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Super-Efficiency and Sensitivity Analysis in DEA for the Case of Exogenously Fixed Inputs

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Abstract

This paper reports on a study of the use of super-efficiency approach in data envelopment analysis (DEA) sensitivity analysis for the case of "exogenously fixed" factors. This issue is important since in any realistic situation there may exist exogenously fixed or non-discretionary factors that are beyond the control of a DMU's management, which also need to be considered. When a DMU under evaluation is not included in the reference set of the original DEA models, the resulting DEA models are called super-efficiency DEA models. In this paper, by means of the modified Banker and Morey's (BM hereafter) model [2], in which the test DMU is excluded from the reference set, we show that super-efficiency score can be decomposed into two data perturbation components of a particular test frontier decision making unit (DMU) and the remaining DMUs. As a result, we are able to determine what perturbations of discretionary data can be tolerated before frontier DMUs become nonfrontier.

1 Introduction

Data envelopment analysis is a non-parametric mathematical programming technique for measuring and evaluating the relative efficiencies of a set of entities, called decision making units (DMUs), with common inputs and outputs. Examples include agricultural productivity, banks, business firms, courts, hospitals, libraries, schools, universities, and others, including as well as the performance of countries, regions, etc. [3]. Being a non-parametric technique, DEA does not require a structural form for the production frontier and can handle multiple outputs quite easily. These attractive properties of the DEA approach have enabled its widespread use across many disciplines. See Seiford [13] and Emrouznejad et al. [8] for a survey of the literature on the development of DEA methodology since its introduction by Charnes et al. [4]. Standard data envelopment analysis implicitly assumes that all inputs and outputs are discretionary, i.e., can be controlled by the management of each DMU and varied at its discretion. However, there may exist exogenously fixed (or non-discretionary) factors that are beyond the control of a DMU's management, which also need to be considered [10,12,15]. On the other hand, data envelopment analysis identifies an empirical efficient frontier of a set of peer decision making units. In data envelopment analysis, extreme efficient units are of primary importance as they define the efficient frontier. The efficient frontier is characterized by the DMUs with an efficiency score of unity. An important problem in the DEA literature is that of ranking those DMUs called efficient by the DEA model, all of which have a score of unity. The super efficiency model involves executing the standard DEA models, but under the assumption that the DMU being evaluated is excluded from the reference set [1,5,6,14,16,18]. For the DEA sensitivity analysis based on the inverse of basis matrix, the reader is referred to [7,11]. Specifically, the super efficiency score in, say, the input-oriented model provides a measure of the proportional increase in the inputs for a DMU that could take place without destroying the "efficient" status of that DMU relative to the frontier created by the remaining DMUs.

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The current research dedicated to apply the super-efficiency approach in data envelopment analysis (DEA) sensitivity analyses, when some inputs are non-discretionary. For this task, we first introduce the BM model [2], then by means of the modified BM model, in which the test DMU is excluded from the reference set, we determine what perturbations of data can be tolerated before frontier DMUs become nonfrontier. The sensitivity analysis approach developed in this paper can be applied to all DMUs on the entire frontier. This study attempts to generalize the results in [9] to a situation where variable percentage data changes are assumed for a test DMU and for the remaining DMUs. We consider the same worst-case analysis as in [9]. It is shown that a particular super-efficiency score can be decomposed into two data perturbation components of a particular test DMU and the remaining DMUs. Necessary and sufficient conditions for preserving a DMU's BM-efficiency classification are developed when variable percentage data changes are applied to all DMUs. Note that in this paper we assume that the factors are either fully discretionary or fully non-discretionary. Also we assume that none of the models have non-discretionary outputs.
The layout of this article is as follows. In Section 2, basic definitions, that will be used in the succeeding sections, are given. In Section 3 we will discuss super-efficiency and sensitivity analysis in the BM model. Section 4 is the main part of this study where we will discuss simultaneous changes in all the discretionary data. Section 5 provides a numerical example from DEA, where some of the ideas of the paper are illustrated. The last section provides a summary and some future research directions.

2 Definitions

The following standard notations and definitions are used in the paper. Consider a set of *n* DMUs, where each DMU_j $(j = 1, 2, \ldots, n)$ uses *m* different discretionary inputs, $x_{ij},\ (i = 1, 2, \ldots, m)$, and *p* different non-discretionary inputs $z_{ij},\ (i = 1, 2, \ldots, p)$, to produce *s* different outputs, $y_{rj},\ (r = 1, 2, \ldots, s)$. We assume that the data set are positive.

Assuming constant returns to scale, the BM model to evaluate the efficiency of any DMU – in the input-oriented case – is given by the following modification of the CCR model:

$$\boxed{\text{BM}_{\texttt{CCR}}}$$

$$\begin{aligned}
\theta^{\texttt{CCR}*}_{\texttt{ND}} = \quad & \min\ \theta & & (1)\\
& s.t.\ \sum_{j=1}^{n} x_{ij}\lambda_j \le \theta x_{io}, & & i \in \mathbf{D}\\
& \sum_{j=1}^{n} z_{ij}\lambda_j \le z_{io} & & i \in \mathbf{ND}\\
& \sum_{j=1}^{n} y_{rj}\lambda_j \ge y_{ro}, & & r = 1, 2, \ldots, s\\
& \lambda_j \ge 0, & & j = 1, 2, \ldots, n.
\end{aligned}$$

Here the symbols **D** and **ND** refer to *Discretionary* and *Non-Discretionary*, respectively. Note that the variable θ is not applied to the non-discretionary input constraints, because these values are exogenously fixed and it is therefore not possible to vary them at the discretion of management. This is recognized by entering all $z_{io},\ i \in \mathbf{ND}$, at their fixed (observed) values. If we add an additional convex constraint of $\sum_{j=1}^{n} \lambda_j = 1$ to (1), we obtain an input-oriented VRS model. Based on the optimal solution of Model (1), we define a DMU as being $\text{BM}_{\texttt{CCR}}$-efficient as follows:

Definition 1. ($\mathbf{BM_{CCR}}$-efficiency) A DMU_o is BM_{CCR}-efficient if and only if it satisfies the following two conditions:
i. $\theta_{\text{ND}}^{\text{CCR}*}=1$,
ii. In all alternative optimal solutions, all discretionary slacks are zero.
Furthermore, if in all alternative optima, all non-discretionary slacks are zero, then DMU_o is called Full-BM_{CCR}-efficient.

Definition 2.(Extreme $\mathbf{BM_{CCR}}$-efficient) A BM_{CCR}-efficient DMU_o is extreme BM_{CCR}-efficient if and only if it has a unique optimal solution in Model (1).

3 Super-efficiency and sensitivity analysis in the BM model

As in Charnes et al. [5], the DMUs can be partitioned into two groups: frontier DMUs and non-frontier DMUs. Furthermore, by Definition 1 the frontier DMUs consist of DMUs in set E (extreme Full-BM_{CCR}-efficient), set E' (Full-BM_{CCR}-efficient but not an extreme point), set E'' (BM_{CCR}-efficient but with non-zero non-discretionary slacks) and set F (weakly BM_{CCR}-efficient or frontier point but with non-zero discretionary slacks).

We may use a super-efficiency non-discretionary DEA model to identify the classification of DMU_o. That is,

$\boxed{\text{BM}^{\text{Super}}}$

$$\begin{aligned} \theta_{\text{ND}}^{sup*} = \quad & \min\ \theta \qquad\qquad (2)\\ & s.t. \sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} \leq \theta x_{io}, \quad i \in \mathbf{D} \\ & \sum_{j=1, j\neq o}^{n} \lambda_j z_{ij} \leq z_{io}, \quad i \in \mathbf{ND} \\ & \sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} \geq y_{ro}, \quad r = 1, 2, \ldots, s \\ & \lambda_j \geq 0, \quad j = 1, 2, \ldots, n; j \neq o \end{aligned}$$

Suppose $\theta_{\text{ND}}^{sup*}$ is the optimal value to (2). Based on Hosseinzadeh et al. ([9], Theorem 4), we have:

i. $\theta_{\text{ND}}^{sup*} > 1$ or the [BM^{Super}] model is infeasible, if and only if $\text{DMU}_o \in E$,
ii. $\theta_{\text{ND}}^{sup*} = 1$ if and only if $\text{DMU}_o \in E' \cup E'' \cup F$, and
iii. $\theta_{\text{ND}}^{sup*} < 1$ if and only if DMU_o is a non-frontier point or DMU_o belongs to the inefficient frontier.

Example 1. Consider a system with 6 units, each unit with two inputs and one output, where the first input is non-controllable. Table **1** exhibits the data and displays the BM-efficiency and the BM-super-efficiency of each unit.

Table 1: Results of the BM model for super efficiency.

DMU	Input 1	Input 2	Output	$\theta_{\text{ND}}^{\text{CCR}*}$	s_1^{-*}	s_2^{-*}	s^{+*}	$\theta_{\text{ND}}^{sup*}$
A	2	3	1	1	0	0	0	1.5
B	4	1	1	1	0	0	0	1.4
C	6	1	1	1	2	0	0	1
D	6	3	1	0.3333	2	0	0	0.3333
E	2	4.5	1	0.6667	0	0	0	0.6667
F	3.5	1.5	1	1	0	0	0	1

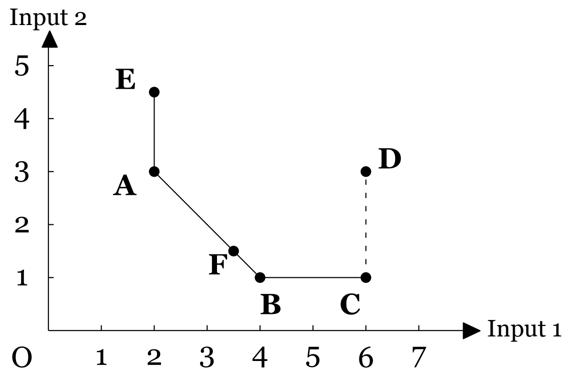

Figure1: The diagram for Example 1.

The results presented in Table **1** indicate that **A** and **B** belong to set E, **F** belongs to set E', **C** belongs to set E'', and **D**, **E** are BM_{CCR}-inefficient.

4 Super-efficiency and non-discretionary data

The extreme DMUs in DEA are of primary importance as they define the DEA frontier. In this section we will discuss the stability of efficiency classification for such units. We consider the general case. That is, we are interested in whether DMU_o will still be a frontier point after data perturbations in all the DMUs. Our discussion is based on a worst-case scenario in which the BM_{CCR}-efficiency of DMU_o declines and the BM_{CCR}-efficiencies of all other $\text{DMU}_j,\ j \neq o$, improve.

Let $\mathbf{D_I} \subseteq \mathbf{D}$ and $\mathbf{O}$ denote, respectively, the discretionary input and output subsets in which we are interested. That is, we consider the data changes in set $\mathbf{D_I}$ and set $\mathbf{O}$. Then the simultaneous data perturbations in discretionary inputs/outputs of DMU_o and all $\text{DMU}_j, j \neq o$, can be written as percentage data perturbation (variation):

for DMU_o for $DMU_j (j \neq o)$:

$$\begin{cases} \widehat{x}_{io} = \alpha_i x_{io}, \quad \alpha_i \geq 1, & i \in \mathbf{D_I} \\ \widehat{x}_{io} = x_{io}, & i \notin \mathbf{D_I} \\ \widehat{z}_{io} = z_{io}, & i \in \mathbf{ND} \\ \widehat{y}_{ro} = \beta_r y_{ro}, \quad 0 < \beta_r \leq 1, & r \in \mathbf{O} \\ \widehat{y}_{ro} = y_{ro}, & r \notin \mathbf{O} \end{cases} \qquad \begin{cases} \widehat{x}_{ij} = \dfrac{x_{ij}}{\widetilde{\alpha}_i}, \quad \widetilde{\alpha}_i \geq 1, & i \in \mathbf{D_I} \\ \widehat{x}_{ij} = x_{ij}, & i \notin \mathbf{D_I} \\ \widehat{z}_{ij} = z_{ij}, & i \in \mathbf{ND} \\ \widehat{y}_{rj} = \dfrac{y_{rj}}{\widetilde{\beta}_r}, \quad 0 < \widetilde{\beta}_r \leq 1, & r \in \mathbf{O} \\ \widehat{y}_{rj} = y_{rj}, & r \notin \mathbf{O} \end{cases}$$

where $(\widehat{.})$ represents adjusted data. Note that the data perturbations represented by α_i and $\widetilde{\alpha}_i$ (or β_r and $\widetilde{\beta}_r$) can be different for each $i \in \mathbf{D_I}$ (or $r \in \mathbf{O}$).

Now we modify Model (2) to the following super-efficiency DEA model, when the same percentage changes of DMU_o and $\text{DMU}_j, j \neq o$, are assumed:

$$\begin{aligned} \theta_{\text{ND}}^{\mathbf{I}^*} = \quad & \min \quad \theta_{\text{ND}}^{\mathbf{I}} && (3) \\ & \text{s.t.} \sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} \leq \theta_{\text{ND}}^{\mathbf{I}} x_{io}, && i \in \mathbf{D_I} \\ & \sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} \leq x_{io}, && i \notin \mathbf{D_I} \\ & \sum_{j=1, j\neq o}^{n} \lambda_j z_{ij} \leq z_{io}, && i \in \mathbf{ND} \\ & \sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} \geq y_{ro}, && r = 1, 2, \ldots, s \\ & \lambda_j \geq 0, j = 1, 2, \ldots, n; j \neq o \end{aligned}$$

By the optimal values of Models (2) and (3), we have:

Lemma 1. If Model (3) is feasible and $\theta_{\text{ND}}^{\text{Sup}^*} > 1$, then $\theta_{\text{ND}}^{\mathbf{I}^*} > 1$.
Proof. See Hosseinzadeh et al. [9] for a proof. ■

Theorem 1. Let Model (3) be feasible and $\theta_{\text{ND}}^{\text{Sup}^*} > 1$. If $1 \leq \alpha_i \widetilde{\alpha}_i < \theta_{\text{ND}}^{\mathbf{I}^*}$, $i \in \mathbf{D_I}$, then DMU_o remains as an extreme BM_{CCR}-efficient point. Furthermore, if equality holds for $\alpha_i \widetilde{\alpha}_i = \theta_{\text{ND}}^{\mathbf{I}^*}$, that is, $1 \leq \alpha_i \widetilde{\alpha}_i \leq \theta_{\text{ND}}^{\mathbf{I}^*}$, then DMU_o remains on the frontier, where $\theta_{\text{ND}}^{\mathbf{I}^*}$ is the optimal value to (3). In other words, any values of α_i and $\widetilde{\alpha}_i$ within this range of variation for both x_{io} and x_{ij} will not affect the BM_{CCR}-efficiency status of DMU_o.

Proof. By Lemma 1, we have $\theta_{\text{ND}}^{\mathbf{I}^*} > 1$. Now suppose $1 \leq \alpha_i \widetilde{\alpha}_i < \theta_{\text{ND}}^{\mathbf{I}^*}$, but $\text{DMU}_{\widehat{o}}$ is not extreme BM_{CCR}-efficient, when $\widehat{x}_{io} = \alpha_i x_{io}$, $\widehat{x}_{ij} = \dfrac{x_{ij}}{\widetilde{\alpha}_i}; i \in \mathbf{D_I}$. Then Model (2) for evaluating $\text{DMU}_{\widehat{o}}$ has an optimal solution $(\widehat{\theta}_{\text{ND}}^{\text{Sup}^*}, \widehat{\lambda}_j^*; j = 1, 2, \ldots, n,\ j \neq o)$ such that $\widehat{\theta}_{\text{ND}}^{\text{Sup}^*} \leq 1$. In the optimal solution, the constraints of Model (2) for evaluating $\text{DMU}_{\widehat{o}}$ are as follows

$$\begin{cases} \displaystyle\sum_{j=1, j\neq o}^{n} \widehat{\lambda}_j^* \widehat{x}_{ij} \leq \widehat{\theta}_{\text{ND}}^{\text{Sup}^*} \widehat{x}_{io}, & i \in \mathbf{D} \\ \displaystyle\sum_{j=1, j\neq o}^{n} \widehat{\lambda}_j^* \widehat{z}_{ij} \leq \widehat{z}_{io}, & i \in \mathbf{ND} \\ \displaystyle\sum_{j=1, j\neq o}^{n} \widehat{\lambda}_j^* \widehat{y}_{rj} \geq \widehat{y}_{ro}, & r = 1, 2, \ldots, s \end{cases}$$

Or equivalently

$$\begin{cases} \sum_{j=1,j\neq o}^{n} \widehat{\lambda}_j^* x_{ij} \leq \widehat{\theta}_{\mathtt{ND}}^{\mathtt{Sup}^*} \alpha_i \widetilde{\alpha}_i x_{io} \leq \widehat{\theta}_{\mathtt{ND}}^{\mathtt{Sup}^*} \alpha_k \widetilde{\alpha}_k x_{io}, & i \in \mathbf{D_I} \\ \sum_{j=1,j\neq o}^{n} \widehat{\lambda}_j^* x_{ij} \leq \widehat{\theta}_{\mathtt{ND}}^{\mathtt{Sup}^*} x_{io} \leq x_{io}, & i \notin \mathbf{D_I} \\ \sum_{j=1,j\neq o}^{n} \widehat{\lambda}_j^* z_{ij} \leq z_{io}, & i \in \mathbf{ND} \\ \sum_{j=1,j\neq o}^{n} \widehat{\lambda}_j^* y_{rj} \geq y_{ro}, & r = 1, 2, \ldots, s, \end{cases}$$

where $\alpha_k \widetilde{\alpha}_k = \max\limits_{i \in \mathbf{D_I}} \{\alpha_i \widetilde{\alpha}_i\}$.

This means that $(\alpha_k \widetilde{\alpha}_k \widehat{\theta}_{\mathtt{ND}}^{\mathtt{Sup}^*}, \widehat{\lambda}_j^*; j = 1, 2, \ldots, n,\ j \neq o)$ is feasible to (3) for evaluating DMU_o.

Moreover $\alpha_k \widetilde{\alpha}_k \widehat{\theta}_{\mathtt{ND}}^{\mathtt{Sup}^*} < \theta_{\mathtt{ND}}^{\mathbf{I}^*} \widehat{\theta}_{\mathtt{ND}}^{\mathtt{Sup}^*} \leq \theta_{\mathtt{ND}}^{\mathbf{I}^*}$, violating the optimality of $\theta_{\mathtt{ND}}^{\mathbf{I}^*}$.

Next suppose $1 \leq \alpha_i \widetilde{\alpha}_i \leq \theta_{\mathtt{ND}}^{\mathbf{I}^*}$, but $\text{DMU}_{\widehat{o}}$ is not a frontier point, when $\widehat{x}_{io} = \alpha_i x_{io}$ and $\widehat{x}_{ij} = \dfrac{x_{ij}}{\widetilde{\alpha}_i}; i \in \mathbf{D_I}$. Then Model (2) for evaluating $\text{DMU}_{\widehat{o}}$ has an optimal solution $(\widehat{\theta}_{\mathtt{ND}}^{\mathtt{Sup}^*}, \widehat{\lambda}_j^*; j = 1, 2, \ldots, n,\ j \neq o)$ such that $\widehat{\theta}_{\mathtt{ND}}^{\mathtt{Sup}^*} < 1$. As can be seen above, $(\alpha_k \widetilde{\alpha}_k \widehat{\theta}_{\mathtt{ND}}^{\mathtt{Sup}^*}, \widehat{\lambda}_j^*; j = 1, 2, \ldots, n,\ j \neq o)$ is a feasible solution to (3). Now we get $\alpha_k \widetilde{\alpha}_k \widehat{\theta}_{\mathtt{ND}}^{\mathtt{Sup}^*} \leq \theta_{\mathtt{ND}}^{\mathbf{I}^*} \widehat{\theta}_{\mathtt{ND}}^{\mathtt{Sup}^*} < \theta_{\mathtt{ND}}^{\mathbf{I}^*}$, which is in contradiction to $\theta_{\mathtt{ND}}^{\mathbf{I}^*}$ being the optimal value of Model (3).

In fact, Theorem 1 gives sufficient conditions for preserving $\text{BM}_{\mathtt{CCR}}$-efficiency. The following theorem implies necessary conditions for preserving $\text{BM}_{\mathtt{CCR}}$-efficiency of an extreme $\text{BM}_{\mathtt{CCR}}$-efficient DMU_o.

Theorem 2. Suppose Model (3) is feasible and $\theta_{\mathtt{ND}}^{\mathtt{Sup}^*} > 1$. If $\alpha_i \widetilde{\alpha}_i > \theta_{\mathtt{ND}}^{\mathbf{I}^*}$ for $i \in \mathbf{D_I}$, then $\text{DMU}_{\widehat{o}}$ is not extreme $\text{BM}_{\mathtt{CCR}}$-efficient, where $\theta_{\mathtt{ND}}^{\mathbf{I}^*}$ is the optimal value of Model (3). ($\text{DMU}_{\widehat{o}}$ represents DMU_o after the perturbations).

Proof. By contradiction we assume that $\text{DMU}_{\widehat{o}}$ is an extreme-$\text{BM}_{\mathtt{CCR}}$-efficient point after the changes in the discretionary inputs of all units with $\alpha_i \widetilde{\alpha}_i > \theta_{\mathtt{ND}}^{\mathbf{I}^*},\ i \in \mathbf{D_I}$. Then $\widehat{\theta}_{\mathtt{ND}}^{\mathtt{Sup}^*} > 1$, where $\widehat{\theta}_{\mathtt{ND}}^{\mathtt{Sup}^*}$ is the optimal value to (2) for evaluating $\text{DMU}_{\widehat{o}}$. Now suppose $(\widehat{\theta}_{\mathtt{ND}}^{\mathbf{I}^*}, \widehat{\lambda}_j^*; j = 1, 2, \ldots, n,\ j \neq o)$ is an optimal solution to (3) for evaluating $\text{DMU}_{\widehat{o}}$. Then, by Lemma 1 we have $\widehat{\theta}_{\mathtt{ND}}^{\mathbf{I}^*} > 1$. Also, in the optimal solution we have

$$\begin{cases} \sum_{j=1,j\neq o}^{n} \widehat{\lambda}_j^* \widehat{x}_{ij} \leq \widehat{\theta}_{\mathtt{ND}}^{\mathbf{I}^*} \widehat{x}_{io}, & i \in \mathbf{D_I} \\ \sum_{j=1,j\neq o}^{n} \widehat{\lambda}_j^* x_{ij} \leq x_{io}, & i \notin \mathbf{D_I} \\ \sum_{j=1,j\neq o}^{n} \widehat{\lambda}_j^* z_{ij} \leq z_{io}, & i \in \mathbf{ND} \\ \sum_{j=1,j\neq o}^{n} \widehat{\lambda}_j^* y_{rj} \geq y_{ro}, & r = 1, 2, \ldots, s \end{cases}$$

Or equivalently

$$\begin{cases} \sum_{j=1,j\neq o}^{n} \widehat{\lambda}_j^* x_{ij} \leq \alpha_i\widetilde{\alpha}_i\widehat{\theta}_{\mathtt{ND}}^{\mathbf{I}^*} x_{io} \leq \alpha_k\widetilde{\alpha}_k\widehat{\theta}_{\mathtt{ND}}^{\mathbf{I}^*} x_{io}, & i \in \mathbf{D_I} \\ \sum_{j=1,j\neq o}^{n} \widehat{\lambda}_j^* x_{ij} \leq x_{io}, & i \notin \mathbf{D_I} \\ \sum_{j=1,j\neq o}^{n} \widehat{\lambda}_j^* z_{ij} \leq z_{io}, & i \in \mathbf{ND} \\ \sum_{j=1,j\neq o}^{n} \widehat{\lambda}_j^* y_{rj} \geq y_{ro}, & r = 1,2,\ldots,s \end{cases}$$

It can be easily verified that $\alpha_k\widetilde{\alpha}_k\widehat{\theta}_{\mathtt{ND}}^{\mathbf{I}^*} = \theta_{\mathtt{ND}}^{\mathbf{I}^*}$, where $\alpha_k\widetilde{\alpha}_k = \max_{i\in\mathbf{D_I}}\{\alpha_i\widetilde{\alpha}_i\}$.

Thus, $\widehat{\theta}_{\mathtt{ND}}^{\mathbf{I}^*} = \dfrac{\theta_{\mathtt{ND}}^{\mathbf{I}^*}}{\alpha_k\widetilde{\alpha}_k} < 1$, violating $\widehat{\theta}_{\mathtt{ND}}^{\mathbf{I}^*} > 1$. ■

The data perturbation can be expressed in a quadratic function, $\alpha_i\widetilde{\alpha}_i = \theta_{\mathtt{ND}}^{\mathbf{I}^*}$. This function gives an upper bound for discretionary input changes. Any data variations fall below this function and above lines $\alpha_i \geq 1$ and $\widetilde{\alpha}_i \geq 1,\ i \in \mathbf{D_I}$ will preserve the frontier status of DMU$_o$.

The above developments consider the input changes in all DMUs. Next, we consider the following modified DEA measure for simultaneous variations of inputs and outputs.

$$\begin{aligned} \Omega_{\mathtt{ND}}^* = \quad & \min \quad \Omega_{\mathtt{ND}} & & (4) \\ & s.t. \sum_{j=1,j\neq o}^{n} \mu_j x_{ij} \leq (1+\Omega_{\mathtt{ND}})x_{io}, & & i \in \mathbf{D_I} \\ & \sum_{j=1,j\neq o}^{n} \mu_j x_{ij} \leq x_{io}, & & i \notin \mathbf{D_I} \\ & \sum_{j=1,j\neq o}^{n} \mu_j z_{ij} \leq z_{io}, & & i \in \mathbf{ND} \\ & \sum_{j=1,j\neq o}^{n} \mu_j y_{rj} \geq (1-\Omega_{\mathtt{ND}})y_{ro}, & & r \in \mathbf{O} \\ & \sum_{j=1,j\neq o}^{n} \mu_j y_{rj} \geq y_{ro}, & & r \notin \mathbf{O} \\ & \mu_j \geq 0, & & j = 1,2,\ldots,n; j \neq o \end{aligned}$$

Note that if DMU$_o$ is a frontier point, then $\Omega_{\mathtt{ND}} \geq 0$.

Theorem 3. Let DMU$_o$ be a frontier point, and let $\Omega_{\mathtt{ND}}^*$ be the optimal value to (4). If $1 \leq \alpha_i\widetilde{\alpha}_i \leq 1+\Omega_{\mathtt{ND}}^*,\ i \in \mathbf{D_I}$, and $1-\Omega_{\mathtt{ND}}^* \leq \beta_r\widetilde{\beta}_r \leq 1,\ r \in \mathbf{O}$, then DMU$_o$ remains as a frontier point .

Proof. By contradiction we assume that $DMU_{\widehat{o}}$ is not a frontier point. Suppose that $\widehat{\Omega}_{\mathtt{ND}}^*$ is the optimal value to (4) for evaluating $DMU_{\widehat{o}}$, then we have $\widehat{\Omega}_{\mathtt{ND}}^* < 0$.

Consider the constraints of Model (4) in the optimal solution as follows:

$$\begin{cases} \sum_{j=1,j\neq o}^{n} \widehat{\mu}_j^* \widehat{x}_{ij} \leq (1+\widehat{\Omega}_{\text{ND}}^*)\widehat{x}_{io}, & i \in \mathbf{D_I} \\ \sum_{j=1,j\neq o}^{n} \widehat{\mu}_j^* \widehat{x}_{ij} \leq \widehat{x}_{io}, & i \notin \mathbf{D_I} \\ \sum_{j=1,j\neq o}^{n} \widehat{\mu}_j^* \widehat{z}_{ij} \leq \widehat{z}_{io}, & i \in \mathbf{ND} \\ \sum_{j=1,j\neq o}^{n} \widehat{\mu}_j^* \widehat{y}_{rj} \geq (1-\widehat{\Omega}_{\text{ND}}^*)\widehat{y}_{ro}, & r \in \mathbf{O} \\ \sum_{j=1,j\neq o}^{n} \widehat{\mu}_j^* \widehat{y}_{rj} \geq \widehat{y}_{ro}, & r \notin \mathbf{O} \end{cases}$$

Or equivalently

$$\begin{cases} \sum_{j=1,j\neq o}^{n} \widehat{\mu}_j^* x_{ij} \leq \alpha_i\widetilde{\alpha}_i(1+\widehat{\Omega}_{\text{ND}}^*)x_{io} \leq [1+(\alpha_k\widetilde{\alpha}_k(1+\widehat{\Omega}_{\text{ND}}^*)-1)]x_{io}, & i \in \mathbf{D_I} \\ \sum_{j=1,j\neq o}^{n} \widehat{\mu}_j^* x_{ij} \leq x_{io}, & i \notin \mathbf{D_I} \\ \sum_{j=1,j\neq o}^{n} \widehat{\mu}_j^* z_{ij} \leq z_{io}, & i \in \mathbf{ND} \\ \sum_{j=1,j\neq o}^{n} \widehat{\mu}_j^* y_{rj} \geq \beta_r\widetilde{\beta}_r(1-\widehat{\Omega}_{\text{ND}}^*)y_{ro} \geq [1-(1-\beta_t\widetilde{\beta}_t(1-\widehat{\Omega}_{\text{ND}}^*))]y_{ro}, & r \in \mathbf{O} \\ \sum_{j=1,j\neq o}^{n} \widehat{\mu}_j^* y_{rj} \geq y_{ro}, & r \notin \mathbf{O} \end{cases}$$

where $\alpha_k\widetilde{\alpha}_k = \max_{i\in\mathbf{D_I}}\{\alpha_i\widetilde{\alpha}_i\}$ and $\beta_t\widetilde{\beta}_t = \min_{r\in\mathbf{O}}\{\beta_r\widetilde{\beta}_r\}$.
Set $\widetilde{\Omega}$= max $\{(\alpha_k\widetilde{\alpha}_k(1+\widehat{\Omega}_{\text{ND}}^*)-1),\ (1-\beta_t\widetilde{\beta}_t(1-\widehat{\Omega}_{\text{ND}}^*))\}$.
Obviously ($\widetilde{\Omega},\ \widehat{\mu}_j^*;\ j=1,2,\ldots,n,\ j\neq o$) is a feasible solution of (4) for DMU$_o$. Therefore, $\Omega_{\text{ND}}^* \leq \widetilde{\Omega}$. Now consider the following two cases:

Case 1: If $\widetilde{\Omega} = (\alpha_k\widetilde{\alpha}_k(1+\widehat{\Omega}_{\text{ND}}^*)-1)$, then from the assumptions we get $1 \leq \alpha_k\widetilde{\alpha}_k \leq 1+\Omega_{\text{ND}}^*$. Since $\widehat{\Omega}_{\text{ND}}^* < 0$, we have $0 < 1+\widehat{\Omega}_{\text{ND}}^* < 1$. Thus, $(1+\widehat{\Omega}_{\text{ND}}^*)\alpha_k\widetilde{\alpha}_k \leq (1+\widehat{\Omega}_{\text{ND}}^*)(1+\Omega_{\text{ND}}^*) < 1+\Omega_{\text{ND}}^*$.
This means that $\widetilde{\Omega} = (1+\widehat{\Omega}_{\text{ND}}^*)\alpha_k\widetilde{\alpha}_k - 1 < \Omega_{\text{ND}}^*$, which is a contradiction.

Case 2: If $\widetilde{\Omega} = (1-\beta_t\widetilde{\beta}_t(1-\widehat{\Omega}_{\text{ND}}^*))$, then from the assumptions we have $1-\Omega_{\text{ND}}^* \leq \beta_t\widetilde{\beta}_t \leq 1$. Since $\widehat{\Omega}_{\text{ND}}^* < 0$, we have $(1-\widehat{\Omega}_{\text{ND}}^*) > 1$. So, $(1-\widehat{\Omega}_{\text{ND}}^*)\beta_t\widetilde{\beta}_t \geq (1-\widehat{\Omega}_{\text{ND}}^*)(1-\Omega_{\text{ND}}^*) > 1-\Omega_{\text{ND}}^*$. This means that $\widetilde{\Omega} = (1-\beta_t\widetilde{\beta}_t(1-\widehat{\Omega}_{\text{ND}}^*)) < \Omega_{\text{ND}}^*$, which is in conflict with $\Omega_{\text{ND}}^* \leq \widetilde{\Omega}$. ■

Theorem 4. Suppose DMU$_o$ is an extreme BM$_{\text{CCR}}$-efficient point, and Ω_{ND}^* is the optimal value to (4). If $\alpha_i\widetilde{\alpha}_i > 1+\Omega_{\text{ND}}^*,\ i \in \mathbf{D_I}$ and $1-\Omega_{\text{ND}}^* > \beta_r\widetilde{\beta}_r,\ r \in \mathbf{O}$, then DMU$_o$ will not remain extreme-BM$_{\text{CCR}}$-efficient.

Proof. This is similar to the proof of Theorem 2 only with some minor modifications and, hence, omitted.

In fact, Theorem 3 gives sufficient conditions for preserving $\mathrm{BM}_{\mathtt{CCR}}$-efficiency, and Theorem 4 implies necessary conditions for preserving $\mathrm{BM}_{\mathtt{CCR}}$-efficiency of an extreme-$\mathrm{BM}_{\mathtt{CCR}}$-efficient DMU_o.

5 An application

To further clarify, we apply the above approach to the data set obtained from 16 hospitals: H1 to H16 (see Table 2). The data is taken from Tone [17]. Each hospital uses four inputs to produce two outputs. Table **2** shows the types of these inputs and outputs.

Table 2: **The types of inputs and outputs for hospitals**

Input	
Doctor	Total hours worked by doctors in the survey period
Nurse	Total hours worked by nurses
Tech. Worker	Total hours worked by technical workers
Office	Total hours worked by office staff
Output	
Outpatient	Total medical insurance points for outpatients
Inpatient	Total medical insurance points for inpatients

Table 3: **Hospitals**

	Input				**Output**	
Hospital	Doctor	Nurse	Tech. Worker	Office	Outpatient	Inpatient
H1	995	6205	1375	2629	4127	1678
H2	917	5898	1379	2047	3721	1277
H3	3178	10049	3615	3511	2706	2051
H4	813	5833	1124	1730	2176	1538
H5	1236	8639	2486	4990	5220	20426
H6	1146	7610	1600	3589	3517	1856
H7	705	5600	1557	3623	2352	20606
H8	2871	11524	2880	2452	1755	1664
H9	1089	8998	1730	2823	4412	2334
H10	2032	9383	2421	4454	5386	2080
H11	1414	10468	2140	3649	5735	2691
H12	1967	11260	2759	3178	6079	2804
H13	1851	9880	2335	4570	5893	2495
H14	3100	15649	5487	2940	5248	3692
H15	5016	18010	4008	3567	7800	4582
H16	1924	12682	2490	2975	6040	3396

In evaluating the efficiency of a hospital, the total hours worked by doctors in the survey period is an important (input) factor. But,"Doctor" is non-controllable and so we apply the BM model in order to evaluate the BM-efficiency and super-$\mathrm{BM}_{\mathtt{CCR}}$-efficieny of the hospitals. The results obtained by applying [$\mathrm{BM}_{\mathtt{CCR}}$], [$\mathrm{BM}^{\mathtt{Super}}$], and Model (3) are given in Table **4**. As can be seen, 10 of the hospitals are $\mathrm{BM_{CCR}}$-efficient (see column 2 in Table **4**). The 3rd Column of Table **4** reports the optimal value to Model (2), $\theta_{\mathtt{ND}}^{sup*}$. It can be seen that Model (2) is infeasible when hospitals $\mathbf{H}_5$, $\mathbf{H}_7$, and $\mathbf{H}_9$ are under evaluation.

Table 4: BM-efficiency and BM-super-efficiency scores.

Hospital	$\theta_{\mathtt{ND}}^{\mathtt{CCR}*}$	$\theta_{\mathtt{ND}}^{sup*}$	$\theta_{\mathtt{ND}}^{\mathbf{I}=\{2,4\}^*}$	Stability Regions $i \in \mathbf{D_I} = \{2,4\}$
H1	1.0000	1.2911	Infeasible	
H2	1.0000	1.1399	1.1399	$1 \leq \alpha_i\widetilde{\alpha}_i < 1.3399$
H3	0.6904	0.6904	0.6904	
H4	1.0000	1.0076	Infeasible	
H5	1.0000	Infeasible	Infeasible	
H6	0.8809	0.8809	0.8126	
H7	1.0000	Infeasible	Infeasible	
H8	0.5558	0.5558	0.5558	
H9	1.0000	infeasible	Infeasible	
H10	0.8630	0.8630	0.8630	
H11	0.9898	0.9898	0.9898	
H12	1.0000	1.0006	1.0006	$1 \leq \alpha_i\widetilde{\alpha}_i < 1.0006$
H13	0.9155	0.9155	0.9155	
H14	1.0000	1.0368	1.0368	$1 \leq \alpha_i\widetilde{\alpha}_i < 1.0368$
H15	1.0000	1.1034	1.1061	$1 \leq \alpha_i\widetilde{\alpha}_i < 1.1061$
H16	1.0000	1.2622	Infeasible	

Now, we apply our sensitivity analysis to some hospitals. Assume that **H2** is under evaluation. From column 4 in Table **4**, if $\mathbf{D_I} = \{2,4\}$ then $\theta_{\mathtt{ND}}^{\mathbf{I}^*} = 1.1339$. Using Theorem 2, if $1 \leq \alpha_i\widetilde{\alpha}_i < 1.1339,\ i \in \{2,4\}$, then **H2** remains as an extreme $\mathrm{BM_{CCR}}$-efficient point when the discretionary inputs $i \in \{2,4\}$ of **H2** change from $\mathbf{x}_2^{\mathbf{D_I}}$ to $\alpha_i\mathbf{x}_2^{\mathbf{D_I}}$ and the discretionary inputs $i \in \{2,4\}$ of other units change from $\mathbf{x}_j^{\mathbf{D_I}}$ to $\dfrac{\mathbf{x}_j^{\mathbf{D_I}}}{\widetilde{\alpha}_i}$. Now consider **H5** as the test DMU. It can be seen that both Models (2) and (3) are infeasible when **H5** is under evaluation. This means that any values $\alpha_i \geq 1$ and $\widetilde{\alpha}_i \geq 1$ of variation will not affect the BM-efficiency status of **H5** when Models (2) and (3) are applied, respectively.

6 Conclusions

The current paper develops a new super-efficiency DEA sensitivity analysis approach when some data are non-controllable. This development is important since in any realistic situation there may exist "exogenously fixed" or non-discretionary factors that are beyond the control of a DMU's management, which also need to be considered. The new sensitivity analysis approach simultaneously considers the data perturbations in all DMUs. The data perturbation in the test DMU can be different from that in the remaining DMUs, where the $\mathrm{BM_{CCR}}$-efficiency of the test DMU is deteriorating while the $\mathrm{BM_{CCR}}$-efficiencies of other DMUs are improving. Necessary and sufficient conditions for preserving a DMU's $\mathrm{BM_{CCR}}$-efficiency classification are developed when various data changes are applied to all DMUs. Because certain super-efficiency DEA models may be infeasible for some extreme-$\mathrm{BM_{CCR}}$-efficient DMUs, some direction for future research includes the study of super-efficiency and DEA sensitivity analysis for such DMUs with non-controllable factors.

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