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C*-Algebra-Valued Fuzzy Metric Spaces in Coupled Fixed Point with Applications

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ABSTRACT

Coupled fixed point has been a major area of research interest in the field of fixed point theory. This work establishes coupled fixed point for C*-algebra-valued contractions in fuzzy metric spaces. Also, application in integral equations to back up our claim was given. The results of our findings are an improvement to some existing works in the literature.

1 Introduction

The introduction of fixed point theorem in metric space and the Banach contraction principle has played a significant role in the field of mathematical analysis and its applications [1]. Hence, numerous theories have developed in extending its notion in many diverse ways. Accordingly, if T is a contraction on a Banach space X, then T has a unique fixed point in X. Many researchers studied the Banach fixed point theorem in diverse ways and show its generalizations and applications. Among them, Bakhtin [2] introduced a very important generalization of the idea of a metric space, which is later used by Czerwick [3-4] to present the findings of their work.

Fixed point theorems have been studied in many contexts, one of which is the fuzzy setting. The concept of fuzzy sets was initially introduced by Zadeh [8] in 1965. To use this concept in topology and analysis, so many authors have extensively developed the theory of fuzzy sets and its applications. One of the most significant and interesting work in fuzzy topology is to appropriately find the definition of fuzzy metric space and its applications. It is well known that a fuzzy metric space is an important generalization of the metric space. Many authors have considered this problem and have introduced it in different ways as there remains considerable literature about

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fixed point properties for mappings defined on fuzzy metric spaces, which have been examined by many researchers (see [9-12]).

Zhu and Xiao [7] and Hu [6] gave a coupled fixed point theorem for contractions in fuzzy metric spaces, and Fang [5] proved some common fixed point theorems under ϕ-contractions for compatible and weakly compatible mappings on Menger probabilistic metric spaces. Moreover, Elagan and Segi Rahmat [13] studied the existence of a fixed point in locally convex topology generated by fuzzy n-normed spaces.

In this paper, our main aim is to generalize and extend the work done in [14] on fixed point theorems in C^*-algebra valued fuzzy metric spaces to coupled fixed point theorems in C^*-algebra valued fuzzy metric spaces with application. The main result proved further generalizes and integrates results present in the literature.

2 Preliminaries

In this section, we will discuss few of the basic concepts of C^*-algebra-valued metric space.

Definition 2.1 [14] Let X be an arbitrary nonempty set, Δ a continuous ι-norm, and M a fuzzy set on $\mathrm{X}^2 \times (0,\infty)$. The 3-tuple $(\mathrm{X}, \mathrm{M}, \Delta)$ is called a fuzzy metric space if the following conditions are satisfied $\forall\ \lambda, \mu, \theta \in \mathrm{X}$ and $\iota, \phi > 0$,

(i) $M(\lambda,\mu,\iota) > 0$,
(ii) $M(\lambda,\mu,\iota) = 1$ if and only if $\lambda = \mu\ \forall\ \iota > 0$,
(iii) $M(\lambda,\mu,\iota) = M(\mu,\lambda,\iota)$,
(iv) $M(\lambda,\mu,\iota)\Delta M(\mu,\theta,\phi) \le M(\lambda,\theta,\iota+\phi)\ \forall\ \iota,\phi > 0$,
(v) $M(\lambda,\mu,\cdot): (0,\infty) \to [0,1]$ is continuous,
(vi) $\lim_{\iota\to\infty} M(\lambda,\mu,\iota) = 1\ \forall\ \lambda,\mu \in \mathrm{X}$.

Definition 2.2 [16] Let $(\mathrm{X}, \mathrm{M}, \Delta)$ be a fuzzy metric space. A sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ in X is said to converge to $\lambda \in \mathrm{X}$ if $\lim_{\iota\to\infty} M(\lambda_n,\mu,\iota) = 1\ \forall\ \iota > 0$.

Lemma 2.1 [16] Let $(\mathrm{X}, \mathrm{M}, \Delta)$ be a fuzzy metric space and $\{\lambda_n\}$, $\{\mu_n\}$ are sequences in X such that $\lambda_n \to \lambda$, $\mu_n \to \mu$ then $M(\lambda_n,\mu_n,\iota) \to M(\lambda,\mu,\iota)$ for every continuity point ι of $M(\lambda,\mu,\cdot)$.

Definition 2.3 [15] Let X be a nonempty set. Suppose that the mapping $\delta_{\mathrm{A}}: \mathrm{X}^2 \to \mathbb{A}$ satisfies:

(i) $\delta_{\mathrm{A}}(\lambda,\mu) > 0_{\mathrm{A}}$,
(ii) $\delta_{\mathrm{A}}(\lambda,\mu) = 0_{\mathrm{A}}$ if and only if $\lambda = \mu$,
(iii) $\delta_{\mathrm{A}}(\lambda,\mu) = \delta_{\mathrm{A}}(\mu,\lambda)$,
(iv) $\delta_{\mathrm{A}}(\lambda,\mu) \le \delta_{\mathrm{A}}(\lambda,\theta) + \delta_{\mathrm{A}}(\theta,\mu)\ \forall\ \lambda,\mu,\theta \in \mathrm{X}$.

Definition 2.4 [15] Let $(\mathrm{X}, \mathbb{A}, \delta_{\mathrm{A}})$ be a C*-algebra-valued metric space. A mapping $\mathrm{T}: \mathrm{X} \to \mathrm{X}$ is called a C*-algebra-valued contraction mapping on X, if exists $\beta \in \mathbb{A}$ with $\|\beta\| < 1$ such that

$$\delta_{\mathrm{A}}(\mathrm{T}\lambda, \mathrm{T}\mu) \le \beta^* \delta_{\mathrm{A}}(\lambda,\mu)\beta,\ \ \forall\ \lambda,\mu,\theta \in \mathrm{X}.$$

3 Main Results

In this part of the research, firstly we defined the concepts of C^*-algebra valued contraction mapping (C^*-AVCM) in metric and fuzzy metric spaces.

Definition 3.1 Let $(\mathrm{X}, \mathbb{A}, \delta)$ be a C^*-algebra valued metric space (C^*-AVMS). A mapping $\Psi: \mathrm{X}^2 \to \mathrm{X}$ is a C^*-algebra valued contraction mapping on X, if $\beta \in \mathbb{A}$ with $\|\beta\| < 1$ such that

$$\delta_{\mathbb{A}}\big(\Psi(\lambda, \mu), \Psi(\mu, \lambda)\big) \leq \beta^* \delta_{\mathrm{A}}(\lambda, \mu)\beta, \quad \forall\, \lambda, \mu \in \mathrm{X}.$$

Definition 3.2 Let X be a nonempty set and $(\mathrm{X}, \mathbb{A}, \mathrm{F}_{\mathrm{A}}, \Delta)$ be a FMS. A mapping $\Psi: \mathrm{X}^2 \to \mathrm{X}$ is said to be a C^*-AVCM if there exists $\beta \in \mathbb{A}$ with $\|\beta\| < 1$ such that

$$\frac{1}{\mathrm{F}_{\mathrm{A}}(\Psi(\lambda, \mu), \Psi(\mu, \lambda), \iota)} - 1 \leq \beta^* \left(\frac{1}{\mathrm{F}_{\mathrm{A}}(\lambda, \mu, \iota)} - 1\right)\beta \tag{1}$$

$\forall\, \lambda, \mu \in \mathrm{X}$ and $\iota > 0$.

Theorem 3.1 Let $(\mathrm{X}, \mathbb{A}, \mathrm{F}_{\mathrm{A}}, \Delta)$ be a Cauchy FMS. A mapping $\Psi: \mathrm{X}^2 \to \mathrm{X}$ is a C^*-AVCM if Ψ has a unique fixed point in X.

Proof. Suppose that $\beta \neq 0_{\mathrm{A}}$,
Let $\lambda_0 \in \mathrm{X}$. Then, the sequence $\{\lambda_n\}_{n\geq 0}$ for $\lambda_{n+1} = \Psi_{\lambda_n} = \Psi^{\mathrm{n}}{}_{\lambda_0}$ and going by C^*-algebra if $\alpha_1, \alpha_2 \in \mathbb{A}^+$ and $\alpha_1 \leq \alpha_2$, hence $\omega^*\alpha_1\omega \leq \omega^*\alpha_2\omega \quad \forall\ \omega \in \mathbb{A}$. Then,

$$\begin{aligned}
\frac{1}{\mathrm{F}_{\mathrm{A}}(\lambda_n, \lambda_{n+1}, \iota)} - 1 &= \frac{1}{\mathrm{F}_{\mathrm{A}}(\Psi(\lambda_{n-1}, \mu_{n-1}), \Psi(\lambda_n, \mu_n), \iota)} - 1 \\
&\leq \beta^* \left(\frac{1}{\mathrm{F}_{\mathrm{A}}(\lambda_{n-1}, \lambda_n, \iota)} - 1\right)\beta \\
&\leq (\beta^*)^2 \left(\frac{1}{\mathrm{F}_{\mathrm{A}}(\lambda_{n-2}, \lambda_{n-1}, \iota)} - 1\right)\beta^2 \\
&\ \ \vdots \\
&\leq (\beta^*)^n \left(\frac{1}{\mathrm{F}_{\mathrm{A}}(\lambda_0, \lambda_1, \iota)} - 1\right)\beta^n \\
&= (\beta^*)^n \mathbb{F} \beta^n,
\end{aligned} \tag{2}$$

where $\mathbb{F} = \frac{1}{\mathrm{F}_{\mathrm{A}}(\lambda_0, \lambda_1, \iota)} - 1$.
If $n + 1 > m$, and on applying triangular inequality of fuzzy metric spaces, we obtain

$$\frac{1}{F_A(\lambda_m, \lambda_{n+1}, \iota)} - 1$$
$$\leq \left[\frac{1}{F_A(\lambda_m, \lambda_{m+1}, \iota)} - 1\right] + \left[\frac{1}{F_A(\lambda_{m+1}, \lambda_{m+2}, \iota)} - 1\right] + \cdots + \left[\frac{1}{F_A(\lambda_{n-1}, \lambda_n, \iota)} - 1\right]$$
$$+ \left[\frac{1}{F_A(\lambda_n, \lambda_{n+1}, \iota)} - 1\right]$$
$$\leq (\beta^*)^m \mathbb{F}\beta^m + (\beta^*)^{m+1}\mathbb{F}\beta^{m+1} + \cdots + (\beta^*)^n \mathbb{F}\beta^n$$
$$= \sum_{j=m}^{n} (\beta^*)^j \mathbb{F}\beta^j$$
$$= \sum_{j=m}^{n} (\beta^*)^j \mathbb{F}^{1/2}\mathbb{F}^{1/2}\beta^j$$
$$= \sum_{j=m}^{n} \left(\mathbb{F}^{1/2}\beta^j\right)^* \left(\mathbb{F}^{1/2}\beta^j\right)$$
$$= \sum_{j=m}^{n} \left|\mathbb{F}^{1/2}\beta^j\right|^2$$
$$\leq \left\|\sum_{j=m}^{n} \left|\mathbb{F}^{1/2}\beta^j\right|^2\right\| \mathrm{T}$$
$$\leq \sum_{j=m}^{n} \left\|\mathbb{F}^{1/2}\right\|^2 \cdot \left\|\beta^j\right\|^2 \mathrm{T}$$
$$\leq \left\|\mathbb{F}^{1/2}\right\|^2 \sum_{j=m}^{n} \left\|\beta^j\right\|^2 \mathrm{T}$$
$$\leq \|\mathbb{F}\| \cdot \frac{\|\beta\|^{2m}}{1 - \|\beta\|} T \longrightarrow 0_A, \qquad as \quad m \to \infty.$$

Then, $\{\lambda_n\}_{n\geq 0}$ is a Cauchy sequence in X with respect to $\mathbb{A}$. Then $(\mathrm{X}, \mathbb{F}, \Delta)$ is Cauchy and for $\lambda \in \mathrm{X}$ such that $\lim_{n\to\infty} \mathbb{F}_A(\lambda_n, \lambda, \iota) = 1$, i.e., $\lim_{n\to\infty} \mathrm{F}_A(\Psi\lambda_{n-1}, \lambda, \iota) = 1$.

Then,

$$0_A \leq \frac{1}{\mathbb{F}_A(\Psi(\lambda, \mu), \lambda, \iota)} - 1$$
$$\leq \left[\frac{1}{\mathbb{F}_A(\Psi(\lambda, \mu), \Psi(\lambda_n, \mu_n), \iota)} - 1\right] + \left[\frac{1}{\mathbb{F}_A(\Psi(\lambda_n, \mu_n), \lambda, \iota)} - 1\right] + \beta^* \left[\frac{1}{\mathbb{F}_A(\lambda, \lambda_n, \iota)} - 1\right]\beta$$
$$+ \left[\frac{1}{\mathbb{F}_A(\lambda_{n+1}, \lambda, \iota)} - 1\right]$$
$$\leq 0_A, \quad as \;\; n \to \infty.$$

Hence, $\Psi(\lambda, \mu) = \lambda$ and similarly, $\Psi(\mu, \lambda) = \mu$, that is, λ and μ are Coupled fixed point of Ψ.

At this point, we are going to show that λ and μ are unique Coupled fixed point. If $\omega_1 \neq \lambda$ and $\omega_2 \neq \mu$ be Coupled fixed point of Ψ, then $\Psi(\omega_1,\omega_2) \neq \Psi(\lambda,\mu)$ and $\Psi(\omega_2,\omega_1) \neq \Psi(\mu,\lambda)$. Now, by the contraction condition (1), we obtain,

$$\begin{aligned}\frac{1}{\mathbb{F}_{\mathrm{A}}(\omega_1,\lambda,\iota)}-1 &\leq \left[\frac{1}{\mathbb{F}_{\mathrm{A}}(\Psi(\omega_1,\omega_2),\Psi(\lambda,\mu),\iota)}-1\right]\\ &\leq \beta^*\left[\frac{1}{\mathbb{F}_{\mathrm{A}}(\omega_1,\lambda,\iota)}-1\right]\beta\\ &\leq \left[\frac{1}{\mathbb{F}_{\mathrm{A}}(\omega_1,\lambda,\iota)}-1\right]\beta^*\beta\\ &\leq \left[\frac{1}{\mathbb{F}_{\mathrm{A}}(\omega_1,\lambda,\iota)}-1\right]\|\beta\|^2.\end{aligned}$$

Similarly,

$$\frac{1}{\mathbb{F}_{\mathrm{A}}(\omega_2,\mu,\iota)}-1 \leq \left[\frac{1}{\mathbb{F}_{\mathrm{A}}(\omega_2,\mu,\iota)}-1\right]\|\beta\|^2.$$

Since $\|\beta\|^2 < 1$, it is a contradiction. Therefore, Ψ has unique coupled fixed point. This completes the proof.

4 Applications

In this section, we apply the main results to the existence of the solution of integral equations. Let $\mathrm{X} = \mathrm{L}^\infty(\ell)$ and $\mathrm{H} = \mathrm{L}^\infty(\ell)$ be a Hilbert space, where ℓ represent a set of Lebesque measurable. For $f,g \in \mathrm{X}$, let $\bar{\mathrm{F}}_{\mathrm{A}}(f,g,\iota) = \Pi_{\frac{\iota}{\iota+|f-g|}}$, where $\Pi_h(\lambda) = h.\lambda \ \forall \ \lambda \in \mathrm{H}$. Suppose that $\bar{\mathrm{F}}_{\mathrm{A}}: \ell^4 \times \mathbb{R}^2 \to \mathbb{R}$ there exists $g: \ell^4 \to \mathbb{R}$ is a continuous function, then $\operatorname{Sup}_{\varphi_1 \in \ell} \int |g(\varphi_1,\omega)| d\omega \leq 1$ and $\alpha \in (0,1)$ for $\varphi_1,\varphi_2 \in \ell$ and for $\mu,\omega \in \mathbb{R}$ we obtain

$$\left|\frac{\iota}{\iota+\bar{\mathrm{F}}(\varphi_1,\varphi_2,\mu)} - \frac{\iota}{\iota+\bar{\mathrm{F}}(\varphi_1,\varphi_2,\omega)}\right| \leq \alpha \left|g(\varphi_1,\varphi_2)\left(\frac{\iota}{\iota+\mu}-\frac{\iota}{\iota+\omega}\right)\right|.$$

Hence, the integral equation has λ^* as its unique solution.

$$\lambda(\varphi_1) = \int_\ell \bar{\mathrm{F}}\big(\varphi_1,\varphi_2,\lambda(\varphi_2)\big)d\varphi_2, \quad \varphi_1 \in \ell. \tag{3}$$

Proof. Let $(\mathrm{X},\mathrm{L}(\mathrm{H}),\mathrm{F},\Delta)$ be a Cauchy C^*-AVFMS with respect to $\mathrm{L}(\mathrm{H})$. Suppose $\Psi: \mathrm{X}^2 \to \mathrm{X}$ be a mapping, we have

$$\Psi(\lambda,\mu)(\varphi_1) = \int_\ell \bar{\mathrm{F}}\big(\varphi_1,\varphi_2,\lambda(\varphi_2)\big)d\varphi_2, \quad \varphi_1 \in \ell$$

Hence,

$$\|\mathbb{F}_{\mathrm{A}}(\Psi(\lambda,\mu),\Psi(\mu,\lambda),\iota)\| = \operatorname*{Sup}_{\|h\|=1}\left(\Pi_{\frac{\iota}{\iota+|(\Psi(\lambda,\mu),\Psi(\mu,\lambda))|}} h, h\right)$$

$$
\begin{aligned}
&= \sup_{\|h\|=1} \int_{\ell} \left(\left| \int_{\ell} \left(\frac{\iota}{\iota + \mathrm{F}\big(\varphi_1, \varphi_2, \lambda(\varphi_2)\big)} - \frac{\iota}{\iota + \mathrm{F}\big(\varphi_1, \varphi_2, \mu(\varphi_2)\big)} \right) \right| d\varphi_2 \right) h(\varphi_1)\overline{h(\varphi_1)} d\varphi_1 \\
&\leq \sup_{\|h\|=1} \int_{\ell} \left(\left| \int_{\ell} \left(\frac{\iota}{\iota + \mathrm{F}\big(\varphi_1, \varphi_2, \lambda(\varphi_2)\big)} - \frac{\iota}{\iota + \mathrm{F}\big(\varphi_1, \varphi_2, \mu(\varphi_2)\big)} \right) \right| d\varphi_2 \right) |h(\varphi_1)|^2 d\varphi_1 \\
&\leq \sup_{\|h\|=1} \int_{\ell} \left(\left| \int_{\ell} \alpha g(\varphi_1, \varphi_2) \left(\frac{\iota}{\iota + \lambda(\varphi_2)} - \frac{\iota}{\iota + \mu(\varphi_2)} \right) \right| d\varphi_2 \right) |h(\varphi_1)|^2 d\varphi_1 \\
&\leq \alpha \sup_{\varphi_1 \in \ell} \int_{\ell} |g(\varphi_1, \varphi_2)| d\varphi_2 \cdot \sup_{\|h\|=1} \int_{\ell} |h(\varphi_1)|^2 d\varphi_1 \cdot \left\| \frac{\iota}{\iota + \lambda(\varphi_2)} - \frac{\iota}{\iota + \mu(\varphi_2)} \right\|_{\infty} \\
&\leq \alpha \left\| \frac{\iota}{\iota + \lambda(\varphi_2)} - \frac{\iota}{\iota + \mu(\varphi_2)} \right\|_{\infty} \\
&\leq \|\beta_1\| \|\mathrm{F_A}(\lambda, \mu, \iota)\|
\end{aligned}
$$

and similarly,

$$\|\mathbb{F}_{\mathrm{A}}(\Psi(\mu, \lambda), \Psi(\lambda, \mu), \iota)\| \leq \|\beta_2\| \|\mathrm{F_A}(\mu, \lambda, \iota)\|.$$

Since, $\|\beta_1\| < 1$ and $\|\beta_2\| < 1$, hence $\lambda^* \in \mathrm{X}$ and $\mu^* \in \mathrm{X}$ are the unique solutions of the integral equation.

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