)$: Trivial since (subbasic) basic open sets are open sets.

$(iv) \Rightarrow (ii)$: Suppose that $f^{-1}(B)$ is *somewhat*-open in X for each $B \in B$ where B is a basis for the topology in Y. Let G be an open set in Y. Then $G = \cup\{B : B \in B^*\}$, for some $B^* \subseteq B$. It follows that $f^{-1}(G) = \cup\{f^{-1}(B) : B \in B^*\}$. Since the arbitrary union of *somewhat*-open sets is *somewhat*-open, $f^{-1}(G)$ is *somewhat*-open in X.

$(v) \Rightarrow (vi)$: Let $A \subseteq X$ and $p \in swCl(A)$. Let G be an open set of Y containing $f(p)$. Then, by assumption, there exists a *somewhat*-open set O of X containing p such that $f(O) \subseteq G$. Since $p \in swCl(A)$, $O \cap A \neq \varnothing$, by [6, Theorem 3.8 (e)]. Thus, $\varnothing \neq f(O \cap A) \subseteq f(O) \cap f(A) \subseteq G \cap f(A)$. This implies that $f(p) \in Cl(f(A))$. Hence, $f(swCl(A)) \subseteq Cl(f(A))$.

$(vi) \Rightarrow (vii)$: Let $B \subseteq Y$ and let $A = f^{-1}(B) \subseteq X$. By assumption, $f(swCl(A) \subseteq Cl(f(A))$. Hence, $swCl(f^{-1}(B)) \subseteq f^{-1}(f(swCl(A))) \subseteq f^{-1}(Cl(f(A))) = f^{-1}(Cl(B))$. Thus, $swCl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$.

$(vii) \Rightarrow (iii)$: Let F be a closed subset of Y. Then, $Cl(F) = F$. By assumption,

$$swCl(f^{-1}(F)) \subseteq f^{-1}(Cl(F)) = f^{-1}(F).$$

Since $f^{-1}(F) \subseteq swCl(f^{-1}(F))$, it follows that $swCl(f^{-1}(F)) = f^{-1}(F)$ and so $f^{-1}(F)$ is *somewhat*-closed in X. ■

Theorem 3.2 Let O be a nonempty *somewhat*-open set in the product space $Y = \Pi\{Y_\alpha : \alpha \in A\}$. Then $p_\alpha(O) = Y_\alpha$ for all but at most finitely many α and $p_\alpha(O)$ is *somewhat*-open in Y_α for every $\alpha \in A$.

Proof. Let $x \in O$. Then there exists a basic open set $U = \langle U_{\alpha_1}, \ldots, U_{\alpha_n}\rangle$ such that $x \in U \subseteq O$. It follows that $p_\alpha(U) \subseteq p_\alpha(O)$ for every $\alpha \in A$. Note that $p_\alpha(U) = p_\alpha(\langle U_{\alpha_1}, \ldots, U_{\alpha_n}\rangle) = Y_\alpha$ for each $\alpha \notin \{\alpha_1, \ldots, \alpha_n\}$. Hence $p_\alpha(O) = Y_\alpha$ for all but at most a finite number of indices in A.

Next, let $\alpha \in A$. Then $p_\alpha(O) = Y_\alpha$ or $p_\alpha(O) \neq Y_\alpha$. If $p_\alpha(O) = Y_\alpha$, then $p_\alpha(O)$ is *somewhat*-open in Y_α. Suppose that $p_\alpha(O) \neq Y_\alpha$. Since $O \neq \varnothing$, $p_\alpha(O) \neq \varnothing$. Since O is a nonempty *somewhat*-open, there exists $y \in O$ and open set $V = \langle V_{\alpha_1}, \ldots, V_{\alpha_n}\rangle$ such that $y \in V = \langle V_{\alpha_1}, \ldots, V_{\alpha_n}\rangle \subseteq O$. Hence, there exists $p_\alpha(Y) \subseteq p_\alpha(O)$ and an open set $p_\alpha(V) = V_\alpha$ such that $p_\alpha(y) \in V_\alpha \subseteq p_\alpha(O)$. This shows that $p_\alpha(O)$ is *somewhat*-open in Y_α. ■

Corollary 3.3 Let X be a topological space, $Y = \Pi\{Y_\alpha : \alpha \in A\}$ a product space, and $f_\alpha : X \to Y_\alpha$ a function for each $\alpha \in A$. Let $f : X \to Y$ be the function defined by $f(x) = \langle f_\alpha(x)\rangle$. Then, f is *somewhat*-continuous on X if and only if each f_α is *somewhat*-continuous for each $\alpha \in A$.

Theorem 3.4 Let $Y = \Pi\{Y_\alpha : \alpha \in A\}$ be a product space. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ be a finite subset of A and $\varnothing \neq O_\alpha \subseteq Y_\alpha$ for each $\alpha \in A$. Then $O = \langle O_{\alpha_1}, \ldots, O_{\alpha_n}\rangle$ is *somewhat*-open in Y if and only if each O_{α_i} is *somewhat*-open in Y_{α_i}.

Proof. Suppose that $O = \langle O_{\alpha_1}, \ldots, O_{\alpha_n}\rangle$ is *somewhat*-open in Y and $\varnothing \neq O_\alpha \subseteq Y_\alpha$ for each $\alpha \in A$. Choose $a_{\alpha_i} \in O_{\alpha_i} = p_{\alpha_i}(O)$ for each $i = 1, 2, \ldots, n$. Then there exists $x = \langle a_\alpha\rangle \in O$ such that $p_{\alpha_i}(x) = a_{\alpha_i}$. Since O is *somewhat*-open in Y, there exists an open set $U = \langle U_{\alpha_1}, \ldots, U_{\alpha_n}\rangle$ such that $x = \langle a_\alpha\rangle \in \langle U_{\alpha_1}, \ldots, U_{\alpha_n}\rangle \subseteq \langle O_{\alpha_1}, \ldots, O_{\alpha_n}\rangle$. Thus, $p_{\alpha_i}(\langle a_\alpha\rangle) \in p_{\alpha_i}(\langle U_{\alpha_1}, \ldots, U_{\alpha_n}\rangle) \subseteq p_{\alpha_i}(\langle O_{\alpha_1}, \ldots, O_{\alpha_n}\rangle)$ implying that $a_{\alpha_i} \in U_{\alpha_i} \subseteq O_{\alpha_i}$. Hence, each O_{α_i} is *somewhat*-open in Y_{α_i}.

Conversely, suppose that each O_{α_i} is *somewhat*-open in Y_{α_i}. Let $O = \langle O_{\alpha_1}, \ldots, O_{\alpha_n}\rangle$. Then, $O \neq \varnothing$. Hence, there exists $x = \langle a_\alpha\rangle \in O$. Then $a_{\alpha_i} \in O_{\alpha_i}$ for each $\alpha_i \in S$. Hence, there exists an open set V_{α_i} such that $a_{\alpha_i} \in V_{\alpha_i} \subseteq O_{\alpha_i}$. Let $V = \langle V_{\alpha_1}, \ldots, V_{\alpha_n}\rangle$. Then V is open in Y such that $x \in V \subseteq O$. Thus, O is *somewhat*-open in Y. ■

Theorem 3.5 Let $X = \Pi\{X_\alpha : \alpha \in A\}$ and $Y = \Pi\{Y_\alpha : \alpha \in A\}$ be product spaces and $f_\alpha : X_\alpha \to Y_\alpha$ be a function for each $\alpha \in A$. If each f_α is *somewhat*-continuous on X_α, then the function $f : X \to Y$ defined by $f(\langle x_\alpha\rangle) = \langle f_\alpha(x_\alpha)\rangle$ is *somewhat*-continuous on X.

Proof. Let $\langle V_\alpha\rangle$ be a subbasic open set in Y. Then $f^{-1}(\langle V_\alpha\rangle) = \left\langle f_\alpha^{-1}(V_\alpha)\right\rangle$. Since f_α is *somewhat*-continuous, $f_\alpha^{-1}(V_\alpha)$ is *somewhat*-open in X_α. If $f_\alpha^{-1}(V_\alpha) = \varnothing$ for some $\alpha \in A$, then $f^{-1}(\langle V_\alpha\rangle) = \varnothing$, so that $f^{-1}(\langle V_\alpha\rangle)$ is *somewhat*-open in X. Assume that $f_\alpha^{-1}(V_\alpha) \neq \varnothing$ for every $\alpha \in A$. Then $f^{-1}(\langle V_\alpha\rangle) \neq \varnothing$. Let $x = \langle X_\alpha\rangle \in f^{-1}(\langle V_\alpha\rangle) = \left\langle f_\alpha^{-1}(V_\alpha)\right\rangle$. Then $X_\alpha \in f_\alpha^{-1}(V_\alpha)$ for every $\alpha \in A$. Hence, there exists an open set O_α such that $X_\alpha \in O_\alpha \subseteq f_\alpha^{-1}(V_\alpha)$. Hence, $\langle O_\alpha\rangle$ is open in X and $x = \langle O_\alpha\rangle \subseteq \left\langle f_\alpha^{-1}(V_\alpha)\right\rangle = f^{-1}(\langle V_\alpha\rangle)$.

Thus, f is *somewhat*-continuous on X. ■

4. *Somewhat*-connectedness

This section gives the relationships between *somewhat*-connectedness, classical connectedness, ω_θ-connectedness, and the ω -connectedness. Also, this section presents a characterization of a *somewhat*-connected space. Denote by $\mathcal{D}$, the topological space $\{0,1\}$ with the discrete topology.

The proof of the following lemma is standard, hence omitted.

Lemma 4.1 Let X be a topological space and $f_A : X \to \mathcal{D}$ the characteristic function of a subset A of X. Then f_A is *somewhat*-continuous if and only if A is both *somewhat*-open and *somewhat*-closed.

Theorem 4.2 Let X be a topological space. Then the following statements are equivalent:

- (i) X is *somewhat*-connected.
- (ii) The only subsets of X that are both *somewhat*-open and *somewhat*-closed are $\varnothing$ and X.
- (iii) No *somewhat*-continuous function from X to $\mathcal{D}$ is surjective.

Proof. $(i)\Rightarrow$ (ii): Suppose that X is *somewhat*-connected and let $G \subseteq X$ which is both *somewhat*-open and *somewhat*-closed. Then $X \setminus G$ is also both *somewhat*-open and *somewhat*-closed. Moreover, $X = G \cup (X \setminus G)$. Since X is *somewhat*-connected, either $G = \varnothing$ or $G = X$.

$(ii)\Rightarrow$ (iii): Suppose that $f : X \to \mathcal{D}$ is a *somewhat*-continuous surjection. Then $f^{-1}(\{0\}) \neq \varnothing, X$. Since $\{0\}$ is open and closed in $\mathcal{D}$, $f^{-1}(\{0\})$ is both *somewhat*-open and *somewhat*-closed in X. This is a contradiction.

$(iii)\Rightarrow$ (i): If $X = A \cup B$, where A and B are disjoint nonempty *somewhat*-open sets, then A and B are also *somewhat*-closed sets. Consider the characteristic function $f_A : X \to \mathcal{D}$ of $A \subseteq X$, which is surjective. By Lemma 4.1, f_A is *somewhat*-continuous. This is a contradiction. Thus X is *somewhat*-connected. ■

Remark 6 *Let X be a topological space. If X is somewhat-connected, then X is connected, but not conversely.*

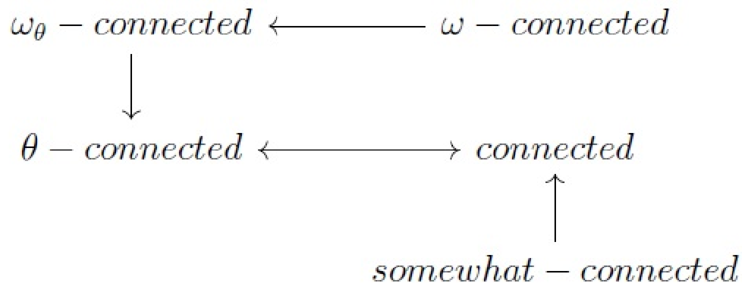
Since every open is *somewhat*-open, *somewhat*-connected implies connected. To show that a connected space is not necessarily *somewhat*-connected, consider $X = \{1,2,3,4\}$ with the topology $T = \{X, \varnothing, \{3\}, \{1,2\}, \{1,2,3\}\}$. Then X is connected but *somewhat*-disconnected since $A = \{1,2\}$ and $B = \{3,4\}$ are *somewhat*-open sets in X and $A \cup B = X$.

Remark 7 *Let X be a topological space. Then*

- (i) *somewhat-connected is not necessarily ω_θ-connected.*
- (ii) *somewhat-connected is not necessarily ω-connected.*

To verify this, consider $X = \{w,x,y,z\}$ with the topology $T = \{X, \varnothing, \{w\}, \{w,x\}, \{w,z\}, \{w,x,z\}\}$. Then $T_{sw} = \{X, \varnothing, \{w\}, \{w,x\}, \{w,z\}, \{w,y\}, \{w,x,y\}, \{w,x,z\}, \{w,y,z\}\}$ which means that X is *somewhat*-connected. But X is ω_θ-disconnected and ω-disconnected since $T_{\omega_\theta} = T_\omega = P(X)$, where $P(X)$ is the power set of X.

Remark 8 *The following diagram holds for any topological space.*



5. Conclusion

The paper has characterized *somewhat*-connectedness and described its connection to the other well-known concepts such as the classical connectedness, the ω_θ-connectedness, and the ω-connectedness. Moreover, the paper has formulated a necessary and sufficient condition for *somewhat*-continuity of a function from an arbitrary space into the product space. This particular result is the counterpart of the known characterization of the ordinary continuity of a function into a product space.

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