Journal of
Linear and Topological Algebra
Vol. 09*, No.* 02*,* 2020, 129- 137

Approximation of endpoints for multi-valued mappings in metric spaces

K. Ullah[a], J. Ahmad[a,*], N. Muhammad[a]

[a] *Department of Mathematics, University of Science and Technology, Bannu 28100 Khyber Pakhtunkha, Pakistan.*

Received 16 March 2020; Revised 19 May 2020; Accepted 21 June 2020.

Communicated by Choonkil Park

Abstract. In this paper, under some appropriate conditions, we prove some Δ and strong convergence theorems of endpoints for multi-valued nonexpansive mappings using modified Agarwal-O'Regan-Sahu iterative process in the general setting of 2-uniformly convex hyperbolic spaces. Our results extend and unify some recent results of the current literature.

Keywords: Endpoint, condition (J), Δ-convergence, strong convergence, hyperbolic space.

2010 AMS Subject Classification: 47H09, 47H10.

1. Introduction

Let D be a nonempty subset of a metric space (X, d) and $t : D \to D$. A point $p \in D$ is called a fixed point of t if $p = t(p)$. t is called nonexpansive if

$$d(t(u), t(v)) \leqslant d(u, v) \text{ for all } u, v \in D.$$

Nonexpansive mappings have many important application in applied sciences. In 1965, Browder [4], Gohde [8] and Kirk [11] independently established that every nonexpansive mapping $t : D \to D$ has a fixed point provided that C is nonempty closed convex and X is uniformly convex. Lim [14] proved multi-valued version of Kirk-Browder-Gohde result

*Corresponding author.
E-mail address: kifayatmath@yahoo.com (K. Ullah); ahmadjunaid436@gmail.com (J. Ahmad); naseermuhammadpst@gmail.com (N. Muhammad).

in 1974. It is indeed natural to iterate a single-valued mapping but for a multi-valued mapping one it is a difficult task. In 2005, Sastry and Babu [19] proved fixed point convergence theorems for multi-valued nonexpansive mappings using modified Mann and Ishikawa iterative processes in the setting of Hilbert spaces. In 2007, Panyanak [17] extended results of Sastry and Babu [19] to the general setting of uniformly convex Banach spaces. In 2008, Song and Wang [26] quickly noted some gaps in the results of Panyanak [17]. After this many results for multi-valued mappings using different iteration schemes in appropriate spaces are proved (see, e.g., [2, 21–25] and others). In 2018, Panyanak [15] initiated study of approximation of endpoints of multi-valued nonexpansive mappings in the setting of Banach spaces. In 2019, Ullah et al. [27] extended some of his results to the setting of CAT(0) spaces. In 2020, Abdeljawad et al. [1] used modified Agarwal-O'Regan-Sahu iteration (which is independent of but faster than Mann and Ishikawa iterative processes) to approximate endpoints of multi-valued nonexpansive mappings in Banach spaces. Let D be a nonempty convex subset of a Banach space $(X, \|.\|)$ and $\alpha_n, \beta_n \in [0, 1]$. The modified Agarwal-O'Regan-Sahu iteration process is defined as follows:

$$\begin{cases} u_1 \in D, \\ v_n = (1-\beta_n)u_n + \beta_n z_n, \\ u_{n+1} = (1-\alpha_n)z_n + \alpha_n z'_n, n \in \mathbb{N}, \end{cases} \tag{1}$$

where $z_n \in Tu_n$ such that $||u_n - z_n|| = R(u_n, Tu_n)$ and $z'_n \in Tv_n$ such that $||v_n - z'_n|| = R(v_n, Tv_n)$.

Under some appropriate conditions, Abdeljawad et al. [1] proved some weak and strong convergence results of endpoints for multi-valued nonexpansive mappings using iterative process (1) in Banach spaces. In this paper, we extend their results to the general setting of hyperbolic spaces.

2. Preliminaries

Throughout the work, we will represent the set of all natural numbers and set of all real numbers respectively by $\mathbb{N}$ and $\mathbb{R}$. Let $X = (X, d)$ be a metric space, D be a nonempty subset of X. For $u \in X$, set

$$d(u, D) = \inf\{d(u, v) : v \in D\},$$

and

$$R(u, D) = \sup\{d(u, v) : v \in D\}.$$

We shall denote the set of all nonempty and compact subsets of D by $\mathcal{K}(D)$. Set

$$H(A, B) = \max\left\{\sup_{u\in A} d(u, B), \sup_{v\in B} d(v, A)\right\}, \text{ for each } A, B \in \mathcal{K}(D).$$

$H(.,.)$ is called a Hausdorff metric on the set $\mathcal{K}(D)$. A multivalued mapping $T : D \to \mathcal{K}(D)$ is called nonexpansive if

$$H(Tu, Tv) \leqslant d(u, v), \text{ for all } u, v \in D.$$

A point $p \in D$ is said to be a fixed point of $T : D \to \mathcal{K}(D)$ if $p \in Tp$ and is said to be an endpoint of $T : D \to \mathcal{K}(D)$ if $Tp = \{p\}$. From now on, we will denote the set of all endpoints and the set of all fixed points of T by $End(T)$ and $Fix(T)$ respectively. Note that, a multivalued mapping $T : D \to \mathcal{K}(D)$ is said to satisfy the endpoint condition if $End(T) = Fix(T)$. For existence of endpoints of multi-valued mappings, we refer the reader to [3, 5, 7, 9, 16, 18, 20].

Definition 2.1 Let D be a nonempty subset of a complete metric space X and $\{u_n\}$ be a bounded sequence in X. The asymptotic radius of $\{u_n\}$ relative to D is the set $r(D, \{u_n\}) = \inf\{\limsup_{n\to\infty} d(u_n, u) : u \in D\}$. Moreover, the asymptotic center of $\{u_n\}$ relative to D is the set $A(D, \{u_n\}) = \{u \in D : \limsup_{n\to\infty} d(u_n, u) = r(D, u_n)\}$.

Definition 2.2 Let D be a nonempty closed convex subset of a complete metric space (X, d) and $u \in D$. Let $\{u_n\}$ be a bounded sequence in X. We say that $\{u_n\}$ Δ-converges to u if $A(D, \{t_n\}) = \{u\}$ for each subsequence $\{t_n\}$ of $\{u_n\}$. In this case we write Δ-$\lim_{n\to\infty} u_n = u$ and call u the Δ-lim of $\{u_n\}$.

Definition 2.3 [13] *A hyperbolic space is a metric space (X, d) together with a function $W : X \times X \times [0, 1] \to X$ such that for all $u, v, z, w \in X$ and $t, s \in [0, 1]$, we have*
$(W1)$ $d(z, W(u, v, t)) \leqslant (1 - t)d(z, u) + td(z, v)$;
$(W2)$ $d(W(u, v, t), W(u, v, s) = |t - s|d(u, v)$;
$(W3)$ $W(u, v, t) = W(v, u, 1 - t)$;
$(W4)$ $d(W(u, z, t), W(v, w, t)) \leqslant (1 - t)d(u, v) + td(z, w)$.

If $u, v \in X$ and $t \in [0, 1]$, we use the notation $(1 - t)u \oplus tv$ for $W(u, v, t)$. It follows from $(W1)$ that

$$d(u, (1 - t)u \oplus tv) = td(u, v) \text{ and } d(v, (1 - t)u \oplus tv) = (1 - t)d(u, v).$$

A nonempty subset D of X is called convex if for each $u, v \in D$, $[u, v] = \{(1 - t)u \oplus tv : t \in [0, 1]\} \subseteq D$.

Definition 2.4 A hyperbolic space (X, d, W) is called uniformly convex if for each $r \in (0, \infty)$ and $\epsilon \in (0, 2]$ there is a $\delta \in (0, 1]$ such that for all $u, v, z \in X$ with $d(u, z) \leqslant r$, $d(v, z) \leqslant r$ and $d(u, v) \geqslant r\epsilon$, we have

$$d\left(\frac{1}{2}u \oplus \frac{1}{2}v\right) \leqslant (1 - \delta)r.$$

A function $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such $\delta = \eta(r, \epsilon)$ for given $r \in (0, \infty)$ and $\epsilon \in (0, 2]$ is called a modulus of uniform convexity. We call η monotone if it is a nonincreasing function of r for every fixed ϵ.

Definition 2.5 [10] Let (X, d) be a uniformly convex hyperbolic space. For each $r \in (0, \infty)$ and $\epsilon \in (0, 2]$, we define

$$\psi(r, \epsilon) = \inf\left\{\frac{1}{2}d^2(u, z) + \frac{1}{2}d^2(v, z) - d^2(\frac{1}{2}u \oplus \frac{1}{2}v, z)\right\}$$

where the infimum is taken over all $u, v, z \in X$ such that $d(u, z) \leqslant r$, $d(v, z) \leqslant r$, and $d(u, v) \geqslant r\epsilon$. We say that (X, d) is 2-uniformly convex if

$$c_M = \inf\left\{\frac{\psi(r,\epsilon)}{r^2\epsilon^2} : r \in (0,\infty), \epsilon \in (0,2]\right\} > 0.$$

Remark 1 *All uniformly convex Banach spaces, CAT(0) spaces and CAT(κ) spaces ($\kappa > 0$ and diam(X) $\leqslant \left(\frac{\frac{\pi}{2}-\epsilon}{\kappa^{\frac{1}{2}}}\right)$ for some $\epsilon \in (0, \frac{\pi}{2})$) are 2-uniformly convex hyperbolic spaces (see [10, 12, 28]).*

Lemma 2.6 [17] Let $\alpha_n, \beta_n \in [0, 1)$ be such that $\lim_{n\to\infty} \beta_n = 0$ and $\sum \alpha_n\beta_n = \infty$. Let $\{\gamma_n\}$ be a sequence of non-negative real numbers such that $\sum \alpha_n\beta_n(1-\beta_n)\gamma_n < \infty$. Then $\{\gamma_n\}$ has a subsequence which converges to 0.

Lemma 2.7 [15] For a multivalued mapping $T : D \to \mathcal{K}(D)$ the following hold.

(i) $d(u, Tu) = 0 \iff u \in Fix(T)$.
(ii) $R(u, Tu) = 0 \iff u \in End(T)$.
(iii) If T is nonexpansive, then the mapping $g : D \to \mathbb{R}$ defined by $g(u) = R(u, Tu)$ is continuous.

Lemma 2.8 [12] Let (X, d) be a 2-uniformly convex hyperbolic space. Then

$$d^2((1-\alpha)u \oplus \alpha v, z) \leqslant (1-\alpha)d^2(u, z) + \alpha d^2(v, z) - 4c_M\alpha(1-\alpha)d^2(u, v),$$

for each $\alpha \in [0, 1]$ and $u, v, z \in X$.

Lemma 2.9 [12] Le X be a complete 2-uniformly convex hyperbolic space X with monotone modulus of uniform convexity, $\emptyset \neq D \subseteq X$ and $T : D \to \mathcal{K}(D)$ be a nonexpansive mapping. Suppose that $\{u_n\}$ is a bounded sequence in D such that $\lim_{n\to\infty} R(u_n, Tu_n) = 0$ and $\{d(u_n, s)\}$ converges for all $s \in End(T)$, then $\omega_\omega(u_n) \subseteq End(T)$. Here, $\omega_\omega(u_n) = \bigcup A(D, \{t_n\})$ where the union is taken over all subsequences $\{t_n\}$ of $\{u_n\}$. Furthermore, $\omega_\omega(u_n)$ is singleton.

3. Convergence theorems in 2-uniformly convex hyperbolic spaces

From now on, X stands for a complete 2-uniformly convex hyperbolic space with monotone modulus of uniform convexity. In this section, under some appropriate conditions, we prove some Δ and strong convergence theorems of endpoints for multi-valued nonexpansive mappings using iterative process (2), which is the modification of (1):

$$\begin{cases} u_1 \in D, \\ v_n = (1-\beta_n)u_n \oplus \beta_n z_n, \\ u_{n+1} = (1-\alpha_n)z_n \oplus \alpha_n z'_n, n \in \mathbb{N}, \end{cases} \tag{2}$$

where $z_n \in Tu_n$ such that $d(u_n, z_n) = R(u_n, Tu_n)$ and $z'_n \in Tv_n$ such that $d(v_n, z'_n) = R(v_n, Tv_n)$.

The following lemma is crucial.

Lemma 3.1 Let D be a nonempty closed convex subset of X and $T : D \to \mathcal{K}(D)$ be a nonexpansive mapping with $End(T) \neq \emptyset$. Let $\{u_n\}$ be the sequence defined by (2). Then

$\lim_{n\to\infty} d(u_n, p)$ exists for all $p \in End(T)$.

Proof. Let $p \in End(T)$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(v_n, p) &\leqslant (1-\beta_n)d(u_n, p) + \beta_n d(z_n, p) \\ &\leqslant (1-\beta_n)d(u_n, p) + \beta_n d(z_n, Tp) \\ &\leqslant (1-\beta_n)d(u_n, p) + \beta_n H(Tu_n, Tp) \\ &\leqslant (1-\beta_n)d(u_n, p) + \beta_n d(u_n, p) \\ &\leqslant d(u_n, p), \end{aligned}$$

which implies

$$\begin{aligned} d(u_{n+1}, p) &\leqslant (1-\alpha_n)d(z_n, p) + \alpha_n d(z'_n, p) \\ &\leqslant (1-\alpha_n)d(z_n, Tp) + \alpha_n d(z'_n, Tp) \\ &\leqslant (1-\alpha_n)H(Tu_n, Tp) + \alpha_n H(Tv_n, Tp) \\ &\leqslant (1-\alpha_n)d(u_n, p) + \alpha_n d(v_n, p) \\ &\leqslant d(u_n, p). \end{aligned}$$

Hence $\{d(u_n, p)\}$ is a non-increasing sequence, which implies $\lim_{n\to\infty} d(u_n, p)$ exists for all $p \in End(T)$. ■

First we establish our Δ-convergence theorem.

Theorem 3.2 Let D be a nonempty closed convex subset of X and $T : D \to \mathcal{K}(D)$ be a nonexpansive mapping with $End(T) \neq \emptyset$. Let $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$ and $\{u_n\}$ be the sequence defined by (2). Then $\{u_n\}$ Δ-converges to an endpoint of T.

Proof. Fix $p \in End(T)$. By Lemma 2.8, we have

$$\begin{aligned} d^2(v_n, p) &\leqslant (1-\beta_n)d^2 d(u_n, p) + \beta_n d^2(z_n, p) - 4c_M\beta_n(1-\beta_n)d^2(u_n, z_n) \\ &\leqslant (1-\beta_n)d^2(u_n, p) + \beta_n H^2(Tu_n, Tp) - 4c_M\beta_n(1-\beta_n)d^2(u_n, z_n) \\ &\leqslant (1-\beta_n)d^2(u_n, p) + \beta_n d^2(u_n, p) - 4c_M\beta_n(1-\beta_n)d^2(u_n, z_n) \\ &\leqslant d^2(u_n, p) - 4c_M\beta_n(1-\beta_n)d^2(u_n, z_n). \end{aligned}$$

Thus

$$\begin{aligned} d^2(u_{n+1}, p) &\leqslant (1-\alpha_n)d^2(z_n, p) + \alpha_n d^2(z'_n, p) - 4c_M\alpha_n(1-\alpha_n)d^2(z_n, z'_n) \\ &\leqslant (1-\alpha_n)H^2(Tu_n, Tp) + \alpha_n H^2(Tv_n, Tp) - 4c_M\alpha_n(1-\alpha_n)d^2(z_n, z'_n) \\ &\leqslant (1-\alpha_n)d^2(u_n, p) + \alpha_n d^2(v_n, p) - 4c_M\alpha_n(1-\alpha_n)d^2(z_n, z'_n) \\ &\leqslant (1-\alpha_n)d^2(u_n, p) + \alpha_n d^2(v_n, p) \\ &\leqslant \alpha_n d^2(u_n, p) + (1-\alpha_n)d^2(u_n, p) - 4c_M\alpha_n\beta_n(1-\beta_n)d^2(u_n, z_n). \end{aligned}$$

Since $4c_M > 0$, it follows that

$$\sum_{n=1}^{\infty} a^2(1-b)d^2(u_n, z_n) \leqslant \sum_{n=1}^{\infty} \alpha_n\beta_n(1-\beta_n)d^2(u_n, z_n) < \infty. \tag{3}$$

Thus $\lim_{n\to\infty} d^2(u_n, z_n) = 0$ and hence

$$\lim_{n\to\infty} R(u_n, Tu_n) = \lim_{n\to\infty} d(u_n, z_n) = 0. \tag{4}$$

By Lemma 3.1, $d(u_n, s)$ converges for all $s \in End(T)$. By Lemma 2.9, $\omega_\omega(u_n)$ is singleton and contained in $End(T)$. This shows that $\{u_n\}$ Δ-converges to an element of $End(T)$. ■

Definition 3.3 [15] Let D be a nonempty subset of X. A mapping $T : D \to \mathcal{K}(D)$ is said to satisfy condition (J) if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$ such that

$$R(u, Tu) \geqslant f(d(u, End(T)))$$

for all $u \in D$.

A mapping $T : D \to \mathcal{K}(D)$ is called semi-compact if for each sequence $\{u_n\}$ in D such that

$$\lim_{n\to\infty} R(u_n, Tu_n) = 0,$$

there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\lim_{k\to\infty} u_{n_k} = q$ for some $q \in D$.

A sequence $\{u_n\}$ in X is called Fejer-monotone with respect to D if

$$d(u_{n+1}, p) \leqslant d(u_n, p),$$

for each $p \in D$ and $n \in \mathbb{N}$.

The following facts are in [6].

Proposition 3.4 Let D be a nonempty closed subset of X and $\{u_n\}$ be a Fejer-monotone sequence with respect to D. Then $\{u_n\}$ converges strongly to an element of D if and only if $\lim_{n\to\infty} d(u_n, D) = 0$.

Now we prove the following strong convergence theorem, which is a generalization of Theorem 2 in [1].

Theorem 3.5 Let D be a nonempty closed convex subset of X and $T : D \to \mathcal{K}(D)$ be a nonexpansive mapping with $End(T) \neq \emptyset$. Let $\alpha_n, \beta_n \in [0, 1)$ be such that $\beta_n \to 0$ and $\sum \alpha_n\beta_n = \infty$ and let $\{u_n\}$ be the sequence defined by (2). If T is semicompact, then $\{u_n\}$ converges strongly to an endpoint of T.

Proof. In view of (3),

$$\sum_{n=1}^{\infty} \alpha_n\beta_n(1-\beta_n)d^2(u_n, z_n) < \infty.$$

By Lemma 2.6, sub-sequences $\{u_{n_j}\}$ and $\{z_{n_j}\}$ of $\{u_n\}$ and $\{z_n\}$ exists respectively, such that $\lim_{j\to\infty} d^2(u_{n_j}, z_{n_j}) = 0$. Hence

$$\lim_{j\to\infty} R(u_{n_j}, Tu_{n_j}) = \lim_{j\to\infty} d(u_{n_j}, z_{n_j}) = 0. \tag{5}$$

On the other hand, T is semi-compact, we may choose by passing through a sub-sequence that $\lim_{j\to\infty} u_{n_j} = q$ for some $q \in D$. Need to show $q \in End(T)$ and $u_n \to q$. By Lemma 2.7(iii) together with (5), we have

$$R(q, Tq) = \lim_{j\to\infty} R(u_{n_j}, Tu_{n_j}) = 0.$$

It follows from Lemma 2.7(ii), that $q \in End(T)$. By Lemma 3.1, $\lim_{n\to\infty} d(u_n, q)$ exists and hence q is the strong limit of $\{u_n\}$. ■

The following strong convergence theorem is a generalization of Theorem 3 in [1].

Theorem 3.6 Let D be a nonempty closed convex subset of X and $T : D \to \mathcal{K}(D)$ be a nonexpansive mapping with $End(T) \neq \emptyset$. Let $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$ and $\{u_n\}$ be the sequence defined by (2). If T satisfies condition (J), then $\{u_n\}$ converges strongly to an endpoint of T.

Proof. Since T satisfies condition (J), by (4) we get that $\lim_{n\to\infty} d(u_n, End(T)) = 0$. Closeness of $End(T)$ follows from the nonexpansiveness of T. In the view of Lemma 3.1, we have $\{u_n\}$ is Fejer-monotone with respect to $End(T)$. By Proposition 3.4, $\{u_n\}$ converges strongly to an element of $End(T)$. ■

4. Example

We now give an example of mapping T, which is semi-compact, nonexpasive and satisfies the condition (J).

***Example* 4.1** Let $X = \mathbb{R}$ and $D = [5, 6]$. Define a multivalued mapping $T : D \to \mathcal{K}(D)$ by $Tu = [5, u]$ for each $u \in D$. Clearly T is semi-compact and nonexpansive with $End(T) = \{5\}$. We show that T satisfies condition (J). For this we define a nondecreasing function f by $f(r) = r$. We prove that $R(u, Tu) \geqslant f(d(u, End(T)))$ for each $u \in D$.

For any $u \in D = [5, 6]$, we have

$$\begin{aligned} f(d(u, End(T))) &= f(d(u, \{5\}) = f(|u - 5|) \\ &= |u - 5| = R(u, Tu). \end{aligned}$$

Hence T satisfies condition (J). By Theorems 3.5 and 3.6, sequence $\{u_n\}$ defined by (2) converges strongly to 5.

5. Conclusion

In the view of above discussions, our underlying spaces is more general than Banach spaces, CAT(0) spaces and CAT(κ) spaces. Moreover, our iterative process is independent of but better than the Ishikawa iterative process. Hence, our presented results extend the corresponding results of Abdeljawad el al. [1] from the setting of Banach spaces to the general setting of hyperbolic spaces. Moreover, our results improve, extend and generalize the results of [15, 27]. Convergence results by using Agarwal-O'Regan-Sahu iteration process for multi-valued mappings were studied in [2, 22–24]. However, we cannot directly apply any results in [2, 22–24] on Example 4.1 because in Example 4.1, T does not satisfy the endpoint condition.

Acknowledgments

We are very much thankful to the reviewers for their constructive comments and suggestions which have been useful for the improvement of this paper.

References

[1] T. Abdeljawad, K. Ullah, J. Ahmad, N. Mlaiki, Iterative approximation of endpoints for multivalued mappings in Banach spaces, J. Funcion Space. (2020), 2020:2179059.
[2] N. Akkasriworn, K. Sokhuma, S-iterative process for a pair of single valued and multi valued mappings in Banach spaces, Thai J. Math. 14 (2016), 21-30.
[3] J. P. Aubin, J. Siegel, Fixed points and stationary points of dissipative multivalued maps, Proc. Am. Math. Soc. 78 (1980), 391-398.
[4] F. E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Nat. Acad. Sci. USA. 54 (1965), 1041-1044.
[5] L. Chen, L. Gao, D. Chen, Fixed point theorems of mean nonexpansive setvalued mappings in Banach spaces, J. Fixed Point Theory Appl. 19 (2017), 2129-2143.
[6] P. Chuadchawna, A. Farajzadeh, A. Kaewcharoen, Convergence theorems and approximating endpoints for multivalued Suzuki mappings in hyperbolic spaces, J. Comp. Anal. Appl. 28 (2020), 903-916.
[7] R. Espinola, M. Hosseini, K. Nourouzi, On stationary points of nonexpansive set-valued mappings, Fixed Point Theory Appl. 236 (2015), 1-13.
[8] D. Gohde, Zum Prinzip der Kontraktiven Abbildung, Math. Nachr. 30 (1965), 251-258.
[9] M. Hosseini, K. Nourouzi, D. O'Regan, Stationary points of set-valued contractive and nonexpansive mappings on ultrametric spaces, Fixed Point Theory 19 (2) (2018), 587-594.
[10] M. A. Khamsi, A. R. Khan, Inequalities in metric spaces with applications, Nonlinear Anal. 74 (2011), 4036-4045.
[11] W. A. Kirk, A fixed point theorem for mappings which do not increase distance, Am. Math. Monthly. 72 (1965), 1004-1006.
[12] T. Laokul, B. Panynak, A generalization of the (CN) inequality and its applications, Carpathian J. Math. 36 (1) (2020), 81-90.
[13] L. Leustean, A quadratic rate of asymptotic regularity for CAT(0) spaces, J. Math. Anal. Appl. 325 (2007), 386-399.
[14] T. C. Lim, A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach spaces, Bull. Am. Math. Soc. 80 (1974), 1123-1126.
[15] B. Panyanak, Approximating endpoints of multi-valued nonexpansive mappings in Banach spaces, J. Fixed Point Theory Appl. 20 (2018), 1-8.
[16] B. Panyanak, Endpoints of multivalued nonexpansive mappings in geodesic spaces, Fixed Point Theory Appl. 147 (2015), 1-11.
[17] B. Panyanak, Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces, Comput. Math. Appl. 54 (2007), 872-877.
[18] S. Reich, Fixed points of contractive functions, Boll. Unione Mat. Ital. 5 (1972), 26-42.
[19] K. P. R. Sastry, G. V. R. Babu, Convergence of Ishikawa iterates for a multivalued mapping with a fixed point, Czechoslovak Math. J. 55 (2005), 817-826.
[20] S. Saejung, Remarks on endpoints of multivalued mappings in geodesic spaces, Fixed Point Theory Appl. 52 (2016), 1-12.
[21] N. Shahzad, H. Zegeye, On Mann and Ishikawa iteration schemes for multivalued maps in Banach spaces, Nonlinear Anal. 71 (2009), 838-844.
[22] K. Sokhuma, On the S-iteration processes for multivalued mappings in some CAT(κ) spaces, Int. J. Math. Anal. 8 (18) (2014), 857-864.
[23] K. Sokhuma, S-iterative process for a pair of single valued and multi-valued nonexpansive mappings, Int. Math. Form. 7 (2012), 839-847.
[24] S. Sopha, W. Phuengrattana, Convergence of the S-iteration process for a pair of single-valued and multi-valued generalized nonexpansive mappings in CAT(κ) spaces, Thai J. Math. 13 (2015), 627-640.
[25] Y. Song, Y. J. Cho, Some notes on Ishikawa iteration for multi-valued mappings, Bull. Korean Math. Soc. 48 (2011), 575-584.
[26] Y. Song, H. Wang, Erratum to "Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces", Comput. Math. Appl. 52 (2008), 2999-3002.

[27] K. Ullah, M. S. U. Khan, N. Muhammad, J. Ahmad, Approximation of endpoints for multivalued nonexpansive mappings in geodesic spaces, Asian-European J. Math. (2019), 2019:2050141.
[28] H. K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal. 16 (1991), 1127-1138.