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Lifting Elements in Coherent Quantales

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Abstract. An ideal I of a ring R is a lifting ideal if the idempotents of R can be lifted modulo I. A rich literature has been dedicated to lifting ideals. Recently, new algebraic and topological results on lifting ideals have been discovered. This paper aims to generalize some of these results to coherent quantales. We introduce the notion of lifting elements in a quantale and a lot of results about them are proven. Some properties and characterizations of a coherent quantale in which any element is a lifting element are obtained. The formulations and the proofs of our results use the transfer properties of reticulation, a construction that assigns to each coherent quantale a bounded distributive lattice. The abstract results on lifting elements can be applied to study some Boolean lifting properties in concrete algebraic structures: commutative rings, bounded distributive lattices, residuated lattices, MV-algebras, BL-algebras, abelian l-groups, some classes of universal algebras, etc.

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1 Introduction

The lifting idempotent property (LIP) is a condition that achieves important classes of rings and ideals. LIP appears whenever we study clean and exchange rings, local and semilocal rings, maximal rings, Gelfand rings, mp–rings, purified rings, etc. (See [1], [36], [40], [43], etc.).

A lifting ideal of a unital ring R is an ideal I such that the idempotents of R can be lifted modulo I: if f is an idempotent of the quotient ring R/I then $f = e/I$, for some idempotent e of R. We say that the ring R has LIP if any ideal of R is a lifting ideal. A remarkable Nicholson's theorem [36] asserts that a commutative ring R has LIP iff R is a clean ring iff R is an exchange ring.

All rings that appear in this paper are commutative.

Inspired by LIP, similar lifting properties were studied in various concrete algebraic structures: bounded distributive lattices [8], [35], residuated lattices [17], [18], abelian l-groups [27], orthomodular lattices [32], MV-algebras and BL-algebras [27], [33], pseudo BL- algebras [5], [6], etc.

On the other hand, two kinds of generalizations of these lifting properties were obtained last decade. Firstly, two lifting properties were introduced for congruences of a congruence modular algebra: Congruence Boolean Lifting Property [20],[22] and Factor Congruence Lifting Property [21]. Secondly, in [10] was defined a lifting property for the elements of a quantale as an abstraction of LIP and of other concrete lifting properties.

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Recently, new interesting results on lifting ring ideals were established in [40], [43]. The aim of this paper is to extend a part of these results to the framework of quantales. We obtain some algebraic and topological properties of the lifting elements in a coherent quantale A, as well as some characterizations of the lifting elements. The main tools for proving the results are the transfer properties of the reticulation $L(A)$ of A (see [16], [10]) and the isomorphism between the Boolean algebra $B(A)$ of the complemented elements in A and the Boolean algebra $Clop(Spec(A))$ of clopen subsets of prime spectrum $Spec(A)$.

Recall from [10] that the reticulation $L(A)$ is a bounded distributive lattice whose prime spectrum $Spec(L(A))$ is isomorphic with the prime spectrum $Spec(A)$ of A. Due to Hochster's theorem [28], for any coherent quantale A one can find a commutative ring R such that the reticulations $L(A)$ and $L(R)$ are isomorphic. Therefore, by using reticulation, one can transfer the properties of lifting ring ideals in R to lifting elements of the quantale A. We will follow a route consisting of two steps: firstly, from commutative rings to bounded distributive lattices, and secondly, from bounded distributive lattices to coherent quantales.

In Section 2 we recall some basic notions and results on the prime spectrum $Spec(A)$ of a quantale A and its Zariski topology, the radical elements, the Boolean center, etc. (see [41], [13], [34]). We prove that the Boolean algebras $B(A)$ and $Clop(Spec(A))$ are isomorphic. Section 3 presents some elementary transfer properties of reticulation.

Section 4 concerns the Boolean Lifting Property (abbreviated LP) in a coherent quantale A, a notion introduced in [10]. We define the lifting elements of A and, by using the reticulation $L(A)$, we prove several properties of them. We describe the clopen subsets of the maximal spectrum $Max(A)$ (endowed with the Zariski topology), then we characterize the situation whenever the Jacobson radical $r(A)$ of A is a lifting element.

The main result of Section 5 is a characterization theorem of lifting elements in a coherent quantale A. If A is the quantale $Id(R)$ of ideals in a commutative ring R we obtain as a particular case the characterization of the lifting ideals in R (see Theorem 3.18 of [43]). Applying our characterization theorem we prove that the join of a regular element and a lifting element is a lifting element. Another consequence is the following result of [10]: a coherent quantale A has LP if and only if A is B-normal.

2 Preliminaries on Quantales

In this section we shall recall some definitions and elementary results in quantale theory. The basic references on quantales are the books [41], [13], [37].

Let us fix a quantale $(A, \bigvee, \wedge, \cdot, 0, 1)$ and denote by $K(A)$ the set of its compact elements. In the usual way, the quantale $(A, \bigvee, \wedge, \cdot, 0, 1)$ is denoted by A. The quantale A is said to be *integral* if the structure $(A, \cdot, 1)$ is a monoid and *commutative*, if the multiplication $\cdot$ is commutative. Recall that a *frame* is a quantale in which the multiplication coincides with the meet (see [30], [39]). The quantale A is said to be *algebraic* if any element $a \in A$ has the form $a = \bigvee X$ for some subset X of $K(A)$. An algebraic quantale A is said to be *coherent* if 1 is a compact element and the set $K(A)$ of compact elements is closed under the multiplication. Coherent frames are defined in a similar way (see [30], [39]). The main example of coherent quantale (resp. coherent frame) is the set $Id(R)$ of ideals of a unital commutative ring R (resp. the set $Id(L)$ of ideals of a bounded distributive lattice L).

Throughout this paper, the quantales are assumed to be integral and commutative. We shall write ab instead of $a \cdot b$.

Each quantale A can be endowed with a residuation operation (= implication) $a \to b = \bigvee\{x | ax \leq b\}$ and with a negation operation $a^{\perp} = a^{\perp_A}$, defined by $a^{\perp} = a \to 0 = \bigvee\{x \in A | ax = 0\}$ (extending the terminology from ring theory [2], $a^{\perp}$ is also called the annihilator of a). Recall from [41] that for all $a, b, c \in A$ the following residuation rule holds: $a \leq b \to c$ if and only if $ab \leq c$, so $(A, \vee, \wedge, \cdot, \to, 0, 1)$ becomes a (commutative) residuated lattice. Particularly, it follows that for any $a \in A$, $a \leq b^{\perp}$ if and only if $ab = 0$. In this paper we shall use without mention some elementary arithmetical properties of residuated lattices [15].

An element $p < 1$ of a quantale A is *m-prime* if for all $a, b \in A$, $ab \leq p$ implies $a \leq p$ or $b \leq p$. The m-prime elements of a quantale extend the notions of prime ideals of a commutative ring and the prime ideals of a bounded distributive lattice. It is well–known that if A is an algebraic quantale, then $p < 1$ is m-prime if and only if for all $c, d \in K(A)$, $cd \leq p$ implies $c \leq p$ or $d \leq p$. Let us recall the following usual notations: $Spec(A)$ is the set of m-prime elements of A and $Max(A)$ is the set of maximal elements of A. If 1 is a compact element then for any $a < 1$ there exists $m \in Max(A)$ such that $a \leq m$. The same hypothesis $1 \in K(A)$ implies that $Max(A) \subseteq Spec(A)$. We remark that the set $Spec(R)$ of prime ideals in a commutative ring R is the prime spectrum of the quantale $Id(R)$ and the set of prime ideals in a bounded distributive lattice L is the prime spectrum of the frame $Id(L)$. Keeping the terminology, we say that $Spec(A)$ is the m–prime spectrum of the quantale A (abbreviated, $Spec(A)$ is the prime spectrum of A).

If R is a ring, then its Jacobson radical is the ideal $J(A) = \bigcap Max(A)$ (cf. [2]). This notion can be generalized to a quantale A: $r(A) = \bigwedge Max(A)$ is the Jacobson radical of A (cf. [10]).

The paper [14] emphasizes various abstract theories of m–prime elements and of corresponding spectra developed in the last decades.

Recall from [41] that the radical $\rho(a)$ of an element a of A is defined by $\rho(a) = \bigwedge\{p \in Spec(A) | a \leq p\}$ (it is clear that this notion generalizes the radical of an ideal in a commutative ring). If $a = \rho(a)$ then a is said to be a radical element of A. The set $R(A)$ of the radical elements of A is a frame [41], [42]. In [10] it is proven that $Spec(A) = Spec(R(A))$ and $Max(A) = Max(R(A))$. The quantale A is semiprime if the meet $\rho(0)$ of all m-prime elements in A is 0.

The following useful lemma extends to quantales a well–known result in ring theory [2].

Lemma 2.1. [34] *Let A be a coherent quantale and $a \in A$. Then the following hold:*

(1) $\rho(a) = \bigvee\{c \in K(A) | c^k \leq a$ for some integer $k \geq 1\}$;

(2) For any $c \in K(A), c \leq \rho(a)$ iff $c^k \leq a$ for some integer $k \geq 1$.

(3) A is semiprime if and only if for any integer $k \geq 1$, $c^k = 0$ implies $c = 0$.

Let A be a quantale such that $1 \in K(A)$, so $Spec(A)$ and $Max(A)$ are non–empty sets. For any $a \in A$, denote $D_A(a) = D(a) = \{p \in Spec(A) | a \not\leq p\}$ and $V_A(a) = V(a) = \{p \in Spec(A) | a \leq p\}$. For all $a, b \in A$ we have $D_A(a \vee b) = D_A(a) \cup D_A(b)$ and $D_A(a \wedge b) = D_A(ab) = D_A(a) \cap D_A(b)$; for any family $(a_i)_{i \in I}$ of A, $D_A(\bigvee_{i \in I} a_i) = \bigcup_{i \in I} D_A(a_i)$. Then $Spec(A)$ is endowed with a topology whose closed sets are $(V(a))_{a \in A}$ [41]. If the quantale A is algebraic then the family $(D(c))_{c \in K(A)}$ is a basis of open sets for this topology. The topology introduced here generalizes the Zariski topology (defined on the prime spectrum $Spec(R)$ of a commutative ring R [2]) and the Stone topology (defined on the prime spectrum $Spec(L)$ of a bounded distributive lattice L [3]). Then this topology will be also called the Zariski topology of $Spec(A)$ and the corresponding topological space will be also denoted by $Spec(A)$. According to [41], if A is a coherent quantale, then $Spec(A)$ is a *spectral space* in the sense of [28], [12].

Let L be a bounded distributive lattice. For any $x \in L$, denote $D(x) = \{P \in Spec(L) | x \notin P\}$ and $V(x) = \{P \in Spec(L) | x \in P\}$. The family $(D(x))_{x \in L}$ is a basis of open sets for the Stone topology on $Spec(L)$ (see [3],[7]).

Let R be a commutative ring. For any element $x \in R$, we shall denote $D(x) = \{P \in Spec(R) | x \notin P\}$ and $V(x) = \{P \in Spec(R) | x \in P\}$. The family $(D(x))_{x \in R}$ is a basis of open sets for the Zariski topology on $Spec(R)$ (see [2], [30]).

An element e of the quantale A is a complemented element if there exists $f \in A$ such that $e \vee f = 1$ and $e \wedge f = 0$. The set $B(A)$ of complemented elements of A is a Boolean algebra (cf. [7], [29]). Then $B(A)$ will be called the Boolean center of the quantale A. For any $e \in B(A)$ we denote by $\neg e$ the complement of e in $B(A)$.

Lemma 2.2. [29] *For all $a, b \in A$ and $e \in B(A)$ the following hold*

(1) If $a \in B(A)$ if and only if $a \vee a^{\perp} = 1$;

(2) $a \wedge e = ae$;

(3) $e \to a = e^{\perp} \vee a$;

(4) If $a \vee b = 1$ and $ab = 0$ then $a, b \in B(A)$;

(5) $(a \wedge b) \vee e = (a \vee e) \wedge (b \vee e)$;

(6) $\neg e = e^{\perp}$ and $e \to a = \neg e \vee a$.

Lemma 2.3. [10] *$B(A) \subseteq K(A)$.*

Lemma 2.4. *If $c \in B(A)$ then $D_A(c)$ is a clopen subset of $Spec(A)$.*

Proof. If $c \in B(A)$ then there exists $d \in B(A) \subseteq K(A)$ such that $c \vee d = 1$ and $cd = 0$. Therefore we have $D_A(c) \cup D_A(d) = D_A(c \vee d) = D_A(1) = Spec(A)$ and $D_A(c) \cap D_A(d) = D_A(cd) = D_A(0) = \emptyset$, so $D_A(c)$ is a clopen subset of $Spec(A)$. $\square$

If X is a topological space then it is well-known that the set $Clop(X)$ of clopen subsets of X is a Boolean algebra. By Lemma 2.2 one can take the map $D_A|_{B(A)} : B(A) \to Clop(Spec(A))$ defined by the assignment $c \mapsto D_A(c)$.

Recall from [31], [1] the following standard result in ring theory: if R is a commutative ring, then the Boolean algebra $B(R)$ of idempotents in R and the Boolean algebra $Clop(Spec(R))$ of clopen subsets of $Spec(R)$ are isomorphic. This lemma is intensively used in commutative algebra and algebraic geometry (see e.g. [31], [1], [30], [43]). The following proposition is a quantale version of the mentioned lemma.

Proposition 2.5. *The map $D_A|_{B(A)} : B(A) \to Clop(Spec(A))$ is a Boolean isomorphism.*

Proof. That $D_A|_{B(A)} : B(A) \to Clop(Spec(X))$ is a Boolean morphism is easy to check and the injectivity of $D_A|_{B(A)}$ follows by observing that for any $e \in B(A)$, $D_A(e) = \emptyset$ implies $e = 0$. By applying Lemma 21 of [10] it results in the surjectivity of $D_A|_{B(A)}$. $\square$

We shall use many times the previous proposition to prove some basic results of this paper.

3 Reticulation of a Coherent Quantale

Let A be a coherent quantale and $K(A)$ the set of its compact elements. On the set $K(A)$ we define the following equivalence relation: for all $c, d \in K(A)$, $c \equiv d$ iff $\rho(c) = \rho(d)$. The quotient set $L(A) = K(A)/\equiv$ is a bounded distributive lattice. For any $c \in K(A)$ denote by $c/\equiv$ its equivalence class. Consider the canonical surjection $\lambda_A : K(A) \to L(A)$ defined by $\lambda_A(c) = c/\equiv$, for any $c \in K(A)$. The pair $(L(A), \lambda_A : K(A) \to L(A))$ (or shortly $L(A)$) will be called the reticulation of A. In [10], [16] was given an axiomatic definition of the reticulation. We remark that reticulation $L(R)$ of a commutative ring R (defined in [30], [42]) is isomorphic with the reticulation $L(Id(R))$ of the quantale $Id(R)$.

For any $a \in A$ and $I \in Id(L(A))$ let us denote $a^* = \{\lambda_A(c) | c \in K(A), c \leq a\}$ and $I_* = \bigvee\{c \in K(A) | \lambda_A(c) \in I\}$. The assignments $a \mapsto a^*$ and $I \mapsto I_*$ define two order - preserving maps $(\cdot)^* : A \to Id(L(A))$ and $(\cdot)_* : Id(L(A)) \to A$. The following lemma collects the main properties of the maps $(\cdot)^*$ and $(\cdot)_*$.

Lemma 3.1. [10] *The following assertions hold*

(1) If $a \in A$ then a^* is an ideal of $L(A)$ and $a \leq (a^*)_*$;

(2) If $I \in Id(L(A))$ then $(I_*)^* = I$;

(3) If $p \in Spec(A)$ then $(p^*)_* = p$ and $p^* \in Spec(L(A))$;

(4) If $P \in Spec((L(A))$ then $P_* \in Spec(A)$;

(5) If $p \in K(A)$ then $c^* = (\lambda_A(c)]$;

(6) If $c \in K(A)$ and $I \in Id(L(A))$ then $c \leq I_*$ iff $\lambda_A(c) \in I$;

(7) If $a \in A$ and $I \in Id(L(A))$ then $\rho(a) = (a^*)_*$, $a^* = (\rho(a))^*$ and $\rho(I_*) = I_*$;

(8) If $c \in K(A)$ and $p \in Spec(A)$ then $c \leq p$ iff $\lambda_A(c) \in p^*$.

Lemma 3.2. [25] *The following assertions hold*

(1) If $(a_i)_{i\in I}$ is a family of elements in A then $(\bigvee_{i\in I} a_i)^* = \bigvee_{i\in I} a_i^*$;

(2) If $a, b \in A$ then $(ab)^* = (a \wedge b)^* = a^* \cap b^*$.

By Lemma 3.1 one can consider the functions $\delta_A : Spec(A) \to Spec(L(A))$ and $\epsilon_A : Spec(L(A)) \to Spec(A)$, defined by $\delta_A(p) = p^*$ and $\epsilon_A(I) = I_*$, for all $p \in Spec(A)$ and $I \in Spec(L(A))$.

Lemma 3.3. ([10] and [24]) *The functions δ_A and ϵ_A are homeomorphisms, inverse to one another.*

We also observe that δ_A and ϵ_A are also order-isomorphisms. In particular, for any m-prime element p of A, we have $p \in Max(A)$ if and only if $p^* \in Max(L(A))$. $Max(A)$ is a topological space as a subspace of $Spec(A)$; the family $(Max(A) \cap D_A(x))_{x\in K(A)}$ is a basis for this topology.
The functions δ_A and ϵ_A are order isomorphisms, therefore the functions $\delta_A\mid_{Max(A)}: Max(A) \to Max(L(A))$ and $\epsilon_A\mid_{Spec(L(A))}: Max(L(A)) \to Max(A)$ are order-isomorphisms.

Corollary 3.4. *The functions $\delta_A|_{Max(A)}$ and $\epsilon_A|_{Max(L(A))}$ are homeomorphisms, inverse to one another.*

The maximal spectrum $Spec(L)$ of a bounded distributive lattice L is a compact $T1$-space (cf. [30], p. 66). By applying the previous corollary it follows that the maximal spectrum $Max(A)$ of a coherent quantale A is a compact $T1$–space.

For a bounded distributive lattice L we shall denote by $B(L)$ the Boolean algebra of the complemented elements of L. It is well-known that $B(L)$ is isomorphic to the Boolean center $B(Id(L))$ of the frame $Id(L)$ (see [7], [30]).

Lemma 3.5. [24] *Assume $c \in K(A)$. Then $\lambda_A(c) \in B(L(A))$ if and only if there exists an integer $n \geq 1$ such that $c^n \in B(A)$.*

Corollary 3.6. [10] *The function $\lambda_A|_{B(A)} : B(A) \to B(L(A))$ is a Boolean isomorphism.*

4 Quantales with Boolean Lifting Property

Let A, B be two quantales. A function $u : A \to B$ is a morphism of quantales if it preserves the arbitrary joins and the multiplication (in this case we have $u(0) = 0$); f is an integral morphism if $f(1) = 1$. If $u(K(A)) \subseteq K(B)$ then we say that u preserves the compacts. If u is an integral quantale morphism that preserves the compacts then it is called a coherent quantale morphism. In a similar manner one defines the frame morphisms, integral frame morphisms, coherent frame morphism, etc. (cf. [30], [39]).

Let $f : R_1 \to R_2$ be a morphism of (unital) commutative rings. If I is an ideal of R_1 then I^e will denote the extension of I to R_2, i.e. the ideal $R_2 f(I)$ generated by $f(I)$ in R_2 (cf. [2], p. 9). Then the function $f^\bullet : Id(R_1) \to Id(R_2)$, defined by $f^\bullet(I) = I^e$, for any $I \in Id(R_1)$, is a coherent quantale morphism.

Let $f : L_1 \to L_2$ be a morphism of bounded distributive lattices. If I is an ideal of L_1 then $f^\bullet(I)$ is the lattice ideal $(f(I)]$ generated by $f(I)$ in L_2. Then the function $f^\bullet : Id(L_1) \to Id(L_2)$, defined by $I \mapsto f^\bullet(I)$, for any $I \in Id(L_1)$, is a coherent frame morphism.

The following result is a straightforward generalization of Lemma 3.8(1) of [11] (for the sake of completeness we shall present its proof).

Lemma 4.1. *If $u : A \to B$ is a surjective coherent morphism of quantales then $u(K(A)) = K(B)$.*

Proof. By the definition of a coherent quantale morphism we have $u(K(A)) \subseteq K(B)$. In order to establish the converse inclusion $K(B) \subseteq u(K(A))$ let us consider an arbitrary element d of $K(B)$ so there exists $x \in A$ such that $d = u(x)$. Since A is a coherent quantale we have $x = \bigvee_{i\in I} c_i$, for some family $(c_i)_{i\in I}$ of compact elements in A, therefore $d = u(\bigvee_{i\in I} c_i) = \bigvee_{i\in I} u(c_i)$. According to $d \in K(B)$ it follows that $d = \bigvee_{i\in J} u(c_i)$, for some finite subset J of I. Denoting $c = \bigvee_{i\in J} c_i$ we have $c \in K(A)$ and $d = u(c)$, so $d \in u(K(A))$. Thus the inclusion $K(B) \subseteq u(K(A))$ is proven. □

Proposition 4.2. [10] *Let $u : A \to B$ be a coherent quantale morphism. Then there exists a morphism of bounded distributive lattices $L(u) : L(A) \to L(B)$ such that the following diagram is commutative*

$$\begin{array}{ccc} K(A) & \xrightarrow{\;u|_{K(A)}\;} & K(B) \\ \Big\downarrow{\scriptstyle \lambda_A} & & \Big\downarrow{\scriptstyle \lambda_B} \\ L(A) & \xrightarrow{\;L(u)\;} & L(B) \end{array}$$

Let R be a commutative ring, $B(R)$ the Boolean algebra of its idempotents, $R(Id(R))$ the frame of radical ideals in A and $B(R(Id(R)))$ the Boolean algebra of complemented elements of $R(Id(R))$.

According to [40], [43], an ideal I of R is said to be a lifting ideal if the canonical ring morphism $R \to F/I$ lifts the idempotents: for any $y \in B(R/I)$ there exists $x \in B(R)$ such that $x/I = y$. We say that R satisfies the lifting idempotent property (LIP) if any ideal of R is a lifting ideal (see [36]). The two Boolean algebras $B(R)$ and $B(R(Id(R)))$ are isomorphic and the condition LIP can be expressed in terms of the frame $R(Id(R))$ (see [4]).

Similarly, following [9] we say that an ideal I of a bounded distributive lattice L satisfies the $Id-Boolean$ $Lifting\ Property$ (abbreviated $Id-BLP$) if the lattice morphism $L \to L/I$ lifts the complemented elements: for any $y \in B(L/I)$ there exists $x \in B(L)$ such that $x/I = y$. If any ideal of L satisfies $Id - BLP$ we say that L satisfies $Id - BLP$.

In what follows we shall generalize the previous lifting properties to the framework of coherent quantales. Firstly, we will develop some preliminary matters.

We fix a coherent quantale A. For any $a \in A$, consider the interval $[a) = \{x \in A | a \le x\}$ and for all $x, y \in [a)_A$, denote $x \cdot_a y = x \cdot y \vee a$. It is easy to see that $[a)_A$ is closed under the new multiplication $\cdot_a$.

Lemma 4.3. [10] *$([a)_A, \bigvee, \wedge, \cdot_a, a, 1)$ is a coherent quantale.*

Let x, y be two elements of the coherent quantale $([a)_A, \bigvee, \wedge, \cdot_a, a, 1)$. Denote by $\to^a$ the implication operation in $[a)_A$ and $x^{\perp_a}$ the annihilator of x in $[a)_A$. The negation of an element $x \in B([a)_A)$ will be denoted by $\neg^a(x)$.

Lemma 4.4. *Assume that x, y are two elements of the coherent quantale $[a)_A$. Then the following hold:*

(1) $x \to^{a} y = x \to y$;

(2) $x^{\perp_a} = x \to a$;

(3) $x \in B(([a)_A)$ *if and only if* $x \vee (x \to a) = 1$;

(4) *If* $e \in B(A)$ *and* $x \in B([e)_A)$ *then* $x \cdot \neg e \in B(A)$ *and* $x = x \cdot \neg e \vee e$;

(5) *If* $f \in B([a)_A)$ *and* $b \in A$ *then* $f \vee b \in B([a \vee b)_A)$.

Proof. The proof of (1) and (2) is easy and (3) follows by Lemma 2.2,(1) and (3). In order to prove (4), assume that $e \in B(A)$ and $x \in B([e)_A)$, so there exists $y \in B([e)_A)$ such that $x \vee y = 1$ and $x \cdot_e y = e$. This last equality implies $xy \leq e$, so $xy \cdot \neg e = 0$. From $x \cdot \neg e \leq x \cdot \neg e$ we get $x \leq \neg e \to x \cdot \neg e$, hence $x \leq x \cdot \neg e \vee e$ (by Lemma 2.2(3)). The converse inequality $x \cdot \neg e \vee e \leq x$ is obvious, so $x = x \cdot \neg e \vee e$. Similarly we have $y = y \cdot \neg e \vee e$.

We observe that $x \cdot \neg e \vee (y \cdot \neg e \vee e) = x \vee y = 1$ and $x \cdot \neg e \cdot (y \cdot \neg e \vee e) = xy \cdot \neg e = 0$, hence $x \cdot \neg e \in B(A)$.

To prove (5), let us assume that $f \in B([a)_A)$, so $f \vee (f \to a) = 1$ (cf. (3)). We remark that $f \to a \leq f \to (a \vee b)$, hence $f \to (a \vee b) = 1$. Thus the following equalities hold:

$(f \vee b) \vee [(f \vee b) \to (a \vee b)] = [f \to (a \vee b)] \wedge [b \to (a \vee b)] = f \to (a \vee b) = 1$.

According to (3), it follows that $f \vee b \in B([a \vee b)_A)$.

□

For an arbitrary $a \in A$, let us consider the function $u_a^A : A \to [a)_A$ defined by $u_a^A(x) = x \vee a$, for any $x \in A$.

Lemma 4.5. *For any* $a \in A$ *the following hold:*

(1) u_a^A *is an integral quantale morphism.*

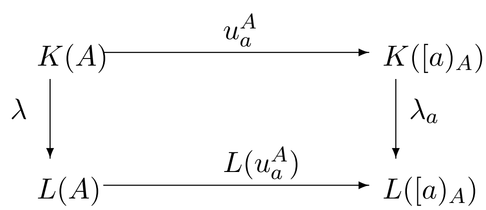
(2) *If* $c \in K(A)$ *then* $u_a^A(c) \in K([a)_A)$.

(3) $u_a^A(K(A)) = K([a)_A)$.

Proof. The first two assertions are proved in [10] and the third follows by Lemma 4.1.

□

Remark 4.6. *According to Lemma 4.5(2), the quantale morphism* u_a^A *preserves the compacts, so applying Proposition 4.2, the following diagram is commutative:*



where $\lambda = \lambda_A$ and $\lambda_a = \lambda_{[a)_A}$.

Proposition 4.7. [10] *For any* $a \in A$*, the bounded distributive lattices* $L([a)_A)$ *and* $L(A)/a^*$ *are isomorphic.*

By Lemma 4.5, u_a^A is a coherent quantale morphism, so we can consider the Boolean morphism $B(u_a^A) = u_a^A|_{B(A)} : B(A) \to B([a)_A)$. The following diagram is commutative:

$$\begin{array}{ccc} B(A) & \xrightarrow{B(u_a^A)} & B([a)_A) \\ \downarrow & & \downarrow \\ K(A) & \xrightarrow{u_a^A|_{K(A)}} & K([a)_A) \end{array}$$

where the vertical arrows are the inclusion maps (cf. Lemma 2.3).

Definition 4.8. [10] *An element $a \in A$ has the (Boolean)* lifting property (LP) *if the Boolean morphism $B(u_a^A)$ is surjective. The quantale A has LP if every element $a \in A$ has LP.*

If I is an ideal of a commutative ring R then it is easy to see that I is a lifting ideal if and only if I has LP in the quantale $Id(R)$. Keeping this terminology, if $a \in A$ has LP we shall say that a is a lifting element of the quantale A.

Similarly, an ideal I of a bounded distributive lattice L has $Id-BLP$ if and only if I has LP in the frame $Id(L)$.

Let A, A' be two coherent quantale and $u : A \to A'$ a coherent quantale morphism. We say that u lifts the complemented elements if for each $e' \in B(A')$ there exists $e \in B(A)$ such that $f(e) = e'$. Then an element a of a quantale A is a lifting element if and only if the quantale morphism $u_a^A : A \to [a)_A$ lifts the complemented elements. If I is an ideal in a commutative ring R and $p_I : A \to A/I$ is the associated ring morphism, then I is a lifting ideal in R if and only if the quantale morphism $p_I^\bullet : Id(R) \to Id(R/I)$ lifts the complemented elements. A similar result for lifting ideals in bounded distributive lattices is valid.

Let A be a coherent quantale. If a is an element of A such that $B([a)_A) = \{a, 1\}$ then it is clear that a is a lifting element. In particular, any minimal non–zero element of A is a lifting element.

Let p be an m–prime element of A. If $x \in B([p)_A)$ then $x \cdot_a \neg^p(x) = p$, so $x = p$ or $\neg^p(x) = p$. Since $Spec([p)_A) = Spec(A) \cap [p)_A$, it results that $p \in Spec([p)_A)$. Thus $B([a)_A) = \{p, 1\}$, hence the m–prime element p is a lifting element. Particularly, any maximal element of A is a lifting element.

Any complemented element e of A is a lifting element. Indeed, if $x \in B([e)_A)$ then, by using Lemma 4.4(4), it follows that $x \cdot \neg e \in B(A)$ and $x = x \cdot \neg e \vee e = u_e^A(x \cdot \neg e)$, i.e. e is a lifting element.

For any $a \in A$ denote $X_a = Spec([a)_A) = \{p \in Spec(A) | p \leq a\}$. We remark that $X_a = X_{\rho(a)}$ and $Clop(X_a) = Clop(X_{\rho(a)})$. According to Proposition 2.5, there exists a Boolean isomorphism $v_a : B([a)_A) \to Clop(X_a)$, defined by $v_a(e) = D_A(e) \cap [a)_A$, for any $e \in B([a)_A)$.

Theorem 4.9. *For any $a \in A$ the following are equivalent:*

(1) a has LP in the quantale A;

(2) $\rho(a)$ has LP in the quantale A;

(3) $\rho(a)$ has LP in the frame $R(A)$.

Proof. (1) $\Leftrightarrow$ (2) Recall that $v_{\rho(a)}$ and v_a are Boolean isomorphisms and $Clop(X_a) = Clop(X_{\rho(a)})$, so there exists a Boolean isomorphism $w : B([\rho(a)_A) \to B([a)_A)$ such that the following diagram is commutative:

$$\begin{array}{ccc} B[\rho(a))_A & \xrightarrow{v_{\rho(a)}} & Clop(X_{\rho(a)}) \\ w \downarrow & & \downarrow id \\ B([a)_A) & \xrightarrow{v_a} & Clop(X_a) \end{array}$$

Consider the Boolean morphisms $v_{\rho(a)} \circ u^A_{\rho(a)} : B(A) \to Clop(X_{\rho(a)})$ and $v_a \circ u^A_a : B(A) \to Clop(X_a)$.

An easy computation shows that for any $e \in B(A)$ the following equalities hold:
$(v_{\rho(a)} \circ u^A_{\rho(a)})(e) = \{p \in Spec(A) | e \vee \rho(a) \not\leq p\} \cap [\rho(a))_A$
$(v_a \circ u^A_a)(e) = \{p \in Spec(A) | e \vee a \not\leq p\} \cap [a)_A$.

We remark that for any $p \in Spec(A)$ we have $a \leq p$ iff $\rho(a) \leq p$ and $e \vee a \not\leq p$ iff $e \vee \rho(a) \not\leq p$, therefore $(v_{\rho(a)} \circ u^A_{\rho(a)})(e) = (v_a \circ u^A_a)(e)$. Since $Clop(X_{\rho(a)}) = Clop(X_a)$ it follows that $v_{\rho(a)} \circ u^A_{\rho(a)} = v_a \circ u^A_a$. But $v_{\rho(a)}$ and v_a are Boolean isomorphisms, hence the following diagram is commutative:

$$\begin{array}{ccc} B(A) & \xrightarrow{B(u^A_{\rho(a)})} & B([\rho(a))_A) \\ & {\scriptstyle B(u^A_a)} \searrow \quad \nearrow {\scriptstyle w} & \\ & B([a)_A) & \end{array}$$

Recall that w is a Boolean isomorphism so $B(u^A_{\rho(a)})$ is surjective iff $B(u^A_a)$ is surjective, therefore $\rho(a)$ has LP iff a has LP.

(2) $\Leftrightarrow$ (3) According to Lemma 19 of [10], the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\rho} & R(A) \\ {\scriptstyle u^A_{\rho(a)}} \downarrow & & \downarrow {\scriptstyle u^{R(A)}_{\rho(\rho(a))}} \\ [\rho(a))_A & \xrightarrow{\rho_{\rho(a)}} & [\rho(\rho(a)))_{R(A)} \end{array}$$

Since $\rho(\rho(a)) = \rho(a)$ we obtain the following commutative diagram in the category of Boolean algebras:

$$\begin{array}{ccc} B(A) & \longrightarrow & B(R(A)) \\ {\scriptstyle B(u^A_{\rho(a)})} \downarrow & & \downarrow {\scriptstyle B(u^{R(A)}_{\rho(a)})} \\ B([\rho(a))_A) & \longrightarrow & B([\rho(a))_{R(A)}) \end{array}$$

where the horizontal arrows are Boolean isomorphisms (in virtue of Proposition 6 of [10]).

From the previous commutative diagram we get that $B(u^A_{\rho(a)})$ is surjective iff $B(u^{R(A)}_{\rho(a)})$ is surjective, so the equivalence of (2) and (3) follows.

$\square$

Corollary 4.10. [10] *Let a be an element of the coherent quantale A. Then a has LP if and only if the ideal a^* of the lattice $L(A)$ has $Id - BLP$.*

Proof. Recall from Lemma 3.1 (7) that $a^* = (\rho(a))^*$. We know from Corollary 2 of [10] that the frames $R(A)$ and $Id(L(R))$ are isomorphic, hence the following properties are equivalent:

- $\rho(a)$ has LP in the frame $R(A)$;
- $(\rho(a))^*$ has LP in the frame $Id(L(R))$;
- a^* has LP in the frame $Id(L(R))$;
- the ideal a^* of the lattice $L(A)$ has $Id - BLP$.

By applying the previous theorem it follows that a has LP if and only if the ideal a^* of the lattice $L(A)$ has $Id - BLP$.

$\square$

Corollary 4.11. [10] *A quantale A has LP if and only if the reticulation $L(A)$ has $Id-BLP$.*

Corollary 4.12. *Let I be an ideal of the reticulation $L(A)$. Then I has $Id-BLP$ if and only if I_* has LP.*

Proof. We know that $I = (I_*)^*$ (cf. Lemma 3.1(2)), hence, by two applications of Corollary 4.10 we obtain: I has $Id-BLP$ iff $(I_*)^*$ has $Id-BLP$ iff I^* has LP. □

Corollary 4.13. *Let a and b be two elements of A such that $\rho(a) = \rho(b)$. Then a has LP if and only if b has LP.*

Proof. By Theorem 4.9, the following equivalences hold: a has LP iff $\rho(a)$ has LP iff $\rho(b)$ has LP iff b has LP. □

Corollary 4.14. *If a is an element of A such that $a \leq \rho(0)$ then a has LP.*

Proof. If $a \leq \rho(0)$ then $\rho(a) = \rho(0)$. It is obvious that 0 has LP. By applying Corollary 4.13 it follows that a has LP. □

In particular, from Corollary 4.14 it follows that $\rho(0)$ is a lifting element.

Following [10], we say that a quantale A is hyperarchimedean if for any $c \in K(A)$ there exists an integer $n \geq 1$ such that $c^n \in B(A)$.

Corollary 4.15. *If the quantale A is hyperarchimedean then any element $a \in A$ has LP.*

Proof. Let a be an element of the hyperarchimedean quantale A. By Theorem 1 of [10], the reticulation $L(A)$ of A is a Boolean algebra. It is straightforward to see that any ideal of a Boolean algebra has $Id-BLP$ (see [9]). Thus the ideal a^* of $L(A)$ has $Id-BLP$, so, by applying Corollary 4.10, it follows that a has LP. □

Remark 4.16. *The equivalence of assertions (1) and (2) of Theorem 4.9 is a quantale generalization of Corollary 3.2 of [43]. Among the consequences of this theorem we mention an important result of [10]: Corollary 4.11is exactly Theorem 2 of [10]. We remark that Theorem 4.9 can be obtained as a corollary of Theorem 2 of [10]. We shall give here a short proof.*

Let a be an element of the quantale A and $L(A)$ the reticulation of A. We know that $a^* = (\rho(a))^*$ (cf. Lemma 3.1(7)) so by a double application of Theorem 2 of [10] the following equivalences hold: a has LP iff a^* has $Id-BLP$ iff $(\rho(a))^*$ has $Id-BLP$ iff $\rho(a)$ has LP.

Theorem 4.17. *Let a and b two elements of the quantale A such that $a \leq b$ and $Max([a)_A) = Max([b)_A)$. If b has LP then a has LP.*

Proof. Firstly we observe that $Max([a)_A) = Max(A) \cap [a)_A$ and $Max([b)_A) = Max(A) \cap [b)_A$. Let us consider an element $x \in B(([a)_A))$, so $x \vee (x \to a) = 1$ (by Lemma 4.4(3)). From $a \leq b$ we get $x \to a \leq x \to b$, hence $1 = x \vee (x \to a) \leq x \vee (x \to b)$. Thus $x \vee b \vee ((x \vee b) \to b) = x \vee b \vee ((x \to b) \wedge (b \to b)) = x \vee (x \to b) = 1$, so $x \vee b \in B([b)_A)$ (cf. Lemma 4.4(3)).

By hypothesis, b has LP, so there exists a complemented element e of A such that $e \vee b = u_b^A(e) = x \vee b$. We shall prove that $V_A(e \vee a) = V_A(x)$.

Firstly, we shall establish the inclusion $V_A(e \vee a) \subseteq V_A(x)$. Assume that $p \in V_A(e \vee a)$, i.e. p is an m–prime element of A such that $e \vee a \leq p$. Consider a maximal element m of A such that $p \leq m$, so $e \leq m$ and $a \leq m$. According to the hypothesis $Max([a)_A) = Max([b)_A)$, from $a \leq m$ we obtain $b \leq m$, hence $x \vee b = e \vee b \leq m$.

Let us assume that $x \not\leq p$. We observe that $p \in Spec([a)_A)$ (because $p \in Spec(A)$ and $a \leq p$), so $x \to a = \neg^a(x) \leq p \leq m$. From $x \in B([a)_A)$, $x \leq m$ and $x \to a \leq m$ we obtain $1 = x \vee (x \to a) \leq m$. This contradicts $m \in Max(A)$, so $x \leq p$, i.e. $p \in V_A(x)$.

In order to prove the converse inclusion $V_A(x) \subseteq V_A(e \vee a)$, assume that $p \in V_A(x)$, so $p \in Spec(A)$ and $x \leq p$. Consider a maximal element m of A such that $p \leq m$, so $a \leq x \leq p \leq m$. By using the hypothesis $Max([a)_A) = Max([b)_A)$, we get $b \leq m$, hence $e \vee b = x \vee b \leq m$.

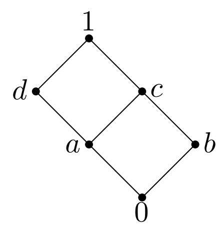
Let us assume that $e \vee a \not\leq p$, so $e \not\leq p$ (because $a \leq p$ and $p \in Spec(A)$). Thus $\neg e \leq p \leq m$, so $1 = e \vee \neg a \leq m$, contradicting $m \in Max(A)$. Then $e \vee a \leq p$, hence $p \in V_A(e \vee a)$.

One remark that $V_{[a)_A}(e \vee a) = V_A(e \vee a) = V_A(x) = V_{[a)_A}(x)$, therefore $D_{[a)_A}(e \vee a) = D_{[a)_A}(x)$. Since $e \vee a, x \in B([a)_A)$ one can apply Proposition 2.5, so from $D_{[a)_A}(e \vee b) = D_{[a)_A}(x)$ we get $u_a^A(e) = e \vee a = x$. Therefore a has LP.

□

One can ask if the element $r(A)$ has or doesn't have LP. The following two examples show that both situations are possible.

Example 4.18. Let us consider the quantale structure defined on the set $A = \{0, a, b, c, d, 1\}$ by the following diagram and table (cf. [8]):

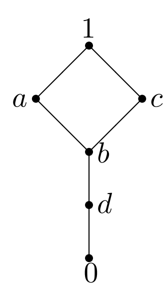


$\odot$	0	1	a	b	c	d
0	0	0	0	0	0	0
1	0	1	a	b	c	d
a	0	a	0	0	0	a
b	0	b	0	b	b	0
c	0	c	0	b	b	a
d	0	d	a	0	a	d

The maximal spectrum of the quantale A is $Max(A) = \{c, d\}$, so $r(A) = c \wedge d = a$. The Boolean center of A is $B(A) = \{0, b, d, 1\}$. We observe that $[r(A))_A = [a)_A = \{a, c, d, 1\}$ is a Boolean algebra, hence $B([r(A))_A) = B([a)_A) = \{a, c, d, 1\}$.

Now we consider the Boolean morphism $B(u^A_{r(A)}) : B(A) \to B([r(A))_A)$ (in fact, $B(u_a^A) : \{0, b, d, 1\} \to \{a, c, d, 1\}$). An easy computation gives $u_a^A(0) = a, u_a^A(b) = a \vee b = c, u_a^A(d) = a \vee d = d, u_a^A(1) = 1$, so $B(u^A_{r(A)})$ is a surjective map. Then $r(A)$ has LP.

Example 4.19. Let us consider the quantale structure defined on the set $A = \{0, a, b, c, d, 1\}$ by the following diagram and table (cf. [8]):



$\odot$	0	1	a	b	c	d
0	0	0	0	0	0	0
1	0	1	a	b	c	d
a	0	a	a	b	b	0
b	0	b	b	b	b	0
c	0	c	b	b	c	d
d	0	d	0	0	d	0

In this case, we have $Max(A) = \{a, c\}$, $r(A) = a \wedge c = b$ and $[r(A))_A = [b)_A = \{b, a, c, 1\}$ is a Boolean algebra. We observe that $B(A) = \{0, 1\}$ and $B([b)_A) = [b)_A = \{b, a, , c, 1\}$. It is clear that $B(u_b^A) : B(A) \to B([b)_A)$ is not surjective, so $r(A) = b$ doesn't have LP.

Corollary 4.20. *Assume that a is an element of A such that $a \leq r(A)$. If $r(A)$ has LP then a has LP.*

Proof. If $a \leq r(A)$ then $Max([a)_A) = Max(A) = Max([r(A))_A)$. By applying Theorem 4.17, it follows that a has LP. $\square$

Theorem 4.21. *For any subset U of $Max(A)$ the following are equivalent:*

(1) U is a clopen subset of $Max(A)$;

(2) There exist $c, d \in K(A)$ such that $c \vee d = 1$, $cd \leq r(A)$ and $U = Max(A) \cap D_A(c)$.

Proof. (1) $\Rightarrow$ (2) Assume that U is a clopen subset of $Max(A)$. We know that $Max(A)$ is compact (cf. Corollary 3.4, $Max(A)$ and $Max(L(A))$ are homeomorphic spaces and by [30], p.66, the maximal spectrum $Max(L(A))$ of the bounded distributive lattice $L(A)$ is compact) and $U, Max(A) - U$ are closed subsets of $Max(A)$, so they are compact. Therefore there exist two positive integers n, m and the compact elements $c_1, \cdots, c_n$, $d_1, \cdots, d_m$ such that $U = \bigcup_{i=1}^{n}(Max(A) \cap D_A(c_i))$ and $Max(A) - U = \bigcup_{j=1}^{m}(Max(A) \cap D_A(d_j))$.

We remark that for all $i = 1, \cdots, n$ and $j = 1, \cdots, m$ the following equalities hold: $Max(A) \cap D_A(c_i d_j) = Max(A) \cap D_A(c_i) \cap D_A(d_j) = \emptyset$. Then for each $m \in Max(A)$ we have $m \notin D_A(c_i d_j)$, hence $c_i d_j \leq m$, therefore $c_i d_j \leq r(A)$.

Denoting $c = \bigvee_{i=1}^{n} c_i$ and $d = \bigvee_{j=1}^{m} d_j$ it results that $U = Max(A) \cap D_A(c)$ and $Max(A) - U = Max(A) \cap D_A(d)$. Then one obtains the equalities $Max(A) = Max(A) \cap (D_A(c) \cup D_A(d)) = Max_A(A) \cap D_A(c \vee d)$. Since $c_i d_j \leq r(A)$ for all i and j it follows that

$cd = \bigvee \{c_i d_j | i = 1, \cdots, n \, ; \, j = 1, \cdots, m\} \leq r(A)$.

Assume by absurdum that $c \vee d < 1$, so $c \vee d \leq m$ for some maximal element m of A. Then $m \notin D_A(c \vee d)$, contradicting $Max(A) = Max(A) \cap D_A(c \vee d)$. We conclude that $c \vee d = 1$.

(2) $\Rightarrow$ (1) Assume that there exist $c, d \in K(A)$ such that $c \vee d = 1$, $cd \leq r(A)$ and $U = Max(A) \cap D_A(c)$. We shall prove the equality $Max(A) \cap D_A(c) = Max(A) \cap V_A(d)$.

Let m be an element of $Max(A) \cap D_A(c)$, hence $c \not\leq m$. Since m is m - prime and $cd \leq r(A) \leq m$ we get $d \leq m$, so $m \in Max(A) \cap V_A(d)$. Thus we obtain the inclusion $Max(A) \cap D_A(c) \subseteq Max(A) \cap V_A(d)$.

In order to prove the converse inclusion $Max(A) \cap V_A(d) \subseteq Max(A) \cap D_A(c)$, let us assume that $m \in Max(A) \cap V_A(d)$, so $d \leq m$. If $c \leq m$ then $1 = c \vee d \leq m$, contradicting that m is a maximal element. It follows that $c \not\leq m$, hence $m \in Max(A) \cap V_A(d)$. Thus the inclusion $Max(A) \cap V_A(d) \subseteq Max(A) \cap D_A(c)$ is established, therefore $Max(A) \cap D_A(c) = Max(A) \cap V_A(d)$. This equality shows that U is a clopen subset of $Max(A)$. $\square$

Corollary 4.22. *If $x \in B([r(A))_A)$ then $Max(A) \cap D_A(x)$ is a clopen subset of $Max(A)$.*

Proof. If $x \in B([r(A))_A)$ then there exists $y \in B([r(A))_A)$ such that $x \vee y = 1$ and $x \cdot_{r(A)} y = r(A)$. By applying the previous theorem to the elements x and y of the quantale $[r(A))_A$, it follows that $Max(A) \cap D_{[r(A))_A}(x)$ is a clopen subset of $Max([r(A))_A)$. We remark that $Max(A) = Max([r(A))_A)$, hence $Max(A) \cap D_A(x) = Max([r(A))_A) \cap D_{[r(A))_A}(x)$, so $Max(A) \cap D_A(x)$ is a clopen subset of $Max(A)$. $\square$

According to this corollary, the assignment $x \mapsto Max(A) \cap D_A(x)$ defines a map $f : B([r(A))_A) \to Clop(Max(A))$.

Proposition 4.23. *The map $f : B([r(A))_A) \to Clop(Max(A))$ is a Boolean isomorphism.*

Proof. Firstly, we prove that f is a Boolean morphism. Assume that x and y are two elements of the Boolean algebra $B([r(a))_A)$. We remark that the following equalities hold: $f(x \vee y) = Max(A) \cap D_A(x \vee y) = (Max(A) \cap (D_A(x) \cup D_A(y)) = Max(A) \cap D_A(x)) \cup (Max(A) \cap D_A(y)) = f(x) \cup f(y)$. Similarly, we have $f(x \wedge y) = f(x) \cap f(y)$. It is clear that $f(r(A)) = Max(A) \cap D_A(r(A)) = \emptyset$ and $f(1) = Max(A)$, hence f is a Boolean morphism.

If $f(x) = \emptyset$, then $Max(A) \cap D_A(x) = \emptyset$, therefore $m \notin D_A(x)$, for any maximal element m of A. It follows that $x \leq m$, for any $m \in Max(A)$, i.e. $x \leq r(A)$. Since $r(A) \leq x$, we get $x = r(A)$. This shows that f is an injective Boolean morphism.

Assume now that U is a clopen subset of $Max(A)$. In accordance with Theorem 4.21 there exist $c, d \in K(A)$ such that $c \vee d = 1$, $cd \leq r(A)$ and $U = Max(A) \cap D_A(c)$.

By using $c \vee d = 1$ and $cd \leq r(A)$ the following equalities hold:

$(c \vee r(A)) \vee (d \vee r(A)) = 1$

$(c \wedge r(A)) \cdot_{r(A)} (d \wedge r(A)) = cd \vee r(A) = r(A)$,

hence $e = c \vee r(A) \in B([r(A))_A)$ (by Lemma 2.2(4)). We have observed that $Max(A) \cap D_A(r(A)) = \emptyset$, therefore we get

$f(e) = Max(A) \cap D_A(e) = (Max(A) \cap D_A(c)) \cup ((Max(A) \cap D_A(r(A)) = Max(A) \cap D_A(c) = U$.

Thus f is a surjective map, so it is a Boolean isomorphism. □

Lemma 4.24. *The Boolean morphism $B(u^A_{r(A)}) : B(A) \to B([r(A))_A)$ is injective.*

Proof. Assume that $e \in B(A)$ and $B(u^A_{r(A)})(e) = r(A)$. Thus $e \vee r(A) = r(A)$, hence $e \leq r(A)$. If $e \neq 0$ then $\neg e \neq 1$, so $\neg e \leq m$, for some maximal element m of A. On the other hand we have $e \leq r(A) \leq m$, hence $1 = e \vee \neg e \leq m$, contradicting that $m \in Max(A)$. It follows that $e = 0$, so $B(u^A_{r(A)})$ is injective. □

Now consider the map $g : B(A) \to Clop(Max(A))$ defined by $g(e) = Max(A) \cap D_A(e)$, for any $e \in B(A)$. It is straightforward that g is a Boolean morphism.

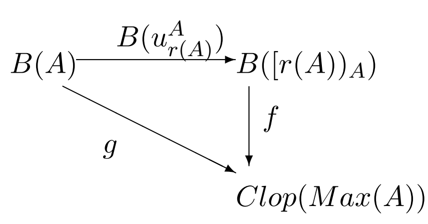
Recall that in general, $r(a)$ does not have LP (see Example 4.19). The following result characterizes the situation whenever $r(A)$ has LP.

Theorem 4.25. *The following properties are equivalent:*

(1) $r(A)$ has LP;

(2) $g : B(A) \to Clop(Max(A))$ is a Boolean isomorphism.

Proof. For any $e \in B(A)$ the following equalities hold: $f(B(u^A_{r(A)})(e)) = f(e \vee r(A)) = Max(A) \cap D_A(e \vee r(A)) = Max(A) \cap (D_A(e) \cup D_A(r(A))) = Max(A) \cap D_A(e) = g(e)$ (because $Max(A) \cap D_A(r(A)) = \emptyset$). It follows that the following diagram is commutative in the category of Boolean algebras:



Recall that f is a Boolean isomorphism (cf. Proposition 4.23) and $B(u^A_{r(A)})$ is injective (cf. Lemma 4.24), therefore, according to the previous commutative diagram it results that g is injective. Then the following equivalences hold: $r(A)$ has LP iff $B(u^A_{r(A)})$ is surjective iff g is surjective iff g is a Boolean isomorphism. □

5 A Characterization Theorem and Its Consequences

We start this section by recalling the following characterization theorem of lifting ideals in a commutative ring R (see Theorems 1.5 and 3.2 of [40] and Theorem 3.18 of [43]).

Theorem 5.1. [43] *If I is an ideal of the commutative ring R then the following are equivalent*

(1) I is a lifting ideal of R;

(2) If J_1, J_2 are two coprime ideals of R such that $J_1J_2 \subseteq I$ then there exists an idempotent e of R such that $e \in I + J_1$ and $\neg e \in I + J_2$;

(3) If J_1, J_2 are two coprime ideals of R such that $J_1J_2 = I$ then there exists an idempotent e of R such that $e \in J_1$ and $\neg e \in J_2$;

(4) If M is a maximal ideal of R and $M^\diamond$ is the ideal of R generated by $\{f \in I | f = f^2\}$ then the quotient ring $R/(I + M^\diamond)$ has no nontrivial idempotents.

We remark that the property (4) of the previous theorem says that the Boolean algebra $B(R/(I + M^\diamond))$ of idempotents of $R/(I + M^\diamond)$ is isomorphic to $L_2 = \{0, 1\}$ (we include the case whenever L_2 is a trivial Boolean algebra).

Theorem 5.1 and some of its consequences were extended in [26] to the lifting congruences in a semidegenerate congruence modular algebra (see Theorem 6.3 of [26]).

In this section we shall prove a new extension of Theorem 5.1. We shall characterize the lifting elements of a coherent quantale. Then we shall present some consequences of Theorem 5.1.

Let A be a coherent quantale. For any element $a \in A$ we shall denote $a^\diamond = \bigvee\{e \in B(A) | e \leq a\}$ (this new element in a quantale was firstly defined in [24]). We remark that in the particular case when A is the quantale $Id(R)$ of the ideals in a commutative ring R we find the ring "diamond construction".

If R is a commutative ring and $I \in Id(R)$ then it is easy to see that $I^\diamond$ is equal to the ideal of R generated by $I \cap B(R)$; if L is a bounded distributive lattice and $I \in Id(L)$ then $I^\diamond$ is equal to the ideal of L generated by $I \cap B(R)$ (i.e. $I^\diamond = [I \cap B(L))$).

Recall that two elements a and b of the quantale A are coprime if $a \vee b = 1$. If a and b are coprime then $ab = a \wedge b$ (see Lemma 2(1) of [10]). For any set Ω we denote by $|\Omega|$ its cardinal number.

The following lemma emphasizes the way in which the reticulation preserves the diamond construction.

Lemma 5.2. *If $a \in A$ then $a^{*\diamond} = a^{\diamond *}$.*

Proof. Firstly, we observe that $a^{*\diamond}$ is the ideal $[a^* \cap B(L(A)))$ of the lattice $L(A)$ generated by $a^* \cap B(L(A))$ and $a^{\diamond *} = \{\lambda_A(c) | c \in K(A), c \leq a^\diamond\}$.

For proving the inclusion $a^{\diamond *} \subseteq a^{*\diamond}$, let x be an element of $a^{\diamond *}$, so $x = \lambda_A(c)$ for some $c \in K(A)$ such that $c \leq a^\diamond$. Since $a^\diamond = \bigvee\{e \in B(A) | e \leq a\}$ and c is compact it follows that there exists $e \in B(A)$ such that $c \leq e \leq a$. Recall that the map λ_A is isotone and preserves the complemented elements. Then $\lambda_A(c) \leq \lambda_A(e)$ and $\lambda_A(e) \in B(L(A)) \cap a^*$, so $x = \lambda_A(c) \in [B(L(A)) \cap a^*) = a^{*\diamond}$. Thus we obtain the inclusion $a^{\diamond *} \subseteq a^{*\diamond}$.

In order to prove the converse inclusion $a^{*\diamond} \subseteq a^{\diamond *}$ it suffices to check that $B(L(A)) \cap a^* \subseteq a^{\diamond *}$. If $x \in B(L(A)) \cap a^*$ then $x = \lambda_A(c)$ for some compact element c of A such that $c \leq a$. By using Lemma 3.5, from $\lambda_A(c) = x \in B(L(A))$ we get $c^n \in B(A)$, for some integer $n \geq 1$. From $c^n \leq a$ and $c^n \in B(A)$ we obtain $c^n \leq a^\diamond$, hence $x = \lambda_A(c) = \lambda_A(c^n) \in a^{\diamond *}$, therefore $a^{*\diamond} \subseteq a^{\diamond *}$.

□

Recall a famous theorem of Hochster [28] (see also [12]): if L is a bounded distributive lattice then there exists a commutative ring R such that the reticulation $L(R)$ of R is isomorphic with L.

Let A be a coherent quantale and $L(A)$ its reticulation. By applying the Hochster theorem one can find a commutative ring R such the reticulations $L(A)$ and $L(R)$ are isomorphic lattices (we shall identify $L(A)$ and $L(R)$). For any element a of A, we know that a^* is an ideal of the bounded distributive lattice $L(A) = L(R)$, so there exists a ring ideal I of R such that $a^* = I^*$ (cf. Lemma 3.1(2)).

This previous construction is a bridge between rings and coherent quantales: by applying the transfer properties of reticulations some results of ring theory can be exported to quantale theory and viceversa. The following propositions are the first illustration of this thesis.

Proposition 5.3. *Keeping the previous notations the following are equivalent*

(1) a is a lifting element of A;

(2) The lattice ideal $a^* = I^*$ has $Id - BLP$;

(3) The ring ideal I of R is a lifting ideal.

Proof. By applying twice Corollary 4.10. □

Proposition 5.4. *Keeping the previous notations the following are equivalent*

(1) The quantale A has LP;

(2) The isomorphic lattices $L(A)$ and $L(R)$ have $Id - BLP$;

(3) The ring R has the Lifting Idempotent Property.

Proof. By applying twice Corollary 4.11. □

Let a be an arbitrary element of the coherent quantale A and m a maximal element of A. Then a^* is an ideal of the lattice $L(A)$ and m^* a maximal ideal of $L(A)$. By using Corollary 3.4 one can find an ideal I of R and a maximal ideal M of the ring R such that $a^* = I^*$ and $m^* = M^*$.

Lemma 5.5. *Keeping the previous notations the following equality holds:*
$|B([a \vee m^\diamond)_A)| = |B(R/(I \vee M^\diamond))|$.

Proof. In accordance with Lemma 3.2(1) we have $(a \vee m^\diamond)^* = a^* \vee m^{\diamond *}$, so by using Proposition 4.7 we get the following isomorphisms in the category of bounded distributive lattices:

$L([a \vee m^\diamond)_A) = L(A)/(a \vee m^\diamond)^* = L(A)/(a^* \vee m^{\diamond *})$

Thus, according to Corollary 3.6, we obtain the following sequence of Boolean isomorphisms:

$B([a \vee m^\diamond)_A) \simeq B(L([a \vee m^\diamond)_A)) \simeq B(L(A)/(a \vee m^\diamond)^*) \simeq B(L(A)/(a^* \vee m^{\diamond *})$.

In a similar way we obtain the Boolean isomorphism: $B(R/(I \vee M^\diamond)) \simeq B(L(R)/(I^* \vee M^{\diamond *}))$.

By Lemma 5.2 and $m^* = M^*$ we have $m^{\diamond *} = m^{*\diamond} = M^{*\diamond} = M^{\diamond *}$, hence the Boolean algebras $B([a \vee m^\diamond)_A)$ and $B(L(R)/(I^* \vee M^{\diamond *}))$ are isomorphic, so their cardinal numbers are equal.
□

Now we are ready to state and prove the following characterization theorem of lifting elements in a coherent quantale.

Theorem 5.6. *Let A be a coherent quantale. For any $a \in A$ the following are equivalent*

(1) a has LP;

(2) If c, d are two coprime elements of A such that $cd \leq a$ then there exists $e \in B(A)$ such that $e \leq a \vee c$ and $\neg e \leq a \vee d$;

(3) If c, d are two coprime elements of A such that $cd = a$ then there exists $e \in B(A)$ such that $e \leq c$ and $\neg e \leq d$;

(4) If m is a maximal element of A then $|B([a \vee m^\diamond)_A)| \leq 2$.

Proof. (1) $\Rightarrow$ (2) Assume that a has LP and c, d are two coprime elements of A such that $cd \leq a$. Then $(a \vee c) \vee (a \vee d) = 1$ and $(a \vee c)(a \vee d) = a \vee cd = a$, so $a \vee c$ and $a \vee d$ are two elements of the Boolean algebra $B([a)_A)$ such that $\neg^a(a \vee c) = a \vee d$.

According to the hypothesis that a has LP there exists $e \in B(A)$ such that $a \vee e = B(u_a^A)(e) = a \vee c$. Since $B(u_a^A)$ is a Boolean morphism it follows that $a \vee d = \neg^a(a \vee c) = \neg^a(B(u_a^A)(e)) = B(u_a^A)(\neg e) = a \vee \neg e$. We conclude that $e \leq a \vee c$ and $\neg e \leq a \vee d$.

(2) $\Rightarrow$ (3) Assume that c, d are two coprime elements of A such that $cd = a$. By the hypothesis (2), there exists $e \in B(A)$ such that $e \leq a \vee c$ and $\neg e \leq a \vee d$. Remark that $c = c(c \vee d) = cd \vee c^2 = a \vee c^2$, hence $a \leq c$. Similarly, we have $a \leq d$, so $e \leq a \vee c = c$ and $\neg e \leq a \vee d = d$, hence the assertion (3) follows.

(3) $\Rightarrow$ (1) Let us consider that $x \in B([a)_A)$ so there exists $y \in B([a)_A)$ such that $y = \neg^a(x)$, hence $x \vee y = 1$ (i.e. x, y are coprime) and $xy \vee a = x \cdot_a y = a$, so $xy \leq a$. Since $a \leq x$, $a \leq y$ and x, y are coprime we have $a \leq x \wedge y = xy$, so $xy = a$. Then one can apply the hypothesis (3), so there exists $e \in B(A)$ such that $e \leq x$ and $\neg e \leq y$.

Recall that u_a^A is a quantale morphism and $u_a^A|_{B(A)} = B(u_a^A)$ is a Boolean morphism. Thus from $e \leq x$ we get $u_a^A(e) \leq u_a^A(x) = x$. On the other hand, $\neg e \leq y$ implies $\neg^a(u_a^A(e)) = u_a^A(\neg e) \leq u_a^A(y) = y$, hence $x = \neg^a(y) \leq u_a^A(e)$. It follows that $x = u_a^A(e) = B(u_a^A)(e)$, so the map $B(u_a^A) : B(A) \to B([a)_A)$ is surjective. Then a has LP.

(1) $\Leftrightarrow$ (4) We shall prove this equivalence by using the reticulation construction in order to obtain a translation of results from lifting ring ideals to lifting elements of the quantale A.

Let R be a commutative ring such that $L(A)$ and $L(R)$ are isomorphic lattices (we will assume that $L(A) = L(R)$). Consider an element a of A; a^* is an ideal of $L(A)$ so one can find an ideal I of R such that $a^* = I^*$. Due to Corollary 3.4, both spaces $Max(A)$ and $Max(R)$ are homeomorphic with $Max(L(A)) = Max(L(R))$. Then for any $m \in Max(A)$ one can find $M \in Max(R)$ such that $m^* = M^*$; conversely, for any $M \in Max(R)$ one can find $m \in Max(A)$ such that $M^* = m^*$.

According to Theorem 5.1, Corollary 4.10 and Lemma 5.5, the following properties are equivalent:

- a has LP (in the quantale A);
- $a^* = I^*$ has $Id - BLP$ (in the lattice $L(A)$);
- I is a lifting ideal of R;
- If M is a maximal ideal of R then $R/(I \vee M^\diamond)$ has no nontrivial idempotents;
- If m is a maximal element of A then $|B([a \vee m^\diamond)_A)| \leq 2$.

Therefore the equivalence (1) $\Leftrightarrow$ (4) follows.

□

Remark 5.7. *The previous proof of the equivalence* (1) $\Leftrightarrow$ (4) *is based on the corresponding equivalence* (1) $\Leftrightarrow$ (4) *in Theorem 5.1. One can pose the problem to do a direct proof of this equivalence, without using Theorem 5.1 and the reticulation. We shall present here a direct proof of implication* (4) $\Rightarrow$ (1).

Let us assume that for any maximal element m of A we have $|B([a \vee m^\diamond)_A)| \leq 2$.

In order to prove that a has LP, consider an element f of $B([a)_A)$. We want to prove that $(f \to a)^\diamond \vee (\neg^a(f) \to a)^\diamond = 1$. Assume by absurdum that $(f \to a)^\diamond \vee (\neg^a(f) \to a)^\diamond < 1$, so that $(f \to a)^\diamond \vee (\neg^a(f) \to a)^\diamond \leq m$, for some maximal element m of A.

According to the hypothesis (4), $B([a \vee m^\diamond)_A) = \{a \vee m^\diamond, 1\}$. By applying Lemma 4.4(5), $f \in B([a)_A)$ implies $f \vee m^\diamond \in B([a \vee m^\diamond)_A)$. We shall distinguish two cases:

(a) $f \vee m^\diamond = a \vee m^\diamond$. We remark that $f \leq a \vee m^\diamond = \bigvee\{a \vee e | e \in B(A), e \leq m\}$ and f is a compact element of $[a)_A$, so there exists $e \in B(A)$ such that $e \leq m$ and $f \leq a \vee e$. By Lemma 2.2(3) we have $f \leq \neg e \to a$, hence $\neg e \leq f \to a$. Since $\neg e \in B(A)$, it follows that $\neg e \leq (f \to a)^\diamond \leq m$, contradicting $e \leq m$ (because $e \leq m$ and $\neg e \leq m$ imply $1 = e \vee \neg e \leq m$ and m is a maximal element of A).

(b) $\neg^{a \vee m^\diamond}(f \vee m^\diamond) = a \vee m^\diamond$. We remark that $B([a \vee m^\diamond)_A) = B([a \vee m^\diamond)_{[a)_A})$ and

$\neg^{a\vee m^\diamond}(f\vee m^\diamond)=\neg^{a\vee m^\diamond}(u^{[a)_A}_{a\vee m^\diamond}(f))=u^{[a)_A}_{a\vee m^\diamond}(\neg^a(f))=\neg^a(f)\vee a\vee m^\diamond=\neg^a(f)\vee m^\diamond$.

Then $\neg^a(f)\vee m^\diamond=a\vee m^\diamond$, *hence* $\neg^a(f)\leq a\vee m^\diamond$. *By using the compactness of* $\neg^a(f)$ *in* $[a)_A$ *and the inequality* $\neg^a(f)\leq\bigvee\{a\vee e|e\in B(A),e\leq m\}$, *we get* $\neg^a(f)\leq a\vee e$, *for some* $e\in B(A)$ *with the property that* $e\leq m$. *Thus* $\neg^a(f)\leq\neg e\to a$, *hence* $\neg e\leq\neg^a(f)\to a$, *resulting* $\neg e\leq(\neg^a(f)\to a)^\diamond\leq m$ *(because of* $\neg e\in B(A)$*). It follows that* $1=e\vee\neg e\leq m$, *contradicting that* $m\in Max(A)$.

In both cases (a) and (b) we obtained a contradiction, therefore the equality $(f\to a)^\diamond\vee(\neg^a(f)\to a)^\diamond=1$ *holds. According to this equality, there exist* $e_1,e_2\in B(A)$ *such that* $e_1\vee e_2=1$, $e_1\leq f\to a$ *and* $e_2\leq\neg^a(f)\to a$.

From $e_1\vee e_2=1$ *and* $e_1\leq f\to a$ *we get* $\neg e_2\leq e_1\leq f\to a$, *therefore* $\neg e_2\vee a\leq f\to a$ *(because* $a\leq f\to a$*). Therefore* $\neg^a(e_2\vee a)=(e_2\vee a)\to a=e_2\to a=\neg e_2\vee a\leq f\to a=\neg^a(f)$. *We observe that* f *and* $e_2\vee a$ *are elements of* $B([a)_A)$, *so from* $\neg^a(e_2\vee a)\leq\neg^a(f)$ *we get* $f\leq e_2\vee a$.

On the other hand, we remark that the inequality $e_2\leq\neg^a(f)\to a$ *implies* $\neg^a(f)\leq e_2\to a=(e_2\vee a)\to a=\neg^a(e_2\vee a)$, *which implies* $e_2\vee a\leq f$. *We conclude that* $f=e_2\vee a=B(u_a^A)(e_2)$, *so* a *has LP.*

Remark 5.8. *It is clear that Theorem 5.6 extends Theorem 5.1 to the abstract framework of quantale theory. On the other hand, Theorem 6.3 of [26] generalizes Theorem 5.1 to a framework of universal algebra: it characterizes the lifting congruences of a semidegenerate congruence modular algebra* M. *We observe that in general Theorem 6.3 of [26] is not a consequence of Theorem 5.6 (because the set* $Con(M)$ *of congruences of the algebra* M *does not have a quantale structure). In the very special case when the commutator operation of* $Con(M)$ *is associative,* $Con(M)$ *becomes a coherent quantale and Theorem 5.6 can be applied, so Theorem 6.3 of [26] can be deduced from our main result.*

An ideal I of a commutative ring R is regular if it is generated by a set of idempotents of A (see [1]). Then I is regular if and only if $I=I^\diamond$. Similarly, an ideal J of a bounded distributive lattice L is regular if it is equal to the ideal $[J\cap B(L))$ generated by $J\cap B(L)$. It is clear that J is regular if and only if $J=J^\diamond$. According to [24], an element a of a quantale A is regular if it is a join of complemented elements. It is obvious that a is regular if and only if $a=a^\diamond$. Of course, $a^\diamond$ is a regular element of A.

The following lemma shows that the function $(\cdot)^*:A\to Id(L(A))$ maps the regular elements of A to regular ideals of $L(A)$.

Lemma 5.9. *If* a *is a regular element of* A *then* a^* *is a regular ideal of* $L(A)$.

Proof. We want to prove that $a^*=[a^*\cap B(L(A)))$. Assume that x is an element of a^*. By Lemma 3.2(1), the following hold:

$a^*=(\bigvee\{e|e\in B(A),e\leq a\})^*=\bigvee\{e^*|e\in B(A),e\leq a\}$

hence there exist $e_1,\cdots,e_n\in B(A)$ such that $e_i\leq a$, for $i=1,\cdots,n$ and $x\in\bigvee_{i=1}^n e_i^*=(\bigvee_{i=1}^n e_i)^*$ (the last equality follows by using Lemma 3.2(1)).

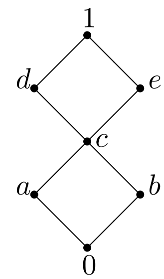
Denoting $e=\bigvee_{i=1}^n e_i$ we have $e\in B(A)$, $e\leq a$ and $x\in e^*$, so there exists $c\in K(A)$ such that $x=\lambda_A(c)$ and $c\leq e$. Therefore $x=\lambda_A(c)\leq\lambda_A(e)$, hence $x\in[a^*\cap B(L(A)))$. We conclude that $a^*\subseteq[a^*\cap B(L(A)))$, and the converse inclusion is obvious.

□

Lemma 5.10. *If* I *is a regular ideal of* $L(A)$ *then there exists a regular element* b *of* A *such that* $I=b^*$.

Proof. Assume that I is a regular ideal of $L(A)$, so $I^\diamond=I$. Let us define $b=(I_*)^\diamond$. Then b is a regular element of A and, by using Lemmas 5.2 and 3.1(2), we get $b^*=((I_*)^\diamond)^*=((I_*)^*)^\diamond=I^\diamond=I$. □

Example 5.11. Let us consider the frame $A=\{0,a,b,c,d,e,1\}$, defined by the following diagram:



We observe that $B(A) = \{0, 1\}$, $B([a)_A) = B(\{a, c, d, e, 1\} = \{a, 1\}$ and $B([b)_A) = B(\{b, c, d, e, 1\} = \{b, 1\}$, therefore a and b are lifting elements of A. On the other hand, $B([c)_A) = B(\{c, d, e, 1\}) = \{c, d, e, 1\}$, hence the Boolean morphism $B(u_c^A) : B(A) \to B([c)_A)$ is not surjective. We conclude that $c = a \vee b$ is not a lifting element of A.

According to the previous example, the join $a \vee b$ of two lifting elements a and b of a coherent quantale A is not necessarily a lifting element of A. The following proposition and its corollary emphasize an important case whenever it happens.

Proposition 5.12. *Let a and b be two elements of the quantale A. If a has LP and b is a regular element of A then $a \vee b$ is a lifting element of A.*

Proof. According to Theorem 5.6, in order to prove that $a \vee b$ is a lifting element of A it suffices to check that for any maximal element m of A we have
$|B([a \vee b \vee m^\diamond)_A)| \leq 2$.

Assume that $m \in Max(A)$. Let R be a commutative ring such that the reticulations $L(R)$ and $L(A)$ of R and A are identical. One can find an ideal I of the ring R and a maximal ideal M of R such that $I^* = a^*$ and $M^* = m^*$.

By Lemma 5.9, b^* is a regular ideal of the lattice $L(A)$, hence, by using Lemma 5.10, one can find a regular ideal J of R such that $J^* = b^*$. We observe that $(I \vee J)^* = I^* \vee J^* = a^* \vee b^* = (a \vee b)^*$ (cf. Lemma 3.2(1)).

An application of Lemma 5.5 for the pairs $(a \vee b, m)$, $(I \vee J, M)$ gives the equality $|B([a \vee b \vee m^\diamond)_A)| = |B(R/(I \vee J \vee M^\diamond))|$.

Since a has LP it results that the ideal $I^* = a^*$ of $L(A)$ has $Id - BLP$ (cf. Corollary 4.10). A new application of Corollary 4.10 shows that I is a lifting ideal of R. Recall that J is a regular ideal of R, so, by using Corollary 3.19 of [43], it follows that $I \vee J$ is a lifting ideal of R.

According to Theorem 5.6, we have $|B(R/(I \vee J \vee M^\diamond))| \leq 2$, therefore we conclude that $|B([a \vee b \vee m^\diamond)_A)| \leq 2$. Thus $a \vee b$ is a lifting element of A. □

We have seen in Section 4 that any complemented of A is a lifting element. This property will be generalized by the following corollary.

Corollary 5.13. *Any regular element of the coherent quantale A is a lifting element of A.*

Proof. Assume that b is a regular element of A. It is obvious that 0 has LP. If one takes $a = 0$ in Proposition 5.12 it follows that $b = 0 \vee b$ has LP. □

Let us consider the quantale $A = \{a, b, c, d, 1\}$ from Example 4.19. We observe that $B(A) = \{0, 1\}$, $B([a)_A) = \{a, 1\}$ and $B([c)_A) = \{c, 1\}$, hence a and c are lifting elements. On the other hand, we have $B([b)_A) = \{b, a, c, 1\}$, so $b = a \wedge c$ is not a lifting element. This example shows that in general the meet of two lifting elements is not a lifting element. The following proposition achieves a particular case whenever the meet of two lifting elements is a lifting element.

Proposition 5.14. *Let a, b be two non-coprime elements of A. If a has LP and $|B([b)_A)| \leq 2$ then $a \wedge b$ has LP.*

Proof. Assume that a, b are two non-coprime elements of A, a has LP and $|B([b)_A)| \leq 2$.

Let R be a commutative ring such that $L(A)$ and $L(R)$ are identical. We can find two ideals I, J of R such that $I^* = a^*$ and $J^* = b^*$. Then I is a lifting ideal of the ring R (by Proposition 5.3).

By using Proposition 4.7 we get $|B([b)_A)| = |B(L(A)/b^*)| = |B(L(R)/J^*)| = |B(R/J)|$.

According to the hypothesis $|B([b)_A)| \leq 2$, we get the inequality $|B(R/J)| \leq 2$, i.e. the quotient ring R/J has no nontrivial idempotents. Since $a \vee b < 1$, we have $a \vee b \leq m$, for some maximal element m of A. One can find a maximal ideal M of R such that $m^* = M^*$. In accordance with Lemma 3.2(1) we have $(I \vee J)^* = I^* \vee J^* = a^* \vee b^* = (a \vee b)^* \subseteq m^* = M^*$, hence, by applying Lemma 3.1(2) we get $I \vee J = ((I \vee J)^*)_* \subseteq (M^*)_* = M$. Thus I, J are non–coprime ideals of the ring R, therefore we can apply Proposition 1.2 of [40] or Proposition 3.2 of [43]. It follows that $I \cap J$ is a lifting ideal of R. By using Lemma 3.2(2) we have $(a \wedge b)^* = a^* \cap b^* = I^* \cap J^* = (I \cap J)^*$, hence $a \wedge b$ is a lifting element of A (by Proposition 5.3).

□

Recall from [36] that a ring R is said to be a clean ring if any element of R is the sum of a unit and an idempotent. An important theorem of [36] asserts that a commutative ring R is a clean ring if and only if any ideal of R is a lifting ideal (i.e. the quantale $Id(R)$ has LP). By Corollary 4.11, the following lemma holds:

Lemma 5.15. *A commutative ring R is a clean ring if and only if the reticulation $L(R)$ of R has $Id - BLP$.*

Proposition 5.16. *Let a be an element of a coherent quantale A such that $a \leq r(A)$. If the quantale $[a)_A$ has LP then A has LP.*

Proof. Let us consider a commutative ring R such that $L(A) = L(R)$ so there exists an ideal I of R such that $a^* = I^*$. Assume that M is a maximal ideal of the ring R, so M^* is a maximal ideal of the lattice $L(A) = L(R)$. Thus one can find a maximal element m of A such that $m^* = M^*$. By the hypothesis, we have $a \leq m$, so $I^* = a^* \subseteq m^* = M^*$. It follows that $I \subseteq \sqrt{I} = (I^*)_* \subseteq (M^*)_* = M$ (by Lemma 3.1,(2) and (7)), therefore $I \subseteq Rad(R)$ (recall that $Rad(R)$ is the Jacobson radical of the ring R).

By using Proposition 4.7 we get the following lattice isomorphisms: $L([a)_A) \simeq L(A)/a^* \simeq L(R)/I^* \simeq L(R/I)$, hence, by applying Corollary 4.11, the following implications hold:

$[a)_A$ has $LP \Rightarrow L([a)_A)$ has $Id - BLP \Rightarrow L(R/I)$ has $Id - BLP \Rightarrow R/I$ is a clean ring.

According to Proposition 1.5 of [36], R is a clean ring. Then the lattice $L(A) = L(R)$ has $Id - BLP$, so A has LP (by Corollary 4.11).

□

Following [10], a quantale A is said to be B-normal if for all coprime elements a, b of A there exist $e, f \in B(A)$ such that $a \vee e = b \vee f = 1$ and $ef = 0$. By using Theorem 5.6 we present here a short proof of the following result of [10]:

Proposition 5.17. *A coherent quantale A has LP if and only if A is B-normal.*

Proof. Assume that A has LP and a, b are two coprime elements of A. Let us denote $x = ab$. By applying the condition (3) of Theorem 5.6 for x, one can find $e \in B(A)$ such that $e \leq a$ and $\neg e \leq b$. Denoting $f = \neg e$ it follows that $a \vee f = b \vee e = 1$ and $ef = 0$, hence A is B–normal.

Conversely, let us suppose that the quantale A is B–normal. Let a, x, y be three elements of A such that x, y are coprime and $a = xy$. Since A is B–normal, there exists $e, f \in B(A)$ such that $x \vee e = b \vee f = 1$ and $ef = 0$. Then $\neg e \rightarrow x = \neg f \rightarrow y = 1$ (cf. Lemma 2.2(3)) and $e \leq \neg f$, therefore $e \leq \neg f \leq y$ and $\neg e \leq x$. Thus the property (3) of Theorem 5.6 is verified, so A has LP.

□

Recall from [37] that a quantale A is normal if for all coprime elements a, b of A there exist $x, y \in A$ such that $a \vee x = b \vee y = 1$ and $xy = 0$.

Proposition 5.18. [10] *If A is a normal coherent quantale then the Jacobson radical $r(A)$ of A is a lifting element.*

Proof. It suffices to verify the condition (3) of Theorem 5.6. Let $c, d \in K(A)$ such that $c \vee d = 1$ and $cd = r(A)$. Since A is normal there exist $x, y \in A$ such that $c \vee x = d \vee y = 1$ and $xy = 0$, so $x \vee y \vee c = x \vee y \vee d = 1$. According to Lemma 2(2) of [10] we have $x \vee y \vee (cd) = 1$, hence $x \vee y \vee r(A) = 1$. By applying Lemma 22 of [10], it follows that $x \vee y = 1$. From $x \vee y = 1$ and $xy = 0$ we get $x, y \in B(A)$ and $y = \neg x$ (cf. Lemma 2.2(4)).

We observe that $x \vee c = 1$ implies $y = y(x \vee c) = xy \vee cy = cy$, so $y \leq c$. Similarly, $y \vee d = 1$ implies $y \leq d$. We have proven that $y \in B(A)$, $y \leq c$ and $\neg y \leq d$, therefore the condition (3) of Theorem 5.6 is verified.

□

6 Concluding Remarks

The study of the coherent quantales with Boolean Lifting Property (LP) and the elements with LP began in [10]. The results of [10] are focused on quantales with LP rather than on elements with LP. The present paper deals with the lifting elements of a coherent quantale (= the elements that satisfy LP).

We proved many algebraic and topological properties of lifting elements in a coherent quantale. Among them we mention a characterization theorem and a lot of its consequences (new or old results). The inspiration points of the paper are the results obtained in [40] and [43] for the lifting ideals in commutative rings.

Two methods are applied to prove the results of this paper. The first one consists in applying a proposition that provides an isomorphism between the Boolean center $B(A)$ of a coherent quantale A and the Boolean algebra of clopen subsets of the prime spectrum $Spec(A)$ of A. The second method uses the transfer properties of the reticulation $L(A)$ of A. According to a Hochster theorem [28], for any coherent quantale A one can find a commutative ring R such that the results on the lifting ideals of R can be transferred to the lifting elements of A.

Our general results can be applied to concrete algebraic structures for which a Boolean Lifting Property can be defined: rings, bounded distributive lattices, commutative l-groups, orthomodular lattices, MV-algebras, BL- algebras, pseudo- BL algebras, residuated lattices, etc.

An important open problem is to extend the construction of reticulation and the study of lifting properties from quantales to more larger classes of multiplicative lattices.

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References

[1] M. Aghajani and A. Tarizadeh, Characterization of Gelfand rings, specially of clean rings and their dual rings, *Results Math.*, 75(3) (2020), 75-125.

[2] M. F. Atiyah and I. G. MacDonald, Introduction to Commutative Algebra, *Addison-Wesley Publ. Comp.,* (1969).

[3] R. Balbes and Ph. Dwinger, Distributive Lattices, *Univ. of Missouri Press*, (1974).

[4] B. Banaschewski, Gelfand and exchange rings: Their spectra in point free topology, *Arab. J. Sci. Eng.*, 25(2C) (2000), 3-22.

[5] B. Barania Nia and A. Borumand Saeid, Classes of pseudo BL-algebras in view of right Boolean lifting property, *Transactions of A. Razmadze Mathematical Institute*, 172 (2018), 146-163.

[6] B. Barania Nia and A. Borumand Saeid, Study of pseudo BL-algebras in view of right Boolean lifting property, *Appl. Appl. Math.*, 13(1) (2018), 354-381.

[7] G. Birkhoff, Lattice Theory, *AMS Collocquium Publ.,* (1967).

[8] D. Cheptea, Lifting properties in classes of universal algebras, *Ph.D.Thesis*, in preparation.

[9] D. Cheptea, G. Georgescu and C. Mureşan, Boolean lifting properties for bounded distributive lattices, *Sci. Ann. Comput. Sci.*, 25 (2015), 29-67.

[10] D. Cheptea and G. Georgescu, Boolean lifting properties in quantales, *Soft Computing*, 24 (2020), 1-13.

[11] T. Demba, On the hull-kernel and inverse topology on frames, *Algebra Universalis*, 70 (2013), 197-212.

[12] M. Dickmann, N. Schwarz and M. Tressl, Spectral Spaces, *Cambridge University Press,* (2019).

[13] P. Eklund, J. G. Garcia, U. Hohle and J. Kortelainen, Semigroups in Complete Lattices: Quantales, Modules and Related Topics, *Springer. USA,* (2018).

[14] A. Facchini, C. A. Pinocchiaro and G. Janelidze, Abstractly constructed prime spectra, *Algebra Universalis*, 83(8) (2022). Available online: https://doi.org/10.107/s00012-021-00764-z.

[15] N. Galatos, P. Jipsen, T. Kowalski and H. Ono, Residuated Lattices: An Algebraic Glimpse at Structural Logics, Studies in Logic and The Foundation of Mathematics, 151, *Elsevier, USA,* (2007).

[16] G. Georgescu, The reticulation of a quantale, *Rev. Roumaine Math. Pures Appl.*, 40(7-8) (1995), 619-631.

[17] G. Georgescu, D. Cheptea and C. Mureşan, Algebraic and topological results on lifting properties in residuated lattices, *Fuzzy Sets Systems*, 271 (2015), 102-132.

[18] G. Georgescu, L. Leuştean and C. Mureşan, Maximal residuated lattices with lifting Boolean center, *Algebra Universalis*, 63 (2010), 83-99.

[19] G. Georgescu and C. Mureşan, Boolean lifting property for residuated lattices, *Soft Computing*, 18 (2014), 2075-2089.

[20] G. Georgescu and C. Mureşan, Congruence Boolean Lifting Property, *J. Multiple-valued Logic and Soft Computing*, 29 (2017), 225-274.

[21] G. Georgescu and C. Mureşan, Factor Congruence Lifting Property, *Studia Logica*, 225 (2017), 1079-2017.

[22] G. Georgescu, L. Kwuida and C. Mureşan, Functorial properties of the reticulation of a universal algebra, *J. Applied Logic*, 8(5) (2021), 1123-1168.

[23] G. Georgescu, Some classes of quantale morphisms, *J. Algebra Number Theory Appl.*, 24(2) (2021), 111-153.

[24] G. Georgescu, Flat topology on the spectra of quantales, *Fuzzy Sets and Systems*, 406 (2021), 22-41.

[25] G. Georgescu, Reticulation of a quantale, pure elements and new transfer properties, *Fuzzy Sets and Systems*, (2021). Available online: https://doi.org/10.1016/j.fss.2021.06.003.

[26] G. Georgescu, New results on Congruence Boolean Lifting Property, *Journal of Algebraic Hyperstructures and Logical Algebras*, 3(1) (2022), 15-34.

[27] A. W. Hager, C. M. Kimber and W. Wm McGovern, Clean unital l-groups, *Math. Slovaca*, 63 (2013), 979-992.

[28] M. Hochster, Prime ideals structures in commutative rings, *Trans. Amer. Math. Soc.*, 142 (1969), 43-60.

[29] P. Jipsen, Generalization of Boolean products for lattice-ordered algebras, *Annals Pure Appl. Logic*, 161 (2009), 224-234.

[30] P. T. Johnstone, Stone Spaces, *Cambridge University Press,* (1982).

[31] A. J. de Jong et al., Stacs project. Available online: http://stacs.math.columbia.edu.

[32] N. A. Kermani, E. Eslami and A. Borumand Saeid, Central lifting property for orthomodular lattices, *Math. Slovaca*, 70(6) (2020), 1307-1316.

[33] L. Leuştean, Representations of Many-valued Algebras, *Editura Universitara, Bucharest,* (2010).

[34] J. Martinez, Abstract ideal theory, Ordered Algebraic Structures, Lecture Notes in Pure and Appl. Math., 99, *Marcel Dekker, New York,* (1985).

[35] C. Mureşan, Algebras of many-valued logic, *Ph.D.Thesis*, Bucharest University, (2009).

[36] W. K. Nicholson, Lifting idempotents and exchange rings, *Transaction of A.M.S.*, 229 (1977), 278-288.

[37] J. Paseka, Regular and normal quantales, *Arch.Math.* (Brno), 22 (1996), 203-210.

[38] J. Paseka and J. Rosicky, Quantales, Current Research in Operational Quantum Logic : Algebras, Categories and Languages, Fund. Theories Phys, vol. 111, *Kluwer Acad. Publ., Dordrecht,* (2000), 245-262.

[39] J. Picado and A. Pultr, Frames and Locales: Topology without points, Frontiers in Mathematics, *Springer, Basel,* (2012).

[40] E. Rostami, S. Hedayat, S. Karimzadeh and R. Nekooei, Comaximal factorization of of lifting ideals, *J. Algebra Appl.*, 19(1) (2021). Available online: https://doi.org/10.1142/S021949882250044x.

[41] K. I. Rosenthal, Quantales and Their Applications, *Longman Scientific and Technical,* (1989).

[42] H. Simmons, Reticulated rings, *J. Algebra*, 66 (1980), 169-192.

[43] A. Tarizadeh and P.K. Sharma, Structural results on lifting, orthogonality and finiteness of idempotents, *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM* 116(1), 54 (2022).

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