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The Category of L-algebras

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Abstract. In this paper, we define and study the category of L-algebras, proving that this category has equalizers, coequalizers, kernel pairs and products. We investigate the existence of injective objects in this category and show that an object in the subcategory of cyclic L-algebras is injective if and only if it is a complete and divisible cyclic L-algebra.

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1 Introduction

The Yang–Baxter equation first appeared in theoretical physics, and in statistical mechanics. Finding solutions of this equation represents a research topic of current interest. W. Rump proved in [19] that every set A with a binary operation $\cdot$ satisfying equation (L) $(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$ corresponds to a solution of the quantum Yang–Baxter equation if the left multiplication is bijective. Equation (L) also appears in algebraic logic, classical or intuitionistic logic, as well as in infinite-valued Łukasiewicz logic (see [20] for details). Based on equation (L), W. Rump developed in [20] the concept of L-algebras, proving that for every L-algebra A there exists a self-similar closure $S(A)$, unique up to isomorphism, with an embedding of A to $S(A)$. The self-similar closure $S(A)$ admits a left group of fractions $G(A)$ with a natural map $A \hookrightarrow S(A) \longrightarrow G(A)$ and, if A is a semiregular L-algebra, then the structure group $G(A)$ is an ℓ-group ([20, Th. 3, Th. 4]). W. Rump and Y. Yang proved that an L-algebra is representable as an interval in an ℓ-group if and only if it is semiregular with the smallest element and bijective negation ([21, Th. 3.11]), and that the pseudo MV-algebras can be characterized as semiregular L-algebras with negation ([30]). Since L-algebras have applications in many areas such as number theory ([24]), group theory ([22], [23], [25]), lattice theory ([26]), the study of these algebras is a topic of great interest nowadays (see for example [7], [14], [28], [29]). The categories of algebras of fuzzy logic have been investigated for Hilbert algebras ([3], [4], [13]), BCI-algebras ([1]), p-semisimple BCI-algebras ([32]), BCH-algebras ([6]), EQ-algebras ([2]), pseudo BCI-algebras ([11], [12]).
Motivated by the fact that the studies on L-algebras are of current interest, in this paper we study the category **Lalg** of L-algebras and prove that the category **Lalg** has equalizers, and coequalizers, kernel pairs and products. We also prove that any coequalizer is surjective and it is a coequalizer of its kernel pair. We construct the product of two particular objects in **Lalg**, and finally we give an example of two objects in **Lalg** having a co-product. We introduce the notion of divisible cyclic L-algebras and prove that the cyclic L-algebras and MV-algebras are categorial equivalent. We also investigate the existence of injective objects

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in the category **Lalg** and prove that $\{1\}$ is the only injective object in this category. The main result consists of proving that an object X in the category **CyLalg** of cyclic L-algebras is injective if and only if X is a complete and divisible cyclic L-algebra.

2 Preliminaries

In this section we recall some basic notions and results regarding L-algebras that we use in this paper (see [20]).

Magma is a structure $(A, \to)$, where $\to$ is a binary operation of a set A. In a magma $(A, \to)$, an element $e \in A$ is a *logical unit* if

$$e \to x = x,\ x \to x = x \to e = e. \tag{U}$$

The logical unit is unique. Indeed, if e, e' are logical units, then $e = e \to e = e'$. We denote the logical unit by 1. Then $(A, \to, 1)$ is called a *unital* magma. A magma $(A, \to)$ is a *cycloid* such that

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z). \tag{L}$$

A *unital cycloid* is a cycloid with logical unit (see [20]). If a unital cycloid $(A, \to, 1)$ satisfies

$$x \to y = y \to x = 1 \text{ implies } x = y, \tag{An}$$

then it is called an *L-algebra*. If an L-algebra $(A, \to, 1)$ satisfies

$$x \to (y \to x) = 1, \tag{K}$$

then it is called a *KL-algebra*. A *CL-algebra* is an L-algebra $(A, \to, 1)$ such that

$$(x \to (y \to z)) \to (y \to (x \to z)) = 1. \tag{C}$$

It follows that in any L-algebra A satisfying condition (C) we have $x \to (y \to z) = y \to (x \to z)$, for all $x, y, z \in A$. Given an L-algebra $(A, \to, 1)$, a binary relation $\leq$ is defined by $x \leq y$ iff $x \to y = 1$, for all $x, y \in A$.

The notion of a self-similar closure was introduced by W. Rump in [20] and it proved to play a crucial role in the study of L-algebras. Let H be a self-similar L-algebra and let A be a subalgebra of H. As we mentioned, H is a left hoop. If the monoid H is generated by A, we call H a *self-similar closure* of A and it is denoted by $S(A)$. According to [20, Th. 3], for any L-algebra A, the self-similar closure $S(A)$ exists and it is unique, up to isomorphism. Obviously, if H is a self-similar left hoop, then $S(H) = H$. By condition (H), any self-similar left hoop H satisfies the *left Ore condition* (for each pair of elements $a, b \in H$, there are $c, d \in H$ such that $ca = db$ - see [20]), hence the self-similar closure $S(A)$ of an L-algebra A has the left Ore condition. Due to the left Ore condition, $S(A)$ admits a *left group of fractions* $G(S(A))$ (consisting of left fractions $x^{-1}y$, for all pairs $x, y \in S(A)$), denoted by $G(A)$. The morphism $A \hookrightarrow S(A) \longrightarrow G(A)$ defines a natural map $q : A \longrightarrow G(A)$ with $q(x) = q(y)$ if and only if there is $c \in S(A)$ such that $cx = cy$ (see [20, Def. 5]). By [20, Prop. 10], if A is a KL-algebra, then $S(A)$ is also a KL-algebra. A monoid H with an additional operation $\to$ is a *left hoop* if the following hold for all $a, b, c \in H$: (E) $a \to a = 1$, (A) $ab \to c = a \to (b \to c)$, (H) $(a \to b)a = (b \to a)b$. It was proved in [20, Prop. 3] that every left hoop is an L-algebra. An L-algebra A is said to be *self-similar* if and only if for any $x \in A$, the map $\rho : \downarrow x = \{y \in X \mid y \leq x\} \longrightarrow A$, defined by $\rho(y) = x \to y$ is a bijection. It is easy to see that ρ is isotone, more precisely, it is monotone increasing. Based on the bijective map ρ we define a new operation on A, namely, for all $x, y \in A$, the product xy is defined as the inverse image of x. In other words, xy is unique, and it is determined by $xy \leq y$ and $y \to xy = x$. By [20, Th. 1], every self-similar L-algebra with the new product operation is a left-hoop. An L-algebra A is called *commutative* if $S(A)$ is commutative as a monoid. According to [20, Prop. 19], $S(A)$ is commutative if and only if A is a KL-algebra and, for all $x, y \in A$:

$$(x \to y) \to y = (y \to x) \to x. \tag{Com}$$

In this case $S(A) \cong G(A)^-$ and $x \vee y = (x \to y) \to y$ holds for all $x, y \in S(A)$. Indeed, since L is a KL-algebra

we have $x \leq (y \to x) \to x = x \vee y$ and $y \leq (x \to y) \to y = x \vee y$. If $u \in L$ such that $x \leq u$ and $y \leq u$, then $u \to y \leq x \to y$, so that $x \vee y = (x \to y) \to y \leq (u \to y) \to y = (y \to u) \to u = 1 \to u = 1$. Similarly, $u \to x \leq y \to x$ and $x \vee y = (y \to x) \to x \leq (u \to x) \to x = (x \to u) \to u = 1 \to u = u$. Hence $x \vee y$ is the lower upper bound of $\{x, y\}$.
Let $(A, \to, 1)$ be an L-algebra. We call $I \subseteq A$ an *ideal* of A if it satisfies the following conditions for all $x, y \in A$ ([20]): (I_0) $1 \in I$; (I_1) $x, x \to y \in I$ imply $y \in I$; (I_3) $x \in I$ implies $y \to x, y \to (x \to y) \in I$. Denote by $\mathcal{ID}(A)$ the set of all ideals of A. Obviously $\{1\}, A \in \mathcal{ID}(A)$.
Let $(A, \to, 1)$ be an L-algebra and let $I \in \mathcal{ID}(A)$. According to [20], [7] we have:
(1) If A satisfies condition (K), then (I_3) can be omitted.
(2) If A satisfies condition (D), then (I_2) can be omitted.
(3) If A satisfies condition (C), then (I_2) and (I_3) can be omitted.
Let A be an L-algebra. For every subset $B \subseteq A$, the smallest ideal of A containing B (i.e. the intersection of all ideals $I \in \mathcal{ID}(A)$ such that $B \subseteq I$) is called the *ideal generated by* B and it will be denoted by $[B)$. If $B = \{x\}$ we write $[x)$ instead of $[\{x\})$. In this case $[x)$ is called a *principal ideal* of A. Let $(A, \to, 1)$ be an L-algebra. Then every ideal I of A defines a congruence:

$$x \sim y \text{ iff } x \to y, y \to x \in I.$$

Conversely, each congruence $\sim$ of A defines an ideal $I := \{x \in X \mid x \sim 1\}$.
A congruence $\sim$ of A is called a *relative congruence* if the quotient algebra $(A/\!\sim, \to, [1]_{\sim})$ is an L-algebra. According to [20, Cor. 1], for an L-algebra X, there is a bijective correspondence between ideals and relative congruences. We denote by $\theta_I = \sim_I$ a relative congruence defined by an ideal I, and $(A/I, \to, [1]_I)$ the corresponding quotient algebra. We write $[x]_{\sim_I} = x/I$ and obviously $I = 1/I$. The function $\pi_I : A \longrightarrow A/I$ defined by $\pi_I(x) = x/I$ for any $x \in A$ is a surjective homomorphism which is called the *canonical projection* from A to A/I. One can easily prove that $\mathrm{Ker}\,(\pi_I) = I$. If A is a self-similar L-algebra and I is an ideal of A, then by [20, Cor. 3] A/I is a self-similar L-algebra.

Let $(A, \to, 1)$ and $(B, \to, 1)$ be two L-algebras. A map $f : A \longrightarrow B$ is called a *morphism* if $f(x \to y) = f(x) \to f(y)$, for all $x, y \in A$. Denote by $\mathrm{HOM}\,(A, B)$ the set of all morphisms from A to B. If $f \in \mathrm{HOM}\,(A, B)$, then $\mathrm{Ker}\,(f) = \{x \in A \mid f(x) = 1\}$ is called the *kernel* of f.
For any $f \in \mathrm{HOM}\,(A, B)$ the following hold: (i) $f(1) = 1$, (ii) $f(x) \leq f(y)$, whenever $x, y \in A$, $x \leq y$, (iii) $\mathrm{Ker}\,(f) \in \mathcal{ID}(A)$.

Proposition 2.1. *Let A, B be two self-similar L-algebras. If $f \in \mathrm{HOM}\,(A, B)$, then $f(xy) = f(x)f(y)$, for all $x, y \in A$.*

Proof. For all $x, y \in A$ we have $xy \leq y$ and $y \to xy = x$. It follows that $f(y) \to f(xy) = f(x)$. On the other hand, $f(x)f(y) \leq f(y)$ and $f(y) \to f(x)f(y) = f(x)$. Since the product is unique we get $f(xy) = f(x)f(y)$. □

3 MV-algebras as L-algebras

We recall the definition and certain results on MV-algebras, and we define the notion of cyclic L-algebras. The main result consists of proving that an algebra $(A, \oplus, 0)$ is an MV-algebra if and only if $(A, \to, 1)$ is a cyclic L-algebra.

Let A be an L-algebra having a smallest element 0, and denote $x^- = x \to 0$, for all $x \in A$. We say that A has a *negation* if the map $^- : A \longrightarrow A$, defined by $x \mapsto x^-$ is bijective. Using the inverse of negation $^-$, denoted by $^\sim$, we define the second implication on A by $x \rightsquigarrow y = y^\sim \to x^\sim$. Clearly, $x \rightsquigarrow 0 = x^\sim$ and

$x^{-\sim} = x^{\sim -} = x$, for any $x \in A$. By [21, Prop. 2.8], if A is a semiregular L-algebra with negation, then $x \leq y$ iff $x^- \geq y^-$. According to [21, Th. 3.8], for any semiregular L-algebra with a negation $(A, \to, 1)$, the structure $A^{op} := (A, \rightsquigarrow, 1)$ is a semiregular L-algebra with negation such that $(A^{op})^{op} = A$. For a semiregular L-algebra with negation A, a product operation $\cdot$ was defined in [21] by $x \cdot y = (x \to y^-)^\sim$, for all $x, y \in A$, and it is proved that $x \cdot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$, for all $x, y, z \in A$ ([21, Prop. 3.2]). Moreover, from $x \to y \leq x \to y$ and $x \rightsquigarrow y \leq x \rightsquigarrow y$ we get $x \leq (x \to y) \rightsquigarrow y$ and $x \leq (x \rightsquigarrow y) \to y$, respectively. It follows that a semiregular L-algebra with negation is a CL-algebra. According to [21, Prop. 3.5], a semiregular L-algebra with negation is a left hoop, so that the operation $\cdot$ is associative. For a semiregular L-algebra with negation A we set:

$$x \wedge y := ((x \to y) \to x^-)^\sim,\ x \vee y = (x^\sim \to y^\sim) \to x,$$

for all $x, y \in A$. It is proved in [21, Prop. 2.9] that $(A, \wedge, \vee)$ is a lattice.

Proposition 3.1. *([8]) Let $(A, \to, 1)$ be a semiregular L-algebra with negation. Then the following hold for all $x, y \in A$:*
(1) $x \cdot 0 = 0 \cdot x = 0$, $x \cdot 1 = 1 \cdot x = x$;
(2) $x^- \cdot x == 0$;
(3) $x \to y = y^- \to x^-$;
(4) $x^- \to y = y^- \to x$;
(5) $y \leq x \to y$.

Let $(A, \to, 0, 1)$ be a semiregular L-algebra with negation. We define the sum of the elements x and y of A:

$$x + y := y^- \to x = x^- \to y.$$

Proposition 3.2. *([8]) Let A be a semiregular L-algebra with negation. Then the following hold for all $x, y \in A$:*
(1) $0 + x = x + 0 = x$;
(2) $1 + x = x + 1 = 1$;
(3) $x + x^- = 1$;
(4) $x \cdot y = (y^- + x^-)^-$;
(5) $x + y = (y^- \cdot x^-)^-$;
(6) $x + y = y + x$.

Proof. The proof is straightforward. □

Definition 3.3. A semiregular L-algebra with negation A is said to be *cyclic* if $x^- = x^\sim$, for all $x \in A$.

If A is cyclic, then we can easily see that $x \vee y = (x \to y) \to y = (y \to x) \to x$, for all $x, y \in A$.
The *MV-algebras* were defined by Chang in 1958 ([5]) as algebraic counterparts of $\aleph_0$-valued Łukasiewicz logic. For details on MV-algebras we refer the reader to [9].
An *MV-algebra* is an algebra $(A, \oplus, ^-, 0)$ with a binary operation $\oplus$, a unary operation $^-$ and a constant 0 satisfying the following equations, for all $x, y, z \in A$: (MV_1) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
(MV_2) $x \oplus y = y \oplus x$;
(MV_3) $x \oplus 0 = x$;
(MV_4) $(x^-)^- = x$;
(MV_5) $x \oplus 0^- = 0^-$;
(MV_6) $(x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x$.
Axioms (MV_1)-(MV_3) state that $(A, \oplus, 0)$ is a commutative monoid. As a consequence, in any MV-algebra A we have $1^- = 0$ and $x \oplus x^- = 1$, for all $x \in A$. We can easily see that the map $x \mapsto x^-$ is bijective. Indeed, if $x_1, x_2 \in A$ with $x_1^- = x_2^-$, then $x_1^{--} = x_2^{--}$, and by (MV_4) we get $x_1 = x_2$. Moreover, since $x = (x^-)^-$, the

map $x \mapsto x^-$ is bijective. If $(A, \oplus, ^-, 0)$ is an MV-algebra, we define the following operations, for all $x, y \in A$: $x \odot y = (x^- \oplus y^-)^-$, $x \to y = x^- \oplus y = (x \odot y^-)^-$, $1 = 0^-$. We can see that $x^- = x \to 0$. A partial order relation $\leq$ is defined on A by $x \leq y$ iff $x^- \oplus y = 1$. Two auxiliary operations $\vee$ and $\wedge$ are defined, by setting $x \vee y = x \oplus y \odot x^- = y \oplus x \odot y^-$ and $x \wedge y = x \odot (y \oplus x^-) = y \odot (x \oplus y^-)$. Then $(A, \wedge, \vee, 0, 1)$ is a lattice.

Lemma 3.4. *If $(A, \oplus, ^-, 0)$ is an MV-algebra, then the following hold for all $x, y \in A$:*
(1) $x \leq y$ *iff* $y^- \leq x^-$;
(2) $(x \to y) \vee (y \to x) = 1$;
(3) $y \to x \odot y = x^- \vee y$.

Proof. (3) Replacing y by y^- in (MV_6), we get $y^- \oplus (y^- \oplus x^-)^- = (x \oplus y)^- \oplus x$, so that $y^- \oplus x \odot y = (x \oplus y)^- \oplus x$. It follows that $y \to x \odot y = (x^- \to y) \to x = x^- \vee y$. $\square$

A monoid $(H, \odot, 1)$ with an additional binary operation $\to$ will be called a *left hoop* if the following are satisfied for $x, y, z \in H$: (h_1) $x \to x = 1$, (h_2) $x \to (y \to z) = x \odot y \to z$, (h_3) $(x \to y) \odot x = (y \to x) \odot y$ ([20, Def. 3]).

Lemma 3.5. *If $(A, \oplus, ^-, 0)$ is an MV-algebra, then $(A, \odot, \to, 1)$ is a left hoop.*

Proof. For all $x, y, z \in A$, we have: $x \to x = x^- \oplus x = 1$, $x \odot y \to z = (x \odot y)^- \oplus z = (y^- \oplus x^-) \odot z = x^- \oplus (y^- \oplus z) = x \to (y \to z)$, and $(x \to y) \odot x = (x^- \oplus y) \odot x = x \wedge y = (y^- \oplus x) \odot y = (y \to x) \odot y$. Hence $(A, \odot, \to, 1)$ is a left hoop. $\square$

Proposition 3.6. *If $(A, \oplus, ^-, 0)$ is an MV-algebra, then $(A, \to, 0, 1)$ is a cyclic L-algebra.*

Proof. We check axioms (U), (L) and (An) from the definition of L-algebras.
Since $x \to x = x^- \oplus x = 1$, $1 \to x = 1^- \oplus x = 0 \oplus x = x$, and $x \to 1 = x^- \oplus 1 = 1$, axiom (U) is satisfied. If $x \to y = y \to x = 1$, then $x^- \oplus y = y^- \oplus x = 1$, so that $x \leq y$ and $y \leq x$. It follows that $x = y$, that is axiom (An) is also verified. Let $x, y, z \in A$. Replacing x by x^- and y by y^- in (MV_6) we get $(x \oplus y^-)^- \oplus y^- = (y \oplus x^-)^- \oplus x^-$, so that $(y \oplus x^-)^- \oplus (x^- \oplus z) = (x \oplus y^-)^- \oplus (y^- \oplus z)$. It follows that $(x \to y)^- \oplus (x \to z) = (y \to x)^- \oplus (y \to z)$, that is $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$, and so, axiom (L) is satisfied. It follows that $(A, \to, 1)$ is an L-algebra. By Lemma 3.5, A is a left hoop and according to [21, Thm. 3.7], an L-algebra with negation is semiregular if and only if it is a left hoop satisfying conditions from Lemma 3.4. We conclude that $(A, \to, 0, 1)$ is a cyclic L-algebra. $\square$

Proposition 3.7. *If $(A, \to, 0, 1)$ is a cyclic L-algebra, then $(A, +, 0)$ is an MV-algebra.*

Proof. We check axioms (MV_1)-(MV_6) from the definition of MV-algebras. Since $(x + y) + z = (x^- \to y) + z = z^- \to (x^- \to y) = x^- \to (z^- \to y) = x + (y + z)$, axiom (MV_1) is satisfied. Axioms (MV_2), (MV_3) and (MV_5) follow from Proposition 3.2(6),(1),(2), respectively, while axiom (MV_4) is true by the definition of negation. Finally, the identity $(x \to y) \to y = (y \to x) \to x (= x \vee y)$ implies (MV_6), so that $(A, +, 0)$ is an MV-algebra. $\square$

Theorem 3.8. *An algebra $(A, \oplus, 0)$ is an MV-algebra if and only if $(A, \to, 1)$ is a cyclic L-algebra.*

Proof. It follows by Propositions 3.6 and 3.7. $\square$

Example 3.9. Consider the set $A = \{0, a, b, 1\}$ and the operation $\to$ given by the following table:

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

.

The structure $(A, \rightarrow, 1)$ is a cyclic L-algebra. The negation $^-$ and the operations $\cdot$, $+$ are given in the tables below.

x	0	a	b	1
x^-	1	b	a	0

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$+$	0	a	b	1
0	0	a	b	1
a	0	a	1	1
b	b	1	b	1
1	1	1	1	1

.

Then $(A, +, 0)$ is an MV-algebra.

Proposition 3.10. *The meets and unions $(\wedge_{MV}, \vee_{MV})$ of an MV-algebra coincide with the meets and unions $(\wedge_L, \vee_L)$ of its corresponding cyclic L-algebra $(A, \rightarrow, 0, 1)$.*

Proof. Recall that:

$$x \wedge_{MV} y = x \odot (y \oplus x^-) = y \odot (x \oplus y^-),\ x \vee_{MV} y = x \oplus y \odot x^- = y \oplus x \odot y^-,$$
$$x \wedge_L y = ((x \rightarrow y) \rightarrow x^-)^- = ((y \rightarrow x) \rightarrow y^-)^-,\ x \vee_L y = (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x,$$

for all $x, y \in A$. Then we have:

$$\begin{aligned} x \wedge_{MV} y &= x \odot (y \oplus x^-) = (x^- \oplus y) \odot x = ((x^- \oplus y) \odot x)^{--} = ((x^- \oplus y)^- \oplus x^-)^- \\ &= ((x \rightarrow y)^- \oplus x^-)^- = ((x \rightarrow y) \rightarrow x^-)^- = x \wedge_L y, \\ x \vee_{MV} y &= y \oplus x \odot y^- = x \odot y^- \oplus y = (x^- \oplus y)^- \oplus y = (x \rightarrow y)^- \oplus y \\ &= (x \rightarrow y) \rightarrow y = x \vee_L y. \end{aligned}$$

Hence the two pairs of lattice operations coincide. □

4 The Category of L-algebras

In this section, we define the category **Lalg** of L-algebras and prove that this category has equalizers, co-equalizers, and kernel pairs. We also prove that any coequalizer is surjective and it is a coequalizer of its kernel pair.

We consider the category of L-algebras, denoted by **Lalg** whose objects are L-algebras and whose morphisms are L-algebras homomorphisms. Denote by **Ob**(Lalg) the class of objects of **Lalg**, and for any $X, Y \in$ **Lalg**, we denote by **Lalg**(X,Y) the class of morphisms of **Lalg**. For details regarding the notions and results of category theory we refer the reader to [17], [18], [16], [3].

In a category $\mathcal{C}$, an object **0** is called an *initial object* if, for every object X of $\mathcal{C}$, there is exactly one morphism from **0** to X. And dually, an object **1** is called a *terminal or final object* if, for every object X, there is exactly one morphism from X to **1**. If an object is simultaneously an initial and a final object, it is called a *nullary object* or a *zero object*.

Proposition 4.1. *The category* **Lalg** *has an initial and final object.*

Proof. We can see that in the category **Lalg**, $\mathbf{0} = \mathbf{1} = (\{1\}, \rightarrow, 1)$ is an initial object as well as a final object. Indeed, for any $X \in \mathbf{Ob}(Lalg)$ there is a unique morphism $f : \{1\} \longrightarrow X$ and there is a unique morphism $f : X \longrightarrow \{1\}$. Hence $\{1\}$ is a nullary object of **Lalg**. □

Generally speaking, if $\mathcal{C}$ is an algebraic category and $X, Y \in \mathbf{Ob}(\mathcal{C})$, then $f \in \mathcal{C}(X, Y)$ is a *monomorphism* if for any $Z \in \mathbf{Ob}(\mathcal{C})$ and $g, h \in \mathcal{C}(Z, X)$ such that $f \circ g = f \circ h$, we have $g = h$. Similarly, if $g \circ f = h \circ f$ implies $g = h$ for any $g, h \in \mathcal{C}(Y, Z)$, then f is called an *epimorphism*.

In this section we extend to the case of **Lalg** some results proved in [11] and [6] for the categories of pseudo-BCI algebras and pseudo-BCH algebras, respectively.

Theorem 4.2. *In the category* **Lalg** *monomorphisms and injective morphisms coincide.*

Proof. Let $X, Y \in \mathbf{Ob}(Lalg)$ and let $f \in \mathbf{Lalg}(X, Y)$ injective. Consider $X' \in \mathbf{Ob}(Lalg)$ and $g, h \in \mathbf{Lalg}(X', X)$ such that $f \circ g = f \circ h$, that is $(f(g(x)) = f(h(x))$, for any $x \in X'$. Since f is injective, we get $f(x) = g(x)$ for all $x \in X'$, hence $g = h$. It follows that f is a monomorphism of **Lalg**. Conversely, suppose that f is a monomorphism, so that $f \circ g = f \circ h$ implies $g = h$. It is enough to prove that $\mathrm{Ker}\,(f) = \{1\}$. Let $\mathrm{Ker}\,(f)$ such that $x \neq 1$, and define $g, h : \mathrm{Ker}\,(f) \longrightarrow X$, by $g(x) = x$, $h(x) = 1$, for all $x \in \mathrm{Ker}\,(f)$. We have $f(x) = f(1) = 1$, hence $f \circ g = f \circ h)$. Since f is a monomorphism, we get $g = h$, a contradiction. Thus $\mathrm{Ker}\,(f) = \{1\}$, that is f is injective. (Indeed, if $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$, we have $f(x_1 \to x_2) = f(x_1) \to f(x_2) = 1$ and $f(x_2 \to x_1) = f(x_2) \to f(x_1) = 1$. It follows that $x_1 \to x_2, x_2 \to x_1 \in \mathrm{Ker}\,(f) = \{1\}$, that is $x_1 \to x_2 = x_2 \to x_1 = 1$. We get $x_1 \le x_2$ and $x_2 \le x_1$, hence by (L_3) we have $x_1 = x_2$). □

Proposition 4.3. *In the category* **Lalg** *surjective morphisms are epimorphisms.*

Proof. Let $X, Y \in \mathbf{Ob}(Lalg)$ and let $f \in \mathbf{Lalg}(\mathbf{X}, \mathbf{Y})$ surjective. Consider $Z \in \mathbf{Ob}(Lalg)$ and $g, h \in \mathbf{Lalg}(\mathbf{Y}, \mathbf{Z})$ such that $g \circ f = h \circ f$. Let $y \in Y$. Since f is surjective, there is $x \in X$ such that $f(x) = y$. It follows that $g(y) = g(f(x)) = h(f(x)) = h(y)$, for all $y \in Y$, that is $g = h$. We conclude that f is an epimorphism in **Lalg**. □

Remark 4.4. The converse of Proposition 4.3 is not always true. Indeed, in [4, Ex. 4.1] is given an example of an epimorphism of Hilbert algebras which is not surjective. Since by [7, Rem. 4.12] any Hilbert algebra is an L-algebra, it follows that not any surjective morphism in **Lalg** is an epimorphism.

We recall that $f \in \mathbf{Lang}(X, Y)$ is a *bimorphism* if it is both monomorphism and epimorphism. If any bimorphism in a category is an isomorphism, the category is called *balanced* or *perfect*.

Corollary 4.5. *The category* **Lang** *is not perfect.*

Proposition 4.6. *Let $f : X \longrightarrow Y$ be an epimorphism of L-algebras. Then $[\mathrm{Im}\,(f)) = Y$.*

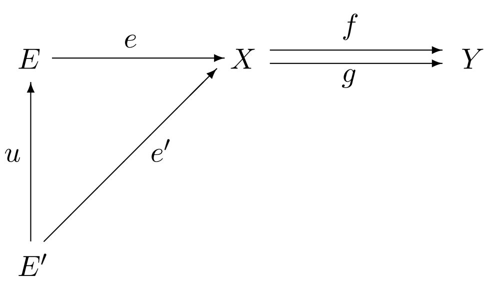
Proof. Let $I = [\mathrm{Im}\,(f))$ and suppose that $I \neq Y$. Consider the map $\mathbf{1}_Y : Y \longrightarrow Y/I$ defined by $\mathbf{1}_Y(x) = 1/I$, for all $x \in Y$. Since $f(x) \in \mathrm{Im}\,(f) \subseteq I$, for any $x \in X$, we have $(\pi_I \circ f)(x) = \pi_I(f(x)) = 1/I = \mathbf{1}_Y(f(x)) = (\mathbf{1}_Y \circ f)(x)$. Hence $\pi_I \circ f = \mathbf{1}_Y \circ f$. On the other hand, $\pi_I(x) = \mathbf{1}_B(x)$ if and only if $x \in I \neq Y$. It follows that f is not an epimorphism, a contradiction. We conclude that $[\mathrm{Im}\,(f)) = Y$. □

Corollary 4.7. *If $f : X \longrightarrow Y$ is an epimorphism of L-algebras such that $\mathrm{Im}\,(f) \in \mathcal{ID}(Y)$, then f is surjective.*

Definition 4.8. A homomorphism $f : X \longrightarrow Y$ of L-algebras satisfying $\mathrm{Im}\,(f) \in \mathcal{ID}(Y)$ is said to be *regular*. A category has *ES property* (*epimorphism surjectivity property*) if all its epimorphisms are surjective.

Corollary 4.9. *The category* **Lalg** *does not have ES property.*

Let $\mathcal{C}$ be a category, and let $X, Y \in \mathbf{Ob}(\mathcal{C})$ and $f, g \in \mathcal{C}(X, Y)$. An *equalizer* of the couple (f, g) is a pair (E, e) with $E \in \mathbf{Ob}(\mathcal{C})$ and $e \in \mathcal{C}(E, X)$ such that:
(i) $f \circ e = g \circ e$;
(ii) if (E', e') is another pair that satisfies (i), then there exists a unique morphism $u \in \mathcal{C}(E', E)$ such that $e' = e \circ u$.



If a couple of morphisms in $\mathcal{C}$ has an equalizer (E, e), then it is unique up to an isomorphism ([3, Rem. 4.2.14]) and e is a monomorphism in $\mathcal{C}$([3, Rem. 4.2.16]). We say that the category $\mathcal{C}$ has equalizers if any couple of morphisms in $\mathcal{C}$ has an equalizer.

Theorem 4.10. *The category* **Lalg** *has equalizers.*

Proof. Let $X, Y \in \mathbf{Ob}(Lalg)$ and let $f, g \in \mathbf{Lalg}(X, Y)$. Then $E = \{x \in X \mid f(x) = g(x)\}$ is a nonempty subalgebra of X and consider the embedding $e : E \longrightarrow X$ $(e(x) = x$, for any $x \in E)$. Obviously, $E \in \mathbf{Ob}(Lalg)$, $e \in \mathbf{Lalg}(E, X)$ and $f \circ e = g \circ e$. Moreover, it is easy to see that e is a monomorfism in **Lalg**. Let $E' \in \mathbf{Ob}(Lalg)$ and let $e' \in \mathbf{Lalg}(E', X)$ such that $f \circ e' = g \circ e'$. Define $u : E' \longrightarrow X$, by $u(x) = e'(x)$ for any $x \in E'$. Since $f(e'(x)) = g(e'(x))$, it follows that $e'(x) \in E$ for all $x \in E'$, hence u is well defined. We have $e(u(x)) = e(e'(x)) = e'(x)$ for any $x \in E'$, so that $e \circ u = e'$. By the fact that e is a monomorphism, it follows that u is unique. We conclude that (E, e) is an equalizer of the couple (f, g), that is **Lalg** has equalizers. $\square$

Corollary 4.11. *If a couple of morphisms in the category* **Lalg** *has an equalizers* (E, e)*, then* e *is injective.*

Proof. It follows by Theorem 4.2, since e is a monomorphism in **Lalg**. $\square$

Example 4.12. Let $X_1 = \{0, 1\}$, $Y_1 = \{0, 1, 2\}$ and consider the following binary operations $\rightarrow_1, \odot_1$ and $\rightarrow_2, \odot_2$ defined on X_1, Y_1, respectively.

$\rightarrow_1$	0	1
0	1	1
1	0	1

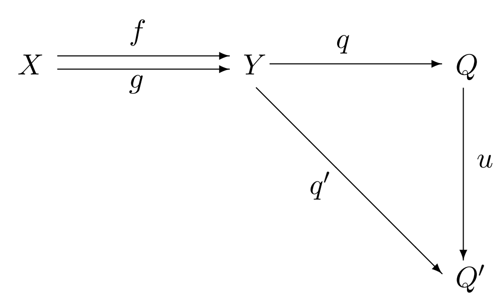
$\odot_1$	0	1
0	0	0
1	0	1

$\rightarrow_2$	0	1	2
0	2	2	2
1	1	2	2
2	0	1	2

$\odot_2$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

Then the structures $(X_1, \odot_1, \rightarrow_1, 1), (Y_1, \odot_2, \rightarrow_2, 1)$ are BL-algebras ([15, Ex. 7.1]), and according to to [7, Prop. 4.7], $X = (X_1, \rightarrow_1, 1), Y = (Y_1, \rightarrow_2, 1)$ are L-algebras. Hence $X, Y \in \mathbf{Ob}(Lalg)$, and let $f, g \in \mathbf{Lalg}(X, Y)$ defined by $f(0) = 0, f(1) = 2$, $g(0) = 1, g(1) = 2$. Consider $E = \{x \in X \mid f(x) = f(y)\} = \{1\} \in \mathbf{Ob}(Lalg)$, and let $e \in \mathbf{Lalg}(E, X)$ defined by $e(x) = x$. Then (E, e) is an equalizer of the pair (f, g).

Let $\mathcal{C}$ be a category, and let $X, Y \in \mathbf{Ob}(\mathcal{C})$ and $f, g \in \mathcal{C}(X, Y)$. A *coequalizer* of the couple (f, g) is a pair (Q, q) with $Q \in \mathbf{Ob}(\mathcal{C})$ and $q \in \mathcal{C}(Y, Q)$ such that:
(i) $q \circ f = q \circ g$;
(ii) if (Q', q') is another pair which satisfies (i), then there exists a unique morphism $u \in \mathcal{C}(Q, Q')$ such that $q' = u \circ q$.



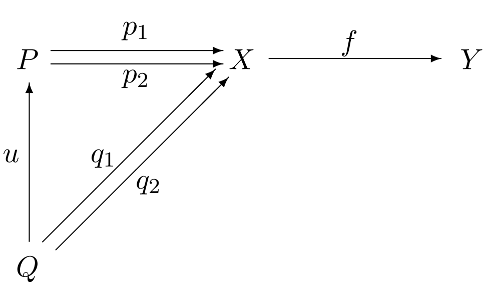
We say that the category $\mathcal{C}$ has coequalizers if any couple of morphisms in $\mathcal{C}$ has a coequalizer.

Theorem 4.13. *The category* **Lalg** *has coequalizers.*

Proof. Let $X, Y \in \mathbf{Ob}(Lalg)$ and let $f, g \in \mathbf{Lalg}(X, Y)$. Denote $Z = \{(f(x), g(x)) \in Y \times Y \mid x \in X\}$ and let $Q = Y/[Z) \in \mathbf{Ob}(Lalg)$ (by [20, Cor. 1]). If $q : Y \longrightarrow Q$ is the canonical projection, then $q \in \mathbf{Lalg}(Y, Q)$, and we prove that (Q, q) is a equalizator for (f, g). Obviously, $(f(x), g(x)) \in \theta_{[Z)}$ for all $x \in X$, so that $(q \circ f)(x) = q(f(x)) = [f(x)]_{\theta_{[Z)}} = [g(x)]_{\theta_{[Z)}} = q(g(x)) = (q \circ g)(x)$, for all $x \in X$. Hence $q \circ f = q \circ g$. Let $Q' \in \mathbf{Ob}(Lalg)$ and let $q' \in \mathbf{Lalg}(Y, Q')$ such that $q' \circ f = q' \circ g$, that is $q'(f(x)) = q'(g(x))$ for all $x \in X$. It follows that $f(x) \to g(x), g(x) \to f(x) \in \mathrm{Ker}\,(q')$, hence $(f(x), g(x)) \in \theta_K$, where $K = \mathrm{Ker}\,(q')$. Thus $Z \subseteq \theta_K$, that is $\theta_{[Z)} \subseteq \theta_K$. Define the morphism $u : Q \longrightarrow Q'$ by $u(y/[Z)) = q'(y)$ (u is well defined, since $y_1/[Z) = y_2/[Z)$ implies $(y_1, y_2) \in \theta_{[Z)} \subseteq \theta_K$, that is $q'(y_1) = q'(y_2)$). Obviously $u \circ q = q'$. Since q is surjective, it is an epimorphism, that is u is unique. We conclude that (Q, q) is a coequalizator for the couple (f, g). □

Example 4.14. Consider $X, Y \in \mathbf{Ob}(Lalg)$ and $f, g \in \mathbf{Lalg}(X, Y)$ from Example 4.12. Let $Z = \{(f(x), g(x)) \in Y \times Y \mid x \in X\} = \{(f(0), g(0)), (f(1), g(1))\} = \{(0, 1), (2, 2)\}$.
Then $[Z) = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 2)\}$ and $Q = Y/[Z) = \{[0] = [1] = \{0, 1\}, [2] = \{2\}\}$. We have $Q \in \mathbf{Ob}(Lalg)$ and let $q \in \mathbf{Lalg}(Y, Q)$ be the canonical projection: $q(0) = q(1) = [0] = [1]$, $q(2) = [2]$. Then (Q, q) is a coequalizator for the pair (f, g).

Let $\mathcal{C}$ be a category, and let $X, Y \in \mathbf{Ob}(\mathcal{C})$ and $f \in \mathcal{C}(X, Y)$. A *kernel pair* of the f is a system (P, p_1, p_2) with $P \in \mathbf{Ob}(\mathcal{C})$ and $p_1, p_2 \in \mathcal{C}(P, X)$ such that:
(i) $f \circ p_1 = f \circ p_2$;
(ii) if (Q, q_1, q_2) is another system which satisfies (i), then there exists a unique morphism $u \in \mathcal{C}(Q, P)$ such that $p_1 \circ u = q_1$ and $p_2 \circ u = q_2$.



We say that the category $\mathcal{C}$ has kernel pairs if any morphisms in $\mathcal{C}$ has a kernel pair.

Theorem 4.15. *The category* **Lalg** *has kernel pairs.*

Proof. Let $X, Y \in \mathbf{Ob}(Lalg)$ and let $f \in \mathbf{Lalg}(X, Y)$. Obvioulsy, the structure $(X \times X, \to, (1, 1))$ is an L-algebra, where $(x_1, y_1) \to (x_2, y_2) = (x_1 \to x_2, y_1 \to y_2)$. Denote $P = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$, and clearly P is an L-subalgebra of $X \times X$, that is $P \in \mathbf{Ob}(Lalg)$. Let $p_1, p_2 : P \longrightarrow X$ be the canonical projections, that is $p_1(x_1, x_2) = x_1$, $p_2(x_1, x_2) = x_2$, for all $(x_1, x_2) \in X \times X$. Obviously $p_1, p_2 \in \mathbf{Lalg}(P, X)$ such that $f \circ p_1 = f \circ p_2$. Consider now $Q \in \mathbf{Ob}(Lalg)$ and $q_1, q_2 \in \mathbf{Lalg}(Q, X)$ such that $f \circ q_1 = f \circ q_2$, and define $u : Q \longrightarrow P$ by $u(x) = (q_1(x), q_2(x))$, for all $x \in Q$. Since $f(q_1(x)) = f(q_2(x))$ implies $(q_1(x), q_2(x)) \in P$ for all $x \in Q$, it follows that u is well defined. Moreover, $u(x_1 \to x_2) = (q_1(x_1 \to x_2), q_2(x_1 \to x_2)) = (q_1(x_1) \to q_1(x_1), q_2(x_1) \to q_2(x_1)) = (q_1(x_1), q_2(x_1)) \to q_1(x_2), q_2(x_2)) = u(x_1) \to u(x_2)$, that is $u \in \mathbf{Lalg}(Q, P)$. For any $x \in Q$, we have $(p_1 \circ u)(x) = p_1(u(x)) = p_1((q_1(x), q_2(x))) = q_1(x)$ and $(p_2 \circ u)(x) = p_2(u(x)) = p_2((q_1(x), q_2(x))) = q_2(x)$, that is $p_1 \circ u = q_1$ and $p_2 \circ u = q_2$. For another $u' \in \mathbf{Lalg}(Q, P)$ such that $p_1 \circ u' = q_1$ and $p_2 \circ u' = q_2$, let $u'(x) = (x_1, x_2)$. From $p_1 \circ u' = p_1 \circ u$ and

$p_2 \circ u' = p_2 \circ u$ we get $p_1(x_1, x_2) = p_1(q_1(x), q_2(x)) = q_1(x)$, $p_2(x_1, x_2) = p_2(q_1(x), q_2(x)) = q_2(x)$, hence $x_1 = q_1(x)$ and $x_1 = q_2(x)$. It follows that $u'(x) = (x_1, x_2) = (q_1(x), q_2(x)) = u(x)$ for all $x \in Q$. Thus u is unique, and we conclude that (P, p_1, p_2) is a kernel pair of f. $\square$

Example 4.16. Consider $X, Y \in \mathbf{Ob}(Lalg)$ and $f \in \mathbf{Lalg}(X, Y)$ from Example 4.12, that is $f(0) = 0$, $f(1) = 2$. We have $X \times X = \{(0,0), (0,1), (1,0), (1,1)\}$ and $P = \{(x_1, x_2) \mid f(x_1) = f(x_2)\} = \{(0,0), (1,1)\}$. Let $p_1, p_2 : P \longrightarrow X$ be the canonical projections, that is $p_1(0,0) = 0$, $p_1(1,1) = 1$, $p_2(0,0) = 0$, $p_2(1,1) = 1$. Then $p_1, p_2 \in \mathbf{Lalg}(P, X)$ and $f \circ p_1 = f \circ p_2$, hence (P, p_1, p_2) is a kernel pair of f.

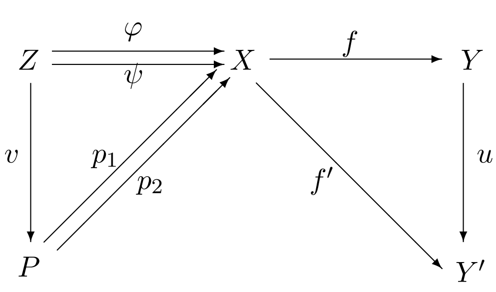
Let $f : X \longrightarrow Y$ be a morphism in the category $\mathcal{C}$, and let $f \in \mathcal{C}(X, Y)$. If there exists $Z \in \mathbf{Ob}(\mathcal{C})$ and $\varphi, \psi \in \mathcal{C}(Z, X)$ such that (Y, f) is a coequalizer of the couple (φ, ψ), then we say that f is a coequalizer in $\mathcal{C}$.

Proposition 4.17. *Any surjective morphism in* **Lalg** *is a coequalizer of its kernel pair.*

Proof. Let $X, Y \in \mathbf{Ob}(Lalg)$ and let $f \in \mathbf{Lalg}(X, Y)$ be a surjective morphism. According to Theorem 4.15, f has a kernel pair (P, p_1, p_2), where $P = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$ and $p_1, p_2 : P \longrightarrow X$ are the canonical projections. We prove that the pair (Y, f) is a coequalizer of (p_1, p_2). Obviously, $f \circ p_1 = f \circ p_2$. Suppose that there exists $Y' \in \mathbf{Ob}(Lalg)$ and $f' \in \mathbf{Lalg}(X, Y')$ auch that $f' \circ p_1 = f' \circ p_2$. Let $y \in Y$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. Consider $u : Y \longrightarrow Y'$ defined by $u(y) = f'(x)$. If $x_1, x_2 \in X$ such that $f(x_1) = f(x_2) = y$, then $(x_1, x_2) \in P$ and $u(y) = f'(x_1) = (f' \circ p_1)(x_1, x_2) = (f' \circ p_2)(x_1, x_2) = f'(x_2)$, so that u is well defined. Consider $y_1, y_2 \in Y$, so that there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. It follows that $f'(x_1) = u(y_1)$, $f'(x_2) = u(y_2)$ and $y_1 \to y_2 = f'(x_1) \to f'(x_2) = f'(x_1 \to x_2)$. We get $u(y_1 \to y_2) = f'(x_1 \to x_2) = f'(x_1) \to f'(x_2) = u(y_1) \to u(y_2)$, so that $u \in \mathbf{Lalg}(Y, Y')$. We can easy chck that $u \circ f = f'$, while u is unique, since f is an epimorphism. We conclude that f is a coequalizer of its pair kernel. $\square$

Proposition 4.18. *Any coequalizer in* **Lalg** *is a coequalizer of its kernel pair.*

Proof. Let $X, Y \in \mathbf{Ob}(Lalg)$ and let $f \in \mathbf{Lalg}(X, Y)$ be a coequalizer in **Lalg**, that is there exists $Z \in \mathbf{Ob}(\mathcal{C})$ and $\varphi, \psi \in \mathcal{C}(Z, X)$ such that f is a coequalizer of the couple (φ, ψ). According to Theorem 4.15, f has a kernel pair (P, p_1, p_2), where $P = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$ and $p_1, p_2 : P \longrightarrow X$ are the canonical projections. We have $f \circ p_1 = f \circ p_2$, so that it is enough to prove that for any other morphism $f' \in \mathbf{Lalg}(X, Y')$ such that $f' \circ p_1 = f' \circ p_2$, there exists a unique morphism $u \in \mathbf{Lalg}(Y, Y')$ such that $f' = u \circ f$. Since (P, p_1, p_2) is a kernel pair of f and $f \circ \varphi = f \circ \psi$, there exists a unique morphism $v \in \mathbf{Lalg}(Z, P)$ such that $\varphi = p_1 \circ v$ and $\psi = p_2 \circ v$.



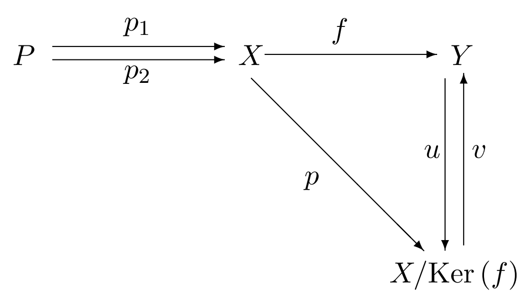
We have $f' \circ \varphi = (f' \circ p_1) \circ v = (f' \circ p_2) \circ v = f' \circ \psi$. Since f is a coequalizer of the couple (φ, ψ), there exists a unique morphism $u \in \mathbf{Lalg}(Y, Y')$ such that $f' = u \circ f$. We conclude that f is a coequalizer of its kernel pair (P, p_1, p_2). $\square$

Lemma 4.19. *Let $X, Y, Z \in \mathbf{Ob}(Lalg)$ and let $f \in \mathbf{Lalg}(X, Y)$, $g \in \mathbf{Lalg}(X, Z)$. If f is surjective and* $\mathrm{Ker}\,(f) \subseteq \mathrm{Ker}\,(g)$, *then there exists a unique morphism $h \in \mathbf{Lalg}(Y, Z)$ such that $h \circ f = g$.*

Proof. According to Theorem 4.15, f has a kernel pair (P, p_1, p_2), where $P = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$ and $p_1, p_2 : P \longrightarrow X$ are the canonical projections. Since f is surjective, by Theorem 4.17, (Y, f) is a coequalizer of (p_1, p_2). For any $(x_1, x_2) \in P$, we have $f(x_1) = f(x_2)$, so that $x_1 \to x_2, x_2 \to x_1 \in \mathrm{Ker}\,(f) \subseteq \mathrm{Ker}\,(g)$, that is $g(x_1) = g(x_2)$. It follows that $g \circ p_1 = g \circ p_2$. Since f is a coequalizer of (p_1, p_2), then there exists a unique morphism $h \in \mathbf{Lalg}(Y, Z)$ such that $h \circ f = g$. $\square$

Theorem 4.20. *Any coequalizer in* **Lalg** *is surjective.*

Proof. Let f be a coequalizer **Lalg**. According to Theorem 4.18, f is a coequalizer of its kernel pair (P, p_1, p_2), where $P = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\} = \{(x_1, x_2) \in X \times X \mid x_1 \to x_2, x_2 \to x_1 \in \mathrm{Ker}\,(f)\}$. Since $\mathrm{Ker}\,(f) \in \mathcal{ID}(X)$, then $X/\mathrm{Ker}\,(f)\} \in \mathbf{Ob}(Lalg)$, and let $p : P \longrightarrow X/\mathrm{Ker}\,(f)$ be the canonical projection. We can see that $(p \circ p_1)(x_1, x_2) = x_1/\mathrm{Ker}\,(f) = x_2/\mathrm{Ker}\,(f) = (p \circ p_2)(x_1, x_2)$, for any $(x_1, x_2) \in X \times X$, that is $p \circ p_1 = p \circ p_2$. Since (Y, f) is a coequalizer of the couple (p_1, p_2), there exists a unique morphism $u : Y \longrightarrow X/\mathrm{Ker}\,(f)$ such that $u \circ f = p$.

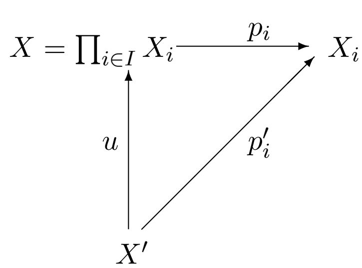


For any $x \in \mathrm{Ker}\,(p)$ we have $p(x) = 1/\mathrm{Ker}\,(f)$, so that $p(x) = x/\mathrm{Ker}\,(f) = 1/\mathrm{Ker}\,(f)$. It follows that $x \in \mathrm{Ker}\,(f)$, that is $\mathrm{Ker}\,(p) \subseteq \mathrm{Ker}\,(f)$. According to Lemma 4.19, there exists a unique morphism $v : X/\mathrm{Ker}\,(f) \longrightarrow Y$ such that $v \circ p = f$. It follows that $(u \circ v) \circ p = u \circ f = p = 1_{X/\mathrm{Ker}\,(f)} \circ p$ and $(v \circ u) \circ f = v \circ p = f = 1_Y \circ f$. But p and f are epimorphisms (p is surjective, while f is a coequalizer), so that $u \circ v = 1_{X/\mathrm{Ker}\,(f)}$ and $v \circ u = 1_Y$. It follows that v is an isomorphism (u the inverse of v, and v the inverse of u), that is v is surjective. Hence $f = v \circ p$ is surjective. $\square$

5 Products and co-products in the Category Lalg

We prove that the category **Lalg** has products, and the subcategory **CLalg** of CL-algebras has co-products. As an example, we construct the product of two objects in **Lalg**, and finally we give an example of two objects in **Lalg** having co-product.

Let $\mathcal{C}$ be a category, and let $(X_i)_{i \in I}$ be a family of objects in $\mathcal{C}$. A *direct product* of the family $(X_i)_{i \in I}$ is a pair $(X, (p_i)_{i \in I})$, with $X \in \mathbf{Ob}(\mathcal{C})$ and $p_i \in \mathcal{C}(X, X_i)$, for any $i \in I$, such that for any other pair $(X', (p'_{i \in I}))$ with $X' \in \mathbf{Ob}(\mathcal{C})$ and $p'_i \in \mathcal{C}(X', X_i)$, there is a unique $u \in \mathcal{C}(X', X)$ such that $p_i \circ u = p'_i$, for any $i \in I$, that is the following diagram is commutative, for any $i \in I$.



If the direct product of a family $(X_i)_{i\in I}$ of objects in $\mathcal{C}$ exists, then it is unique up to an isomorphism ([3, Rem. 4.6.2]), and it is denoted by $\prod_{i\in I} X_i$. The map $p_j : \prod_{i\in I} X_i \longrightarrow X_j$ will be called the j-th *canonical projection*. We say that a category $\mathcal{C}$ *has products* if there exists the direct product of any family of objects in $\mathcal{C}$.

Theorem 5.1. *The category* **Lalg** *has products.*

Proof. Let $(X_i)_{i\in I}$ be a family of objects in **Lalg** and let $X = \prod_{i\in I} X_i$ be the set of all maps $f : I \longrightarrow \bigcup_{i\in I} X_i$ such that $f(i) \in X_i$ for all $i \in I$. If 1_i and $\to_i$ are the logical unit and the implication in L-algebra X_i, we consider the map $1 : I \longrightarrow \bigcup_{i\in I} X_i$ defined by $1(i) = 1_i$. For any $f, g \in X$ define the operation $\to$ on X by $(f \to g)(i) = f(i) \to_i g(i)$, for all $i \in I$. It is easy to check that $(X, \to, 1)$ is an L-algebra, so that it is an object in **Lalg**. For $i \in I$, the projection $p_i : X \longrightarrow X_i$ is defined by $p_i(f) = f(i)$, for all $f \in X$. For any $X' \in \mathbf{Ob}(Lalg)$ and $p'_i \in \mathbf{Lalg}(X', X_i)$, define $u : X' \longrightarrow X$ by $(u(x))(i) = p'_i(x)$, for all $x \in X'$ and $i \in I$. Then we have $(u(x \to y))(i) = p'_i(x) \to_i p'_i(y) = (u(x))(i) \to_i (u(y))(i)$, for all $x, y \in X'$ and $i \in I$, that is u is an L-algebras homomorphism. Moreover, $(p_i \circ u)(x) = p_i(u(x)) = (u(x))(i) = p'_i(x)$, for all $x \in X'$, that is $p_i \circ u = p'_i$. Suppose that there exists another morphism $v : X' \longrightarrow X$ such that $p_i \circ v = p'_i$ for all $i \in I$. It follows that $(p_i \circ v)(x) = p'_i(x) = (p_i \circ u)(x)$ for all $i \in I$ and $x \in X'$. Hence $(v(x))(i) = (u(x))(i)$ for all $i \in I$, so that $v(x) = u(x)$ for all $x \in X'$, that is $v = u$. We conclude that the category **Lalg** has products. □

Example 5.2. Let $X_1 = \{a, b, c, 1_1\}$, $X_2 = \{0, x, y, 1_2\}$ and let $\to_1, \to_2$ be binary operation on X_1, X_2 given in the following tables.

$\to_1$	a	b	c	1_1
a	1_1	a	c	1_1
b	1_1	1_1	c	1_1
c	1_1	1_1	1_1	1_1
1_1	a	b	c	1_1

$\to_2$	0	x	y	1_2
0	1_2	1_2	1_2	1_2
x	y	1_2	y	1_2
y	x	x	1_2	1_2
1_2	0	x	y	1_2

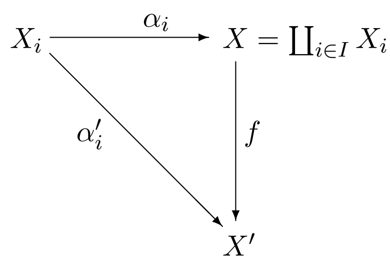
Then $(X_1, \to_1, 1_1), (X_2, \to_2, 1_2) \in \mathbf{Ob}(Lalg)$, and let $I = \{1, 2\}$. Let $X = \{f_1, f_2, \ldots, f_{15}, 1\}$ be the set all functions $f : I \longrightarrow X_1 \cup X_2$ with $f(1) \in X_1$, $f(2) \in X_2$, and define $p_i : X \longrightarrow X_i$, by $p_i(f) = f(i)$, for $i \in I$ (see the tables below).

i	1	2
$f_1(i)$	a	0
$f_2(i)$	a	x
$f_3(i)$	a	y
$f_4(i)$	a	1_2
$f_5(i)$	b	0
$f_6(i)$	b	x
$f_8(i)$	b	y
$f_8(i)$	b	1_2
$f_9(i)$	c	0
$f_{10}(i)$	c	x
$f_{11}(i)$	c	y
$f_{12}(i)$	c	1_2
$f_{13}(i)$	1_1	0
$f_{14}(i)$	1_1	x
$f_{15}(i)$	1_1	y
$1(i)$	1_1	1_2

f	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	1
$p_1(f)$	a	a	a	a	b	b	b	b	c	c	c	c	1_1	1_1	1_1	1_1
$p_2(f)$	0	0	0	0	x	x	x	x	y	y	y	y	1_2	1_2	1_2	1_2

We have $(X, \rightarrow, 1) \in \mathbf{Ob}(Lalg)$ and $X_1 \prod X_2 = (X, p_1, p_2)$.

Let $\mathcal{C}$ be a category, and let $(X_i)_{i\in I}$ be a family of objects in $\mathcal{C}$. A *co-product* (also called a *direct sum*) of the family $(X_i)_{i\in I}$ is a pair $((\alpha_i)_{i\in I}), X)$, with $X \in \mathbf{Ob}(\mathcal{C})$ and $\alpha_i \in \mathcal{C}(X_i, X)$, for any $i \in I$, such that for any other pair $((\alpha'_{i\in I}), X')$ with $X' \in \mathbf{Ob}(\mathcal{C})$ and $\alpha'_i \in \mathcal{C}(X_i, X')$, there is a unique $f \in \mathcal{C}(X, X')$ such that $f \circ \alpha_i = \alpha'_i$, for any $i \in I$, that is the following diagram is commutative, for any $i \in I$.



If the co-product of a family $(X_i)_{i\in I}$ of objects in $\mathcal{C}$ exists, then it is unique up to an isomorphism ([3, Rem. 4.6.7]), and it is denoted by $\coprod_{i\in I} X_i$. The map $\alpha_j : X_j \longrightarrow \coprod_{i\in I} X_i$ will be called the j-th *canonical injection*. We say that a category $\mathcal{C}$ *has co-products* if there exists the co-product of any family of objects in $\mathcal{C}$.

We give the following example using an idea from [2].

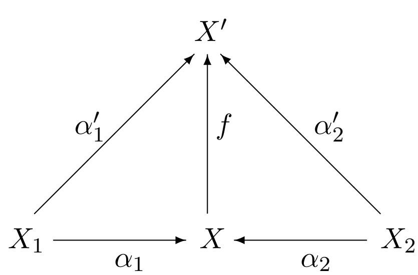
Example 5.3. Let $X_1 = \{u, a, b, 1\}$, $X_2 = \{0, u\}$, $X = X_1 \cup X_2$ and let $\rightarrow_1, \rightarrow_2, \rightarrow$ be binary operations on X_1, X_2, X given in the following tables.

$\rightarrow_1$	u	a	b	1
u	1	1	1	1
a	u	1	1	1
b	u	b	1	1
1	u	a	b	1

$\rightarrow_2$	0	u
0	u	u
u	0	u

$\rightarrow$	0	u	a	b	1
0	1	1	1	1	1
u	0	1	1	1	1
a	0	u	1	1	1
b	0	u	b	1	1
1	0	u	a	b	1

Then $(X_1, \rightarrow_1, u), (X_2, \rightarrow_2, 1), (X, \rightarrow, 1) \in \mathbf{Ob}(Lalg)$, and let $I = \{1, 2\}$. Define $\alpha_1 : X_1 \longrightarrow X$ by by $\alpha_1(x) = x$ for all $x \in X_1$, and $\alpha_2 : X_2 \longrightarrow X$, by $\alpha_2(x) = x$ for all $x \in X_2$. Then (X, α_1, α_2) is the co-product of X_1 and X_2.



Indeed, suppose that X' is another L-algebra with two homomorphisms $\alpha'_1 : X_1 \longrightarrow X'$, $\alpha'_2 : X_2 \longrightarrow X'$.

$$f(x) \quad = \begin{cases} \alpha'_1(x) & x \in X_1 \\ \alpha'_2(x) & x \in X_2. \end{cases}$$

Since α'_1 and α'_2 are homomorphism, then f is homomorphism. We can easily check that $f \circ \alpha_1 = \alpha'_1$ and $f \circ \alpha_2 = \alpha'_2$. Suppose that there exists another homomorphism $g : X \longrightarrow X'$ such that $g \circ \alpha_1 = \alpha'_1$ and $g \circ \alpha_2 = \alpha'_2$, that is $g(\alpha_1(x)) = f(\alpha_1(x))$ for all $x \in X_1$ and $g(\alpha_2(x)) = f(\alpha_2(x))$ for all $x \in X_2$. It follows that $g(x) = f(x)$ for all $x \in X$, hence f is unique. We conclude that (X, α_1, α_2) is the co-product of X_1 and X_2.

Example 5.4. Consider the elements $0 \leq c \leq u \leq a \leq b \leq 1$ and the sets $X_1 = \{u, a, b, 1\}$, $X_2 = \{0, u\}$, $X = X_1 \cup X_2$, $Y = \{0, c, u\}$. Let $\rightarrow_1, \rightarrow_2, \rightarrow, \rightarrow'$ be binary operations on X_1, X_2, X, Y given in the following tables.

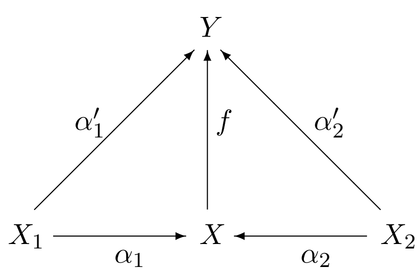
$\rightarrow_1$	u	a	b	1
u	1	1	1	1
a	u	1	1	1
b	u	b	1	1
1	u	a	b	1

$\rightarrow_2$	0	u
0	u	u
u	0	u

$\rightarrow$	0	u	a	b	1
0	1	1	1	1	1
u	0	1	1	1	1
a	0	u	1	1	1
b	0	u	b	1	1
1	0	u	a	b	1

$\rightarrow'$	0	c	u
0	u	u	u
c	0	u	u
u	0	c	u

Then $(X_1, \rightarrow_1, 1), (X_2, \rightarrow_2, u), (X, \rightarrow, 1), (Y, \rightarrow', u) \in \mathbf{Ob}(Lalg)$. Let $\alpha_1 \in \mathbf{Lalg}(X_1, X)$ and $\alpha_2 \in \mathbf{Lalg}(X_2, X)$ defined by $\alpha_1(u) = a$, $\alpha_1(a) = \alpha_1(b) = \alpha_1(1) = 1$, $\alpha_2(0) = b$, $\alpha_2(u) = 1$. We show that the pair (X, α_1, α_2) is not a co-product of the family (X_1, X_2).
Consiter $\alpha'_1 \in \mathbf{Lalg}(X_1, Y)$ and $\alpha'_2 \in \mathbf{Lalg}(X_2, Y)$ defined by $\alpha'_1(u) = c$, $\alpha'_1(a) = \alpha_1(b) = \alpha_1(1) = u$, $\alpha'_2(0) = c$, $\alpha'_2(u) = u$. We must prove that there exists $f \in \mathbf{Lalg}(X, Y)$ such that $f \circ \alpha_1 = \alpha'_1$ and $f \circ \alpha_2 = \alpha'_2$.



The homomorphisms $\mathbf{Lalg}(X, Y)$ are given in the following table.

x	0	u	a	b	1
$f_1(x)$	0	c	u	u	u
$f_2(x)$	0	u	u	u	u
$f_3(x)$	c	u	u	u	u
$f_4(x)$	u	u	u	u	u

For any $i = 1, 2, 3, 4$, we have $(f_i \circ \alpha_1)(u) = f_i \circ (\alpha_1(u)) = f_i(a) = u \neq c = \alpha'_1(u)$, and $(f_i \circ \alpha_2)(0) = f_i \circ (\alpha_2(0)) = f_i(b) = u \neq c = \alpha'_2(0)$. It follows that $f \circ \alpha_1 \neq \alpha'_1$ and $f \circ \alpha_2 \neq \alpha'_2$, for all $f \in \mathbf{Lalg}(X, Y)$, so that the pair (X, α_1, α_2) is not a co-product of the family (X_1, X_2).

A category $\mathbf{C}'$ is a subcategory of a category $\mathbf{C}$ if the following conditions are satisfied: (i) $\mathbf{Ob}(C') \subseteq \mathbf{Ob}(C)$; (ii) $\mathbf{C}'(X, Y) \subseteq \mathbf{C}(X, Y)$, for all $X, Y \in \mathbf{Ob}(C')$; (iii) the composition of any two morphisms in $\mathbf{C}'$ is the same as their composition in $\mathbf{C}$; (iv) $\mathbf{1}_X$ is the same in $\mathbf{C}'$ as in $\mathbf{C}$, for all $X \in \mathbf{Ob}(C')$ ([3, Def. 4.1.3]). We can easily check that the category **CLalg** of CL-algebras is a subcategory of **Lalg**.

Theorem 5.5. *The subcategory* **CLalg** *of CL-algebras has co-products.*

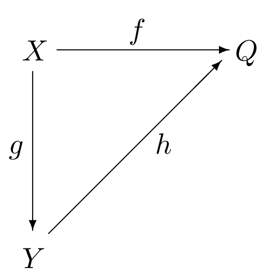
Proof. According to [7, Prop. 2.3], any CL-algebra is a BCK-algebra, so that **CLalg** is also a subcategory of the category **BCK** of BCK-algebras. It was proved in [31] that the category **BCK** has co-products, hence the sucategory **CLalg** also has co-products. □

Open problem 5.6. Investigate whether the category **Lalg** has co-products or not.

6 On the Injective Objects in the Category Lalg

In this section, we introduce the notion of divisible cyclic L-algebras and prove that the cyclic L-algebras and MV-algebras are categorial equivalent. The main result consists of proving that an object X in the category **CyLalg** of cyclic L-algebras is injective if and only if X is a complete and divisible cyclic L-algebra.
Using an idea from [12] we prove that $\{1\}$ is the only injective object in the category **Lalg**.

An object Q in a category **C** is called *injective* if for any morphism $f : X \longrightarrow Q$ and any monomorphism $g : X \longrightarrow Y$, there is a morphism $h : Y \longrightarrow Q$ such that $h \circ g = f$.



A *retraction* of a morphism $f : X \longrightarrow Y$ is a morphism $g : Y \longrightarrow X$ such that $f \circ g = Id_Y$. If f has a retraction, then f is a monomorphism ([3, Def. 4.2.6, Prop. 4.2.7]).

Lemma 6.1. *Let* $(X, \rightarrow, 1)$ *be an L-algebra and let* $0 \notin X$*. Then* $(X \cup \{0\}, \rightarrow, 1)$ *is an L-algebra with* 0 *as the smallest element, where* $x \rightarrow 0 = 0$, $0 \rightarrow x = 1$, $0 \rightarrow 0 = 1$, *for any* $x \in X$.

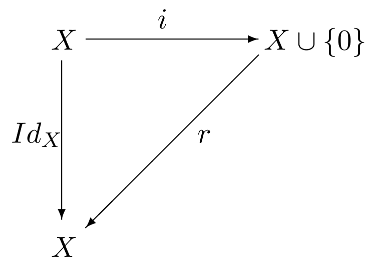
Proof. The proof is straightforward. □

Lemma 6.2. $\{1\}$ *is an injective object in* **Lalg**.

Proof. Obviously, if $f : X \longrightarrow \{1\}$ is a morphism, then $f(x) = 1$, for all $x \in X$. For any monomorphism $g : X \longrightarrow Y$, define the morphism $h : Y \longrightarrow \{1\}$, by $h(y) = 1$, for all $y \in Y$. Then, for any $x \in X$ we have $(h \circ g)(x) = h(g(x)) = 1 = f(x)$, that is $h \circ g = f$. Hence $\{1\}$ is an injective object in **Lalg**. □

Theorem 6.3. *An object* X *in* **Lalg** *is injective if and only if* $X = \{1\}$.

Proof. By Lemma 6.2, $\{1\}$ is an injective object in **Lalg**. Conversely, assume that X is an injective object in **Lalg**. Consider the L-algebra $X \cup \{0\}$ from Lemma 6.1 and let $i : X \longrightarrow X \cup \{0\}$ be the inclusion map. Obviously i is injective, so that i is a monomorphism. Since X is an injective object, there exists a retraction $r : X \cup \{0\} \longrightarrow X$ such that $r \circ i = Id_X$.



Then $r(x) = x$ for any $x \in X$, and let $y = r(0)$. It follows that $y = r(0) = r(y \to 0) = r(y) \to r(0) = y \to y = 1$, that is $r(0) = 1$. For any $x \in X$, we have $1 = r(0) = r(0 \to x) = r(0) \to r(x) = 1 \to x = x$. We conclude that $X = \{1\}$. $\square$

Theorem 6.4. *The cyclic L-algebras and MV-algebras are categorial equivalent.*

Proof. Denote by **MValg** and **CyLalg** the categories of MV-algebras and cyclic L-algebras, respectively. In order to prove the categorial equivalence, with the notations from Section 3 we define two functors $\mathbf{\Phi} : \mathbf{MValg} \longrightarrow \mathbf{CyLalg}$, $\mathbf{\Psi} : \mathbf{CyLalg} \longrightarrow \mathbf{MValg}$. by $\mathbf{\Phi}(X, \oplus, 0) = (X, \to, 0, 1)$, $\mathbf{\Psi}(X, \to, 0, 1) = (X, \oplus, 0)$, $\mathbf{\Phi}(f)(x) = f(x)$, $\mathbf{\Psi}(g)(x) = g(x)$, for any $(X, \oplus, 0) \in \mathbf{Ob}(MValg)$, $(X, \to, 0, 1) \in \mathbf{Ob}(CyLalg)$, $f \in \mathbf{MValg}(X, Y)$, $g \in \mathbf{CyLalg}(X, Y)$, $x \in X$. By Theorem 3.8, $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are mutually inverse, hence **MValg** and **CyLalg** are categorial equivalent. $\square$

Let $(X, \oplus, 0)$ be an MV-algebra. For any $x \in X$ and $n \in \mathbb{N}$, define $0x = 0$ and $nx = x \oplus (n-1)x$, for $n \geq 1$. An MV-algebra X is called *divisible* if for any $a \in X$ and for any $n \in \mathbb{N}$, there is $x \in X$ such that $nx = a$ and $a^- \oplus (n-1)x = x^-$.

Theorem 6.5. *([27]) For any MV-algebra X the following are equivalent:*
(a) X is an injective object in the category **MValg***;*
(b) X is complete and divisibile MV-algebra.

Definition 6.6. A cyclic L-algebra $(X, \to, 0, 1)$ is called *divisible* if its corresponding MV-algebra $(X, \oplus, 0)$ is divisible.

Theorem 6.7. *For any cyclic L-algebra X the following are equivalent:*
(a) X is an injective object in the category **CyLalg***;*
(b) X is a complete and divisibile cyclic L-algebra.

Proof. It follows by Theorems 6.7 and 6.4. $\square$

7 Concluding Remarks

Studying the L-algebras is a topic of great current interest; motivated by this fact, in this paper we define and study the category **Lalg** of L-algebras. We prove that this category has equalizers, coequalizers, kernel pairs and products, and we investigate the existence of injective objects in **Lalg**. We prove that an object of the subcategory of cyclic L-algebras is injective if and only it is a complete and divisible cyclic L-algebra. It was proved in [7, Rem. 4.12] that any Hilbert algebra is an L-algebra, so the category **Halg** of Hilbert algebras is a subcategory of **Lalg**. According to [13], the category **Halg** has co-products. We give an example of two L-algebras having a co-product, but we leave as an open problem whether the category of L-algebras has co-products or not.

Dvurečenskij and Zahiri studied the epicomplete objects in the category of MV-algebras ([10]), and they found a relation between injective MV-algebras and epicomplete MV-algebras. As another topic of research, one could investigate the epicomplete objects in various subcategories of **Lalg**.

Conflict of Interest: The author declares that there are no conflict of interest.

References

[1] S. S. Ahn, On injective BCK-algebras, *Honam Math. J.,* 29 (2007), 289-297.

[2] N. Akhlaghinia, R. A. Borzooei, M. A. Kologani and X. L. Xin, On the category of EQ-algebras, *Bull. Sect. Logic*, https://doi.org/10.18778/0138-0680.2021.01, (2021).

[3] B. Buşneag, Categories of algebraic logic, *Editura Academiei Romane, Bucharest,* (2006).

[4] B. Buşneag and M. Ghiţă, Some properties of epimorphisms of Hilbert algebras, *Centr. Eur. J. Math.,* 8 (2010), 41-52.

[5] C. C. Chang, Algebraic analysis of many-valued logic, *Trans. Amer. Math. Soc.,* 88 (1958), 467-490.

[6] M. A. Chaudhry and H. Fakhar-Ud-Din, Some categorical aspects of BCH-algebras, *IJMMS,* 27 (2003), 1739-1750.

[7] L. C. Ciungu, Results in L-algebras, *Algebra Univ.,* 82(1) (2021), Paper no. 7.

[8] L. C. Ciungu, Quantifiers on L-algebras, *Math. Slovaca*, accepted for publication.

[9] A. Dvurečenskij and S. Pulmannova, New trends in Quantum Structures, *Kluwer Academic Publishers, Dordrecht, Ister Science, Bratislava,* (2000).

[10] A. Dvurečenskij and O. Zahiri, On epicomplete MV-algebras, *FLAP,* 5 (2018), 165-184.

[11] G. Dymek, On the category of pseudo BCI-algebras, *Demonstr. Math.,* 4 (2013), 631-644.

[12] G. Dymek, An injective pseudo-BCI algebra is trivial, *Discuss. Math. Gen. Algebra Appl.,* 39 (2019) 221-229.

[13] M. Ghiţă, Some categorical properties of Hilbert algebras, *An. Univ. Craiova Ser. Mat. Inform.,* 36 (2009), 95-104.

[14] X. J. Hua, State L-algebras and derivations of L-algebras, *Soft Comput.,* 25 (2021), 4201-4212.

[15] A. Iorgulescu, Classes of BCK-algebras - Part III, *Preprint Series of the Institute of Mathematics of the Romanian Academy,* 3 (2004), 1-42.

[16] B. Mitchell, Theory of Categories, *Pure and Applied Mathematics, vol. XVII, Academic Press, New York,* (1965).

[17] N. Popescu and A. Radu, Theory of categories and sheaves, *Editura Ştiinţifică, Bucharest, (in Romanian),* (1971).

[18] L. Popescu and N. Popescu, Theory of categories, *Editura Academiei Romane, Bucharest and Sijthoff & Noordhoff International Publishers,* (1979).

[19] W. Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang–Baxter equations, *Adv. Math.,* 193 (2005), 40-55.

[20] W. Rump, L-algebras, self-similarity, and ℓ-groups, *J. Algebra,* 320 (2008), 2328-2348.

[21] W. Rump and Y. Yang, Intervals in ℓ-groups as L-algebras, *Algebra Univ.,* 67 (2012), 121-130.

[22] W. Rump, Right ℓ-groups, geometric Garside groups, and solutions of the quantum Yang–Baxter equation, *J. Algebra,* 439 (2015), 470-510.

[23] W. Rump, The structure group of an L-algebra is torsion-free, *J. Group Theory,* 20 (2017), 309-324.

[24] W. Rump, The L-algebra of Hurwitz primes, *J. Number Theory,* 190 (2018), 394-413.

[25] W. Rump, Von Neumann algebras, L-algebras, Baer*-monoids, and Garside groups, *Forum Math.,* 30 (2018), 973-995.

[26] W. Rump, The structure group of a generalized orthomodular lattice, *Stud. Logica,* 106 (2018), 85-100.

[27] E. Turunen, Mathematics Behind Fuzzy Logic, *Physica-Verlag,* (1999).

[28] Y. Wu, J. Wang and Y. Yang, Lattice-ordered effect algebras and L-algebras, *Fuzzy Sets Syst.,* 369 (2019), 103-113.

[29] Y. Wu and Y. Yang, Orthomodular lattices as L-algebras, *Soft Comput.,* 24 (2020), 14391-14400.

[30] Y. Yang and W. Rump, Pseudo-MV algebras as L-algebras, *J. Mult.-Valued Logic Soft Comput.,* 19 (2012), 621-632.

[31] H. Yutani, Colimits in the category of BCK-algebras, *Math. Japonica,* 30 (1985), 527-534.

[32] S. M. A. Zaidi and S. Khan, The Category of p-semisimple BCI-algebra, *Kyungpook Math. J.,* 43 (2003), 279-291.

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