Journal of
Linear and Topological Algebra
Vol. 09*, No.* 03*,* 2020*,* 231- 236

Algebraic and topological aspects of quasi-prime ideals

M. Aghajani[a,*], A. Tarizadeh[a]

[a] *Department of Mathematics, Faculty of Basic Sciences, P. O. Box 55136-553, University of Maragheh, Maragheh, Iran.*

Received 29 June 2020; Revised 22 September 2020, Accepted 24 September 2020.

Communicated by Hamidreza Rahimi

Abstract. In this paper, we define the new notion of quasi-prime ideal which generalizes at once both prime ideal and primary ideal notions. Then a natural topology on the set of quasi-prime ideals of a ring is introduced which admits the Zariski topology as a subspace topology. The basic properties of the quasi-prime spectrum are studied and several interesting results are obtained. Specially, it is proved that if the Grothendieck t-functor is applied on the quasi-prime spectrum then the prime spectrum is deduced. It is also shown that there are the cases that the prime spectrum and quasi-prime spectrum do not behave similarly. In particular, natural topological spaces without closed points are obtained.

Keywords: Quasi-prime ideal, connected component, t-functor.

2010 AMS Subject Classification: 13A99, 14A05, 18A99, 54A05, 54A10.

1. Introduction

We call an ideal of a commutative ring a *quasi-prime ideal* if its radical is a prime ideal. We obtained this definition independently and without awareness of the previous work [4]. We were informed that this notion was already introduced and studied in [4] under the name of quasi-primary ideal. Then we took a look through this former work. But luckily, except quasi-prime notion, there is no overlapping between our work and [4]. The quasi-prime notion generalizes both prime ideal and primary ideal notions. So in a given ring, we have the following inclusions:

$$Prime\, ideals \subseteq Primary\, ideals \subseteq Quasi - prime\, ideals.$$

*Corresponding author.
E-mail address: aghajani14@gmail.com (M. Aghajani); ebulfez1978@gmail.com (A. Tarizadeh).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

Note that there are quasi-prime ideals which are not primary ideals, see Example 2.2.

We denote the set of quasi-prime ideals of A by $\mathrm{Sq}(A)$. Then we equip this set with a natural topology whose basis opens are of the form $U_f = \{\mathfrak{q} \in \mathrm{Sq}(A) : f \notin \sqrt{\mathfrak{q}}\}$ where $f \in A$. The space $\mathrm{Sq}(A)$ is called the quasi-prime spectrum of A. It is shown that the quasi-prime spectrum satisfies in all of the conditions of a spectral space except the uniqueness of generic point, see Theorems 2.4 and 2.11. This topology is a natural generalization of the Zariski topology, i.e. the prime spectrum $\mathrm{Spec}(A)$ is a dense subspace of $\mathrm{Sq}(A)$. It is worth mentioning that in [1, 3, 5, 7, 8] various algebraic and topological aspects of the spectral topologies (Zariski and flat topologies) are investigated and several deep results are obtained. For the definition of spectral space see e.g. [5].

In some cases, the topology on quasi-prime spectrum behaves quite different from the Zariski topology. For instance, $\mathrm{Sq}(\mathbb{Z})$ has no closed points. By contrast, all of the points of $\mathrm{Spec}(\mathbb{Z})$ except the zero ideal are closed points. This gives us natural topological spaces without closed point. It may be of interest to the readers that in [6], after some effort a scheme is built that whose underlying space has no closed points.

One of the motivations of this generalization (quasi-prime ideal notion) was to find topological spaces without closed points. Maybe the ideas of [3] was another motivation to introduce the quasi-prime notion. Because in the paper [3], the radical ideals of a given ring are studied from algebraic and topological aspects and various interesting results are obtained, the radical ideal notion ($I = \sqrt{I}$) also generalizes the prime ideal notion. If $\mathrm{Rd}(A)$ is the set of proper radical ideals of a ring A then it is easy to see that $\mathrm{Spec}(A) = \mathrm{Sq}(A) \cap \mathrm{Rd}(A)$. Note that $\mathrm{Rd}(A)$ is a spectral space endowed with a topology whose subbasis opens are of the form $\{I \in \mathrm{Rd}(A) : f \notin I\}$ with $f \in A$, for more information see [3].

In the present paper, we study the basic properties of the quasi-prime spectrum. Indeed, Theorem 2.4, Proposition 2.6, Theorem 2.8, Theorem 2.11, Theorem 3.2, Theorem 3.6 and Theorem 4.1 are the main results of this paper. In Theorem 3.6, the connected components of the quasi-prime spectrum are characterized. In Theorem 4.1, it is shown that the prime spectrum can be canonically recovered from the quasi-prime spectrum by applying the Grothendieck t-functor.

2. Quasi-prime spectrum

Definition 2.1 An ideal $\mathfrak{q}$ of a ring A is called a *quasi-prime* ideal of A if $\sqrt{\mathfrak{q}}$ is a prime ideal of A.

Recall that a proper ideal $\mathfrak{q}$ of a ring A is called a primary ideal of A if $fg \in \mathfrak{q}$ and $f \notin \mathfrak{q}$ for some $f, g \in A$, then $g \in \sqrt{\mathfrak{q}}$. This is equivalent to the statement that if $fg \in \mathfrak{q}$ and $f \notin \sqrt{\mathfrak{q}}$ then $g \in \mathfrak{q}$. Note that if $\mathfrak{q}$ is a primary ideal of A and we have $fg \in \mathfrak{q}$, $f \notin \mathfrak{q}$ and $g \notin \mathfrak{q}$ then $f, g \in \sqrt{\mathfrak{q}}$.

***Example* 2.2** Every primary ideal is quasi-prime, but the converse is not necessarily true. As a specific example, let A be the polynomial ring $k[x, y, z]$ modulo $I = (xy - z^2)$ where k is a field. Then $\mathfrak{p} = (x+I, z+I)$ is a prime ideal of A since $A/\mathfrak{p} \simeq k[y]$, but $\mathfrak{q} = \mathfrak{p}^2$ is a quasi-prime ideal which is not a primary ideal, because $(x + I)(y + I) = (z + I)^2 \in \mathfrak{q}$ but $x + I \notin \mathfrak{q}$ and $y + I \notin \sqrt{\mathfrak{q}}$.

***Example* 2.3** If $\mathfrak{q}$ is a quasi-prime ideal of a ring A then $\mathfrak{q}^n$ is a quasi-prime ideal of A for all $n \geqslant 1$. There are also non-primary quasi-prime ideals which are not as the power of a prime ideal. As an example, let A be the polynomial ring $k[x, y, z, t]$ modulo $I = (xy-z^2)$

where k is a domain. Then $\mathfrak{p} = (z + I, t + I)$ is a prime of A since $A/\mathfrak{p} \simeq k[x, y]$, but $\mathfrak{q} = (z + I, t^2 + I)$ is a non-primary quasi-prime of A which is not also as a power of a prime ideal.

Let $\varphi : X \to Y$ be a function where X is a set and Y is a topological space. Let $\mathcal{B}$ be a basis for the opens of Y. Then there exists a unique topology over X such that the set $\{\varphi^{-1}(B) : B \in \mathcal{B}\}$ forms a basis for its opens.
By applying the above basic fact to the map $\gamma : \mathrm{Sq}(A) \to \mathrm{Spec}(A)$ given by $\mathfrak{q} \rightsquigarrow \sqrt{\mathfrak{q}}$, then we get a unique topology over $\mathrm{Sq}(A)$ such that the collection of $U_f = \{\mathfrak{q} \in \mathrm{Sq}(A) : f \notin \sqrt{\mathfrak{q}}\}$ with $f \in A$ forms a basis for its opens. The set $\mathrm{Sq}(A)$ equipped with this topology is called the quasi-prime spectrum (or, the space of quasi-primes) of A. Clearly $U_f \cap \mathrm{Spec}(A) = D(f)$ for all $f \in A$. Hence, $\mathrm{Spec}(A)$ is a dense subspace of $\mathrm{Sq}(A)$. Recall that $D(f) = \{\mathfrak{p} \in \mathrm{Spec}(A) : f \notin \mathfrak{p}\}$.

Theorem 2.4 Every basis open U_f is quasi-compact. In particular, $\mathrm{Sq}(A)$ is quasi-compact.

Proof. It suffices to show that every open covering of U_f by the basis opens has a finite refinement. Hence let $U_f = \bigcup\limits_{i \in I} U_{g_i}$ where $g_i \in A$ for all i. It follows that $f \in \sqrt{(g_i : i \in I)}$. Thus there exists a finite subset J of I such that $f \in \sqrt{(g_i : i \in J)}$. We show that $U_f = \bigcup\limits_{i \in J} U_{g_i}$. If $\mathfrak{q} \in U_f$ then there exists some $i \in J$ such that $g_i \notin \sqrt{\mathfrak{q}}$. It follows that $\mathfrak{q} \in U_{g_i}$. ■

The proof of Theorem 2.4 also shows that $D(f)$ is quasi-compact for all $f \in A$.

Corollary 2.5 If X is a subspace of $\mathrm{Sq}(A)$ such that $\mathrm{Max}(A) \subseteq X$, then X is quasi-compact. In particular, the primary spectrum (space of primary ideals) of A is quasi-compact.

Proof. If U is an open of $\mathrm{Sq}(A)$ such that $X \subseteq U$, then $U = \mathrm{Sq}(A)$. Because if $\mathfrak{q}$ is a quasi-prime ideal of A, then there exist a maximal ideal $\mathfrak{m}$ of A and an element $f \in A$ such that $\mathfrak{q} \subseteq \mathfrak{m}$ and $\mathfrak{m} \in U_f \subseteq U$. It follows that $\mathfrak{q} \in U_f$. Thus by Theorem 2.4, X is quasi-compact. ■

Proposition 2.6 If either every prime ideal of A is a maximal ideal or A is a domain with Krull dimension one, then every quasi-prime ideal of A is a primary ideal.

Proof. If $\mathfrak{q}$ is a quasi-prime ideal of A, then $\sqrt{\mathfrak{q}}$ is either the zero ideal or a maximal ideal of A. Therefore $\mathfrak{q}$ is a primary ideal of A, since if J is an ideal of a ring R such that $\sqrt{J}$ is a maximal ideal of R then J is a primary ideal of R. Because let $f, g \in R$ such that $fg \in J$ and $f \notin \sqrt{J}$. Then $f + J$ is an invertible element of R/J since $\mathrm{Spec}(R/J) = \{\sqrt{J}/J\}$. It follows that $Rf + J = R$. Thus there are $a \in R$ and $b \in J$ such that $1 = af + b$. Therefore $g = afg + bg \in J$. □ ■

It seems that the converse of Proposition 2.6 also holds, though we have no idea that how to prove it. But, in outside of such rings the assertion is not true. As an example, the ring $A/\mathfrak{p}^2$ has Krull dimension one but its zero ideal is a quasi-prime ideal which is not a primary ideal, for A and $\mathfrak{p}$ see Example 2.2.

Corollary 2.7 If A is a PID, then the quasi-prime ideals of A are precisely of the form $\mathfrak{p}^n$, where $\mathfrak{p} \in \mathrm{Spec}(A)$ and $n \geqslant 1$ a natural number.

Theorem 2.8 If $\mathfrak{q}$ is a quasi-prime ideal of A, then $\overline{\{\mathfrak{q}\}} = \{\mathfrak{p} \in \mathrm{Sq}(A) : \mathfrak{q} \subseteq \sqrt{\mathfrak{p}}\}$.

Proof. Let $\mathfrak{p} \in \overline{\{\mathfrak{q}\}}$ and $f \in \mathfrak{q}$. If $f \notin \sqrt{\mathfrak{p}}$ then $\mathfrak{p} \in U_f$. It follows that $\mathfrak{q} \in U_f$, a contradiction. Conversely, assume that $\mathfrak{q} \subseteq \sqrt{\mathfrak{p}}$. If $\mathfrak{p} \notin \overline{\{\mathfrak{q}\}}$ then there exists some $f \in A$ such that $\mathfrak{p} \in U_f$ but $\mathfrak{q} \notin U_f$. Hence there exists a natural number $n \geqslant 1$ such that $f^n \in \mathfrak{q}$. It follows that $f \in \sqrt{\mathfrak{p}}$. But this is contradiction since $\mathfrak{p} \cap S_f = \emptyset$. ∎

Unlike the prime spectrum, a maximal ideal is not necessarily a closed point of the quasi-prime spectrum. As a specific example, if p is a prime number, then $\{p^n\mathbb{Z} : n \geqslant 1\}$ is the closure of $\{p\mathbb{Z}\}$ in $\mathrm{Sq}(\mathbb{Z})$. In fact, $\mathrm{Sq}(\mathbb{Z})$ has no closed point.

In the space $\mathrm{Sq}(A)$ generic points are not unique:

Corollary 2.9 If $\mathfrak{q}$ is a quasi-prime ideal of A, then $\overline{\{\mathfrak{q}\}} = \overline{\{\sqrt{\mathfrak{q}}\}}$.

Corollary 2.10 Let E be a closed subset of $\mathrm{Sq}(A)$. If $\mathfrak{q} \in E$, then $\sqrt{\mathfrak{q}} \in E$.

It can be shown the closed subsets of $\mathrm{Sq}(A)$ are precisely of the form $\mathcal{V}(I) = \{\mathfrak{q} \in \mathrm{Sq}(A) : I \subseteq \sqrt{\mathfrak{q}}\}$, where I is an ideal of A. Clearly $\mathcal{V}(I) \cap \mathrm{Spec}(A) = V(I)$. If $f \in A$, then $\mathcal{V}(f) = \{\mathfrak{q} \in \mathrm{Sq}(A) : f \in \sqrt{\mathfrak{q}}\}$. If $\mathfrak{p}$ is a quasi-prime of A, then by Theorem 2.8, $\mathcal{V}(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$.

Recall that a topological space is said to be an irreducible space if it is non-empty and can not be written as the union of two proper closed subsets.

Theorem 2.11 Every irreducible and closed subset of $\mathrm{Sq}(A)$ has a generic point.

Proof. Let Z be an irreducible and closed subset of $\mathrm{Sq}(A)$. There exists an ideal J of A such that $Z = \mathcal{V}(J)$. By Theorem 2.8, it suffices to show that J is a quasi-prime of A. Clearly $J \neq A$ since Z is non-empty. Let $f, g \in A$ such that $fg \in J$. We have $Z = \big(\mathcal{V}(f) \cap Z\big) \cup \big(\mathcal{V}(g) \cap Z\big)$.
It follows that either $Z \subseteq \mathcal{V}(f)$ or $Z \subseteq \mathcal{V}(g)$. Thus either $f \in \sqrt{J}$ or $g \in \sqrt{J}$. ∎

The irreducible components of $\mathrm{Sq}(A)$ are precisely of the form $\mathcal{V}(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A. If $\mathfrak{q}$ and $\mathfrak{q}'$ are quasi-prime ideals of a ring A such that $\sqrt{\mathfrak{q}} = \sqrt{\mathfrak{q}'}$, then clearly $\mathfrak{q} \cap \mathfrak{q}'$ is a quasi-prime ideal of A.

3. Connected components

A subspace Y of a topological space X is called a retraction of X if there exists a continuous map $\gamma : X \to Y$ such that $\gamma(y) = y$ for all $y \in Y$. Such a map γ is called a retraction map.

Lemma 3.1 The prime spectrum of a ring is a retraction of its quasi-prime spectrum.

Proof. Let A be a ring. Then the map $\gamma : \mathrm{Sq}(A) \to \mathrm{Spec}(A)$ given by $\mathfrak{q} \rightsquigarrow \sqrt{\mathfrak{q}}$ is a retraction map. ∎

The map γ of Lemma 3.1 is also an open map, since $\gamma(U_f) = D(f)$ for all $f \in A$.

Remark 1 *There is a fundamental result due to Grothendieck which states that the map $f \rightsquigarrow D(f)$ is a bijection from the set of idempotents of A onto the set of clopen (both open and closed) subsets of* $\mathrm{Spec}(A)$, *see [2, Tag 00EE]. Under the light of this result we obtain that:*

Theorem 3.2 The map $f \rightsquigarrow U_f$ is a bijection from the set of idempotents of A onto the set of clopens of $\mathrm{Sq}(A)$.

Proof. By Remark *1*, it suffices to show that the map $U \rightsquigarrow \gamma^{-1}(U)$ is a bijection from the set of clopens of $\operatorname{Spec} A$ onto the set of clopens of $\operatorname{Sq} A$, for γ see Lemma 3.1. Assume that $\gamma^{-1}(U) = \gamma^{-1}(V)$. If $\mathfrak{p} \in U$ then $\mathfrak{p} \in \gamma^{-1}(U)$ and so $\gamma(\mathfrak{p}) = \mathfrak{p} \in V$. Hence $U = V$. It remains to show that this map is surjective. If U is a clopen of $\operatorname{Sq} A$, then $U \cap \operatorname{Spec} A$ is a clopen of $\operatorname{Spec} A$. We have $U = \gamma^{-1}(U \cap \operatorname{Spec} A)$. Because if $\mathfrak{q} \in U$, then by Corollary 2.10, $\gamma(\mathfrak{q}) = \sqrt{\mathfrak{q}} \in U \cap \operatorname{Spec} A$. Conversely, if $\mathfrak{q} \in \gamma^{-1}(U \cap \operatorname{Spec} A)$, then there exists some $f \in A$ such that $\sqrt{\mathfrak{q}} \in U_f \subseteq U$. It follows that $\mathfrak{q} \in U$. ■

Corollary 3.3 The space $\operatorname{Sq}(A)$ is connected if and only if A has no nontrivial idempotents.

Proof. It is an immediate consequence of Theorem 3.2. ■

Proposition 3.4 If $\varphi : A \to B$ is a morphism of rings, then the induced map $\varphi^* : \operatorname{Sq}(B) \to \operatorname{Sq}(A)$ given by $\mathfrak{q} \rightsquigarrow \varphi^{-1}(\mathfrak{q})$ is continuous.

Proof. If $\mathfrak{q}$ is a quasi-prime ideal of B, then $\varphi^{-1}(\mathfrak{q})$ is a quasi-prime ideal of A because $\varphi^{-1}(\sqrt{\mathfrak{q}}) = \sqrt{\varphi^{-1}(\mathfrak{q})}$. Hence φ^* is well-defined. It is continuous since $(\varphi^*)^{-1}(U_f) = U_{\varphi(f)}$ for all $f \in A$. ■

Note that if S is a multiplicative subset of A, then the map $\pi^* : \operatorname{Sq}(S^{-1}A) \to \operatorname{Sq}(A)$ induced by the canonical ring map $A \to S^{-1}A$ is not injective and $\operatorname{Im} \pi^* \subseteq \{\mathfrak{q} \in \operatorname{Sq}(A) : \mathfrak{q} \cap S = \emptyset\}$. Specially $\operatorname{Im} \pi^* \subseteq U_f$ where $\pi : A \to A_f$ is the canonical map.

Lemma 3.5 If I is an ideal of a ring A, then the map $\pi : \operatorname{Sq}(A/I) \to \operatorname{Sq}(A)$ induced by the canonical ring map $A \to A/I$ is injective and $\operatorname{Im} \pi = \mathcal{V}(I)$.

Proof. It is an easy exercise. ■

An ideal of A is said to be a regular ideal of A if it is generated by a subset of idempotent elements of A. Each maximal element of the set of proper regular ideals of A (ordered by inclusion) is called a max-regular ideal of A. By the Zorn's Lemma, every proper regular ideal of A is contained in a max-regular ideal of A. It is well known that a regular ideal M is a max-regular ideal of A if and only if A/M has no nontrivial idempotents, see [7, Lemma 3.19]. It is also well known that the connected components of $\operatorname{Spec} A$ are precisely of the form $V(M)$ where M is a max-regular ideal of A, see [7, Theorem 3.17]. We have then the following result.

Theorem 3.6 The connected components of $\operatorname{Sq}(A)$ are precisely of the form $\mathcal{V}(M)$ where M is a max-regular ideal of A.

Proof. If N is a max-regular ideal of A, then by Corollary 3.3, $\operatorname{Sq} A/N$ is connected. Thus by Proposition 3.4 and Lemma 3.5, $\mathcal{V}(N)$ is connected. Now let C be a connected component of $\operatorname{Sq}(A)$. Then $\gamma(C)$ is contained in a connected component of $\operatorname{Spec}(A)$, for γ see Lemma 3.1. Thus there exists a max-regular ideal M of A such that $\gamma(C) \subseteq V(M)$. It follows that $C \subseteq \mathcal{V}(M)$. Thus $C = \mathcal{V}(M)$ since $\mathcal{V}(M)$ is connected. Conversely, let N be a max-regular ideal of A. Then there exists a connected component C of $\operatorname{Sq} A$ such that $\mathcal{V}(N) \subseteq C$. We observed that there exists a max-regular ideal M of A such that $C = \mathcal{V}(M)$. It follows that $\sqrt{M} \subseteq \sqrt{N}$. Thus $M \subseteq N$ since M is a regular ideal. This implies that $M = N$ because M is max-regular and N is regular ideal. ■

4. t-functor

There exists a covariant functor due to Grothendieck from the category of topological spaces to itself. It is called the t-functor. This functor has geometric applications and builds a bridge between the classical algebraic geometry and modern algebraic geometry. In what follows we shall introduce this functor. If X is a topological space, then the points of $t(X)$ are the irreducible and closed subsets of X. The closed subsets of $t(X)$ are precisely of the form $t(Y)$, where Y is a closed subset of X. If $f : X \to X'$ is a continuous map of topological spaces, then the function $t(f) : t(X) \to t(X')$ given by $Z \rightsquigarrow \overline{f(Z)}$ is well-defined and continuous. There exists also a canonical continuous map $X \to t(X)$ defined by $x \rightsquigarrow \overline{\{x\}}$. We have then the following result which states that there is a one to one correspondence between the irreducible closed subsets of $\mathrm{Sq}(A)$ and the prime ideals of A.

Theorem 4.1 The space $t\big(\mathrm{Sq}(A)\big)$ is canonically homeomorphic to $\mathrm{Spec}(A)$.

Proof. Let Z be an irreducible and closed subset of $\mathrm{Sq}(A)$. By Theorem 2.11, there exists a quasi-prime $\mathfrak{q}$ of A such that $Z = \overline{\{\mathfrak{q}\}}$. We then define $\varphi : t\big(\mathrm{Sq}(A)\big) \to \mathrm{Spec}(A)$ as $Z \rightsquigarrow \sqrt{\mathfrak{q}}$. We show that it is a homeomorphism. The map φ is injective, see Corollary 2.9. If $\mathfrak{p}$ is a prime ideal of A then $Z := \mathcal{V}(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ is an irreducible and closed subset of $\mathrm{Sq}(A)$ and $\varphi(Z) = \mathfrak{p}$. The map φ is continuous because $\varphi^{-1}\big(D(f)\big) = t\big(\mathrm{Sq}(A)\big) \setminus t\big(\mathcal{V}(f)\big)$ for all $f \in A$. It remains to show that φ is a closed map. Let $Y = \mathcal{V}(I)$ be a closed subset of $\mathrm{Sq}(A)$ where I is an ideal of A. Then $\varphi\big(t(Y)\big) = V(I)$. ■

Acknowledgments

We would like to thank the referee for careful reading of the paper and for his/her valuable suggestions and comments which improved the paper. This paper is published as part of a research project supported by the research affairs office of the university of Maragheh.

References

[1] M. Aghajani, A. Tarizadeh, Characterizations of Gelfand rings specially clean rings and their dual rings, Results Math 75, 2020:125.
[2] A. J. de Jong et al., Stacks Project, see http://stacks.math.columbia.edu.
[3] C. A. Finocchiaro, M. Fontana, D. Spirito, A topological version of Hilbert's Nullstellensatz, J. Algebra 461 (2016), 25-41.
[4] L. Fuchs, On quasi-primary ideals, Acta Sci. Math. 11 (3) (1947), 174-183.
[5] M. Hochster, Prime ideal structure in commutative rings, Trans. Amer Math. Soc. 142 (1969), 43-60.
[6] K. Schwede, Gluing schemes and a scheme without closed points, Recent progress in arithmetic and algebraic geometry, Contemp. Math. 386 (2005), 157-172.
[7] A. Tarizadeh, Flat topology and its dual aspects, Comm. Algebra 47 (1) (2019), 195-205.
[8] A. Tarizadeh, Zariski compactness of minimal spectrum and flat compactness of maximal spectrum, J. Algebra Appl. 18, 2019:1950202.