Essential Norm of Composition Operator as a Mapping on Weighted Bergman Space with Regular Weight

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ABSTRACT

The main purpose of this paper is to establish a unified approach to some criteria for the boundedness and some quantities for the essential norm of composition operator acting on the weighted Bergman space $\mathcal{A}^p_\omega$, where the weight function ω belongs to the class $\mathcal{R}$ of regular weights. We express the essential norm in terms of the generalized Nevanlinna counting function associated with analytic self map φ and regular weight ω.

1 Introduction

Let $\mathbb{D}$ denote the unit disk in the complex plane $\mathbb{C}$, and $\mathcal{H}ol(\mathbb{D})$ denote the class of all analytic functions on $\mathbb{D}$.

A positive integrable function ω over $\mathbb{D}$, is called a weight function or simply a weight. It is radial, if $\omega(z) = \omega(|z|)$, for all $z \in \mathbb{D}$. The distortion function of a radial weight $\omega : [0,1) \to (0,\infty)$ is defined by

$$\psi_\omega(r) = \frac{1}{\omega(r)} \int_r^1 \omega(s)ds, \quad 0 \leqslant r < 1,$$

and was introduced by siskakis in [15]. A radial weight ω is called regular, if ω is continuous and its distortion function satisfies

$$\psi_\omega(r) \asymp (1-r), \quad 0 \leqslant r < 1.$$

The class of all regular weights is denoted by $\mathcal{R}$. The standard weights $\omega_\alpha(r) = (1-r)^\alpha$ are regular for $-1 < \alpha < \infty$. For the theory of the regular weights, see [10].

As usual, for two positive real-valued functions f_1 and f_2 we write $f_1 \preceq f_2$, if there exist a positive constant C independent of the parameters such that $f_1 \leqslant C f_2$, and from $f_1 \asymp f_2$ is understood that both $f_1 \preceq f_2$ and $f_2 \preceq f_1$ hold.

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If $\omega \in \mathcal{R}$, several observations on [10, Lemma1.1] and its proof, implies that

$$\frac{\omega(r)}{\omega(t)} \asymp 1, \quad 0 \leqslant r \leqslant t \leqslant r + s(1-r) < 1, \tag{1.1}$$

where the constants of comparison depend on $s \in [0,1)$ and ω. This implies

$$\psi_\omega(r) \geqslant C(1-r), \quad 0 \leqslant r < 1,$$

for some constant $C > 0$. However does not imply the existence of $C > 0$ such that $\psi_\omega(r) \leqslant C(1-r)$, $0 \leqslant r < 1$. Also a radial continuous weight ω is regular if and only if there exist $-1 < k < h < \infty$ and $\delta \in (0,1)$ such that

$$\frac{\omega(r)}{(1-r)^h} \nearrow \infty, \quad \frac{\omega(r)}{(1-r)^k} \searrow o, \quad r \in [\delta, 1).$$

These weights without the continuity assumption in the range $0 < k < h < \infty$ are known as normal weights [14].

Here $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$, $a \in \mathbb{D}$, denotes the automorphism of $\mathbb{D}$ which interchanges the origin and the point a. Such a map satisfies the identity

$$|\sigma_a'(z)| = \frac{1-|\sigma_a(z)|^2}{1-|z|^2} = \frac{1-|a|^2}{|1-\bar{a}z|^2}.$$

The pseudohyperbolic distance from z to a is defined as $\rho(z,a) = |\sigma_a(z)|$ and the pseudohyperbolic disc of center $a \in \mathbb{D}$ and radius $r \in (0,1)$ is denoted by $\Delta(a,r) = \{z : \rho(z,a) < r\}$.

The weight ω (that is not necessarily radial neither continuous) is an invariant weight, if for all $r \in (0,1)$ there exists a constant $C = C(r) \geqslant 1$ such that $C^{-1}\omega(a) \leqslant \omega(z) \leqslant C\omega(a)$, for all $z \in \Delta(a,r)$. The class of all invariant weight is denoted by $\mathcal{I}nv$. In other words, $\omega \in \mathcal{I}nv$ if $\omega(z) \asymp \omega(a)$ in $\Delta(a,r)$. Note that radial invariant weights are neately characterized by the condition (1.1), and thus $\mathcal{R} \cap \mathcal{I}nv = \mathcal{R}$.

We recall that the Carleson square $S(I)$ associated with an interval $I \subset \mathbb{T}$ is the set

$$S(I) = \{re^{i\theta} \in \mathbb{D} : e^{i\theta} \in I, 1-|I| \leqslant r < 1\},$$

where $|I|$ denots the Lebesgue measure of the measurable set I. For each $z \in \mathbb{D} \setminus \{0\}$ define the interval $I_z = \{e^{i\theta} : |arg(ze^{-i\theta})| \leqslant \frac{1-|z|}{2}\}$, and denote $S(z) = S(I_z)$.

For each radial weight ω, the associated weight ω^* is defined by

$$\omega^*(z) = \int_{|z|}^{1} \omega(s) \log \frac{s}{|z|} s ds, \quad z \in \mathbb{D} \setminus \{0\}.$$

Here, as usual, $\omega(E) = \int_E \omega(z) dA(z)$ for each measurable set $E \subset \mathbb{D}$. Pelaez and Rattya [10, Lemma 1.6] established that if ω be a radial weight, then

$$\omega(T(z)) \asymp \omega^*(z), \quad |z| > \frac{1}{2}, \tag{1.2}$$

and for a regular weight ω,

$$\omega(T(z)) \asymp \omega(S(z)), \quad z \in \mathbb{D},$$

where, the nontangential regions are defined by

$$\Gamma(a)=\{z\in\mathbb{D}:|\theta-argz|<\frac{1}{2}(1-\frac{|z|}{r})\},\quad a=re^{i\theta}\in\bar{\mathbb{D}}\setminus\{0\},$$

and the tents

$$T(z)=\{a\in\mathbb{D}:z\in\Gamma(a)\},\quad z\in\mathbb{D}.$$

Namely, if ω is a regular weight, we have

$$\omega^*(r)\asymp\omega(S(r))=\frac{1-r}{\pi}\int_r^1\omega(s)sds\leqslant\frac{1-r}{\pi}\psi_\omega(r)\omega(r)\asymp(1-r)^2\omega(r), \tag{1.3}$$

where $r\geqslant\frac{1}{2}$.

For $0<p<\infty$, and a weight ω, the weighted Bergman space A^p_ω consists of those $f\in\mathcal{H}ol(\mathbb{D})$ for which

$$\|f\|^p_{A^p_\omega}:=\int_{\mathbb{D}}|f(z)|^p\omega(z)dA(z)<\infty,$$

where $dA(z)=\frac{dxdy}{\pi}$ stands for the normalized Lebesgue area measure on $\mathbb{D}$. As usual for the standard radial weights $\omega(z)=(1-|z|^2)^\alpha$, $\alpha>-1$, A^p_ω is the well known classical Bergman space $\mathcal{A}^p_\alpha(\mathbb{D})$ (see, e.g., [3]). Let $\omega\in\mathcal{R}$, and let $C_1=C_1(\omega)>0$ and $C_2=C_2(\omega)>C_1$ such that

$$C_1(1-r)\leqslant\psi_\omega(r)\leqslant C_2(1-r),\quad 0\leqslant r<1.$$

This inequality shows that the differentiable functions $f_{C_1}(r)=(1-r)^{-\frac{1}{C_1}}\int_r^1\omega(s)ds$ and $f_{C_2}(r)=(1-r)^{-\frac{1}{C_2}}\int_r^1\omega(s)ds$ are increasing and decreasing on [0,1) respectively. Therefore

$$(1-r)^{-\frac{1}{C_1}}\preceq\int_r^1\omega(s)ds\preceq(1-r)^{-\frac{1}{C_2}}.$$

The above result with $\psi_\omega(r)\asymp(1-r)$ gives

$$\mathcal{A}^p_\alpha(\mathbb{D})\subset A^p_\omega(\mathbb{D})\subset\mathcal{A}^p_\beta(\mathbb{D}),$$

for $\alpha=C_2^{-1}-1$ and $\beta=C_1^{-1}-1$. This explains that for regular weight ω, the weighted Bergman space A^p_ω lies between two classical weighted Bergman spaces. Of course the weight,

$$\omega(r)=|\sin(\log\frac{1}{1-r})|(1-r)^\alpha+(1-r)^\beta,$$

establish that if ω is a radial continuous weight such that the chain of inclusions above is satisfied for some $-1<\alpha<\beta<\infty$, then ω does not need to be regular [10].

Note that when $\omega\in\mathcal{R}$, ω is comparable to the differentiable weight $\frac{\int_r^1\omega(s)ds}{(1-s)}$, so, by using [9, Theorem 1.1], we deduce

$$\|f\|^p_{A^p_\omega} \asymp |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p(1-|z|)^p\omega(z)dA(z), \quad f \in \mathcal{H}ol(\mathbb{D}). \tag{1.4}$$

Another equivalent form of the norm on A^p_ω is

$$\|f\|^p_{A^p_\omega} = p^2 \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2\omega^*(z)dA(z) + \omega(\mathbb{D})|f(0)|^p, \tag{1.5}$$

in partiqular,

$$\|f\|^2_{A^2_\omega} = 4\|f'\|^2_{A^2_{\omega^*}} + \omega(\mathbb{D})|f(0)|^2,$$

see [10, Theorem 4.2].

Let φ be a holomorphic self map of $\mathbb{D}$. For $f \in \mathcal{H}ol(\mathbb{D})$, denote the composition $f \circ \varphi$ by $C_\varphi f$ and call C_φ the composition operator induced by φ.

Let X and Y be Banach spaces and $\|.\|$ denots the usual operator norm. For a bounded linear operator $T : X \to Y$, the essential norm $\|T\|_{e,X\to Y}$ is defined to be the distance from T to the set of the compact operators K, namely,

$$\|T\|_{e,X\to Y} = \inf\{\|T-K\| : K \quad is \quad compact \quad from \quad X \quad into \quad Y\}.$$

Clearly, T is compact if and only if $\|T\|_{e,X\to Y} = 0$. Thus, the essential norm is related to the compactness problem of concrete operators.

Recently, the essential norm of the composition operator between spaces of analytic functions was investigated (see, [2, 4, 7]). So, by building on those foundations, the present paper continues this line of research.

Shapiro [13] expressed the essential norm of the composition operator $C_\varphi : A^2_\alpha \to A^2_\alpha$ in terms of the generalized Nevanlinna counting function $N_{\varphi,\gamma}$, where for analytic self map φ and $\gamma > 0$, $N_{\varphi,\gamma}$ is defined by

$$N_{\varphi,\gamma}(z) = \sum_{\xi\in\varphi^{-1}\{z\}} [\log(\frac{1}{|\xi|})]^\gamma, \quad z \in \mathbb{D} \setminus \{\varphi(0)\}.$$

Thus $N_{\varphi,1}$ is the natural Nevanlinna counting function N_φ.

Smith [16] used these generalized Nevanlinna counting functions to characterize those ☐ that induce bounded and compact composition operators between weighted Bergman and Hardy spaces. Also Luo and Chen [8] used generalized Nevanlinna counting functions to computed essential norms of $C_\varphi : A^p_\alpha(\mathbb{D}) \to A^q_\beta(\mathbb{D})$. Gonzaleaz and others in [5] presented a unified approach to some criteria for the boundedness and compactness of composition operators between classical weighted Bergman spaces. We generalize these results for weighted bergman spaces with regular weights. An important ingredient in our study, is the use of $N_{\varphi,\omega}$, the generalized Nevanlinna counting function associated with φ, ω as follows:

Let φ be an analytic self map of $\mathbb{D}$ and ω be a regular weight with associated weight ω^*. The generalized Nevanlinna counting function associated with φ, ω is defined by

$$N_{\varphi,\omega}(z) = \sum_{\varphi(\xi)=z,\xi\in\mathbb{D}} \omega^*(\xi), \quad z \in \mathbb{D} \setminus \{\varphi(0)\}.$$

Define $N_{\varphi,\omega}(z) = 0$ when $z = \varphi(0)$, and recall that $N_{\varphi,\omega}(z) = 0$, when $z \in \mathbb{D} \setminus \varphi(\mathbb{D})$ [11]. The weight $\omega_0(r) = 1$ is a regular weight, and $\omega_0^*(r) \asymp (1-r)$, so

$$N_{\varphi,\omega_0}(z) = \sum_{\varphi(\xi)=z,\xi\in\mathbb{D}} \omega_0^*(\xi) \asymp \sum_{\varphi(\xi)=z,\xi\in\mathbb{D}} (1-|\xi|) \asymp N_{\varphi,1}(z).$$

We have the following general change of variable formula that is used in order to obtain an equivalent form of the norm on A^p_ω.

Lemma 1.1. *[1] Let φ be a nonconstant analytic function in $\mathbb{D}$ and f, h be nonnegative measurable functions on $\mathbb{C}$ with respect to area measure. Then*

$$\int_{\mathbb{D}} (f \circ \varphi) h |\varphi'|^2 dA(z) = \int_{\varphi(\mathbb{D})} f(\xi) \left(\sum_{\varphi(z)=\xi} h(z) \right) dA(\xi) \tag{1.6}$$

Note that, throughout the remainder of this paper, C will stand for a positive constant and may differ from one occurrence to the other and we denote by $D(z,r)$ the disc of radius r centered at z.

2 Main results

We mainly characterize the essential norm of the composition operators between weighted bergman spaces with regular weight. It is also worth noting that Shapiro [13], proved that for every analytic self-map φ and for every disc Δ of center ξ which does not contain $\varphi(0)$, we have

$$N_\varphi(\xi) \leqslant \frac{1}{A(\Delta)} \int_\Delta N_\varphi(z) dA(z). \tag{2.1}$$

In order to prove the theorems, we shall need the following lemma. The proof relies on the fact (2.1).

Lemma 2.1. *Let φ be an analytic self map of $\mathbb{D}$ and ω be a regular weight. If $\varphi(0) = 0$, then the generalized Nevanlinna counting function $N_{\varphi,\omega}$ satisfies the sub-mean value property: for every $r > 0$ and every $\xi \in \mathbb{D}$ such that $D(\xi, r) \subset \mathbb{D} \setminus D(0, \frac{1}{2})$ we have*

$$N_{\varphi,\omega}(\xi) \leqslant \frac{1}{r^2} \int_{D(\xi,r)} N_{\varphi,\omega}(z) dA(z).$$

Proof. Let $\varphi_s(z) = \varphi(sz)$. Using (2.1), we have

$$\begin{aligned} N_{\varphi,\omega}(\xi) = \sum_{\varphi(z)=\xi, z\in\mathbb{D}} \omega^*(z) &= \sum_{\varphi(z)=\xi, z\in\mathbb{D}} \int_{|z|}^1 s\omega(s)\log\frac{s}{|z|}ds \\ &= \int_0^1 (\sum_{\varphi(z)=\xi, |z|\leqslant s} \log\frac{s}{|z|})s\omega(s)ds \\ &= \int_0^1 N_{\varphi_s}(\xi)s\omega(s)ds \\ &\leqslant \int_0^1 (\frac{1}{r^2}\int_{D(\xi,r)} N_{\varphi_s}(z)dA(z))s\omega(s)ds \\ &= \frac{1}{r^2}\int_{D(\xi,r)} (\int_0^1 s\omega(s)N_{\varphi_s}(z)ds)dA(z) \end{aligned}$$

$$\begin{aligned} &\leqslant \frac{1}{r^2}\int_{D(\xi,r)} (\int_0^1 (\sum_{\varphi(\xi)=z, |\xi|\leqslant s} \log\frac{s}{|\xi|})s\omega(s)ds)dA(z) \\ &= \frac{1}{r^2}\int_{D(\xi,r)} \sum_{\varphi(\xi)=z, |\xi|\leqslant s} (\int_{|\xi|}^1 s\omega(s)\log\frac{s}{|\xi|}ds)dA(z) \\ &= \frac{1}{r^2}\int_{D(\xi,r)} (\sum_{\varphi(\xi)=z, |\xi|\leqslant s} \omega^*(\xi))dA(z) \\ &= \frac{1}{r^2}\int_{D(\xi,r)} N_{\varphi,\omega}(z)dA(z) \end{aligned}$$

□

In order to prove the next theorem, we need the well-known estimate (see [6]):

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^c}{|1-\bar{a}z|^{2+c+d}} \asymp \frac{1}{(1-|a|^2)^d}, \quad d>0, c>-1. \tag{2.2}$$

Theorem 2.1. *Let φ be an analytic self map of $\mathbb{D}$ such that $\varphi(0)=0$ and let ω be a regular weight. Then*

$$N_{\varphi,\omega}(z) = O(\omega^*(z)), \quad |z|\to 1, \tag{2.3}$$

if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\sigma_a'(z)|^2}{\omega(S(z))} N_{\varphi,\omega}(z)dA(z) < \infty. \tag{2.4}$$

and moreover,

$$N_{\varphi,\omega}(z) = o(\omega^*(z)), \quad |z|\to 1, \tag{2.5}$$

if and only if

$$\lim_{|a|\to 1}\int_{\mathbb{D}}\frac{|\sigma_a'(z)|^2}{\omega(S(z))}N_{\varphi,\omega}(z)dA(z)=0. \tag{2.6}$$

More precisely,

$$\limsup_{|z|\to 1}\frac{N_{\varphi,\omega}(z)}{\omega^*(z)}\asymp\limsup_{|z|\to 1}\int_{\mathbb{D}}\frac{|\sigma_z'(w)|^2}{\omega(S(w))}N_{\varphi,\omega}(w)dA(w). \tag{2.7}$$

Proof. Suppose that (2.3) is satisfied. Then there exists a positive constant C, depending only on r_0, such that for all $r\in(r_0,1)$, by using (1.2), and choose $d=2, c=0$ in (2.2), we deduse

$$\begin{aligned}\int_{\mathbb{D}\setminus D(0,r)}\frac{|\sigma_a'(z)|^2}{\omega(S(z))}N_{\varphi,\omega}(z)dA(z)&\leqslant C\int_{\mathbb{D}\setminus D(0,r)}\frac{|\sigma_a'(z)|^2}{\omega(S(z))}\omega^*(z)dA(z)\\&\asymp C\int_{\mathbb{D}\setminus D(0,r)}\frac{|\sigma_a'(z)|^2}{\omega(S(z))}\omega(S(z))dA(z)\\&\asymp C\int_{\mathbb{D}\setminus D(0,r)}|\sigma_a'(z)|^2dA(z)\\&\preceq(1-|a|^2)^2\int_{\mathbb{D}\setminus D(0,r)}\frac{1}{|1-\bar{a}z|^4}dA(z)\\&\asymp 1.\end{aligned}$$

Conversely, suppose that (2.4) is satisfied. For $a\in\mathbb{D}$ consider the function $\phi=\sigma_a\circ\varphi$. Then $\phi:\mathbb{D}\to\mathbb{D}$ is analytic and

$$N_{\sigma_a\circ\varphi,\omega}(0)=\sum_{(\sigma_a\circ\varphi)(z)=0}\omega^*(z)=\sum_{\varphi(z)=\sigma_a(0)}\omega^*(z)=\sum_{\varphi(z)=a}\omega^*(z)=N_{\varphi,\omega}(a).$$

As well as, by Lemma 2.1 we have

$$N_{\sigma_a\circ\varphi,\omega}(0)\leqslant 4\int_{D(0,\frac{1}{2})}N_{\sigma_a\circ\varphi,\omega}(z)dA(z). \tag{2.8}$$

Since $\omega\in\mathcal{I}nv$, we have $\omega(a)\asymp\omega(z)$ for all $z\in\Delta(a,\frac{1}{2})$. Therefore, the facts $1-|a|\asymp 1-|z|$ and $\omega(a)\asymp\omega(z)$ when

$z \in \Delta(a, \frac{1}{2})$, also the conditions (1.2), (1.3), (2.1), (2.8) imply that:

$$\begin{aligned} N_{\varphi,\omega}(a) &\leqslant 4 \int_{D(0,\frac{1}{2})} N_{\sigma_a \circ \varphi,\omega}(u) dA(u) \\ &= 4 \int_{\Delta(a,\frac{1}{2})} N_{\varphi,\omega}(z) |\sigma'_a(z)|^2 dA(z) \\ &\asymp \int_{\Delta(a,\frac{1}{2})} N_{\varphi,\omega}(z) \frac{|\sigma'_a(z)|^2}{\omega(S(z))} (1-|z|^2)^2 \omega(z) dA(z) \\ &\asymp (1-|a|^2)\omega(a) \int_{\Delta(a,\frac{1}{2})} \frac{|\sigma'_a(z)|^2}{\omega(S(z))} N_{\varphi,\omega}(z) dA(z) \\ &\asymp \omega^*(a) \int_{\Delta(a,\frac{1}{2})} \frac{|\sigma'_a(z)|^2}{\omega(S(z))} N_{\varphi,\omega}(z) dA(z). \end{aligned} \tag{2.9}$$

for a sufficiently close to 1. The equivalence of the conditions (2.5) and (2.6) can be proved in a similar style. Suppose that $A := \limsup_{|z|\to 1} \frac{N_{\varphi,\omega}(z)}{\omega^*(z)}$. Using (2.9), we have

$$\limsup_{|z|\to 1} \frac{N_{\varphi,\omega}(z)}{\omega^*(z)} \preceq \limsup_{|z|\to 1} \int_{\mathbb{D}} \frac{|\sigma'_z(w)|^2}{\omega(S(w))} N_{\varphi,\omega}(w) dA(w).$$

To prove the reverse inequality, given $\varepsilon > 0$, there exists $r_\varepsilon \in (0,1)$ such that for all $|z| \geqslant r_\varepsilon$ we have $\frac{N_{\varphi,\omega}(z)}{\omega^*(z)} \leqslant A+\varepsilon$. Therefore for $r = \max\{\frac{1}{2}, r_\varepsilon\}$, we deduce

$$\begin{aligned} \int_{\mathbb{D}} \frac{|\sigma'_z(w)|^2}{\omega(S(w))} N_{\varphi,\omega}(w) dA(w) &\preceq \int_{D(0,r)} \frac{|\sigma'_z(w)|^2}{\omega(S(w))} N_{\varphi,\omega}(w) dA(w) + A + \varepsilon \\ &\preceq \frac{(1-|z|^2)^2}{(1-r)^4} \int_{D(0,r)} \frac{N_{\varphi,\omega}(w)}{\omega(S(w))} dA(w) + A + \varepsilon \\ &\preceq \frac{(1-|z|^2)^2}{\omega(r)(1-r)^6} \int_{D(0,r)} N_{\varphi,\omega}(w) dA(w) + A + \varepsilon, \end{aligned}$$

and it implies that

$$\limsup_{|z|\to 1} \int_{\mathbb{D}} \frac{|\sigma'_z(w)|^2}{\omega(S(w))} N_{\varphi,\omega}(w) dA(w) \preceq \limsup_{|z|\to 1} \frac{N_{\varphi,\omega}(z)}{\omega^*(z)}.$$

□

In this note the main result of the paper be relevent to essential norm of C_φ. However, before proving this result, we have to make certain the boundedness of C_φ. The following lemma explains the appearance of the non-negative free parameter γ in Theorems 2.2 and 2.3.

Lemma 2.2. *[10] Let ω be a regular weight, then there exists $\gamma_0 = \gamma_0(\omega)$ such that*

$$\int_{\mathbb{D}} \frac{\omega(z)}{|1-\bar{a}z|^{\gamma+1}} dA(z) \asymp \frac{\int_{|a|}^1 \omega(r)dr}{(1-|a|)^\gamma} \asymp \frac{\omega(a)}{(1-|a|)^{\gamma-1}}$$

for all $\gamma > \gamma_0$.

Theorem 2.2. *Let* ω *be a regular weight and* φ *be an analytic self map of* $\mathbb{D}$ *such that* $\varphi(0) = 0$. *Then* C_φ *is bounded on* $\mathcal{A}^p_\omega$ *if and only if*

$$\sup_{z\in\mathbb{D}} \frac{N_{\varphi,\omega}(z)}{\omega^*(z)} < \infty. \tag{2.10}$$

Proof. Assume that (2.10) is satisfied. Then, since $N_{\varphi,\omega}(z) = 0$ when $z \notin \varphi(\mathbb{D})$, the change-of-variable formula (1.6) and equivalent norm (1.5), imply that:

$$\begin{aligned}
\|C_\varphi f\|^p_{\mathcal{A}^p_\omega} &\asymp p^2 \int_{\mathbb{D}} |f(\varphi(z))|^{p-2}|f'(\varphi(z))|^2|\varphi'(z)|^2\omega^*(z)dA(z) + \omega(\mathbb{D})|f\circ\varphi(0)|^p \\
&= |f(0)|^p\omega(\mathbb{D}) + p^2 \int_{\varphi(\mathbb{D})} |f(z)|^{p-2}|f'(z)|^2 N_{\varphi,\omega}(z)dA(z) \\
&= |f(0)|^p\omega(\mathbb{D}) + p^2 \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2 N_{\varphi,\omega}(z)dA(z) \\
&\preceq |f(0)|^p\omega(\mathbb{D}) + p^2 \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2 \omega^*(z)dA(z) \\
&\asymp \|f\|^p_{\mathcal{A}^p_\omega}.
\end{aligned} \tag{2.11}$$

Conversely, let $\gamma = \gamma(\omega)$ be the constant in Lemma 2.2, for each $a \in \mathbb{D}$ and $1 < p < \infty$ set

$$F_{a,p}(z) = \frac{(1-|a|)^{\frac{\gamma+1}{p}}}{(1-\bar{a}z)^{\frac{\gamma+1}{p}}\,\omega(S(a))^{\frac{1}{p}}}$$

By [10, Lemma 2.4] we have

$$\|F_{a,p}\|^p_{\mathcal{A}^p_\omega} \asymp 1. \tag{2.12}$$

Namely, if $|a|$ is close enough to 1, $D(a, \frac{1-|a|}{2}) \subset \mathbb{D}\setminus D(0, \frac{1}{2})$. Using the well known fact $\omega(S(a)) \asymp \omega(T(a)) \asymp \omega^*(z)$,

$|a| > \frac{1}{2}$, and also the Lemma 2.1, we deduce

$$\begin{aligned}
\|C_\varphi F_{a,p}\|_{\mathcal{A}^p_\omega}^p &\asymp p^2 \int_{\mathbb{D}} |F_{a,p}(\varphi(z))|^{p-2} |(F_{a,p}(\varphi(z)))'|^2 \omega^*(z) dA(z) \\
&= p^2 \int_{\varphi(\mathbb{D})} |F_{a,p}(z)|^{p-2} |F'_{a,p}(z)|^2 N_{\varphi,\omega}(z) dA(z) \\
&= p^2 \int_{\varphi(\mathbb{D})} \frac{|a|^2 (1-|a|)^{\gamma+1} (\gamma+1)^2}{p^2 \omega(S(a)) |1-\bar{a}z|^{\gamma+3}} N_{\varphi,\omega}(z) dA(z) \\
&= \frac{|a|^2 (1-|a|)^{\gamma+1} (\gamma+1)^2}{\omega(S(a))} \int_{\mathbb{D}} \frac{N_{\varphi,\omega}(z)}{|1-\bar{a}z|^{\gamma+3}} dA(z) \\
&\geqslant \frac{|a|^2 (1-|a|)^{\gamma+1} (\gamma+1)^2}{\omega(S(a))(1-|a|)^{\gamma+3}} \int_{D(a,\frac{1-|a|}{2})} N_{\varphi,\omega}(z) dA(z) \\
&\geqslant \frac{|a|^2 (\gamma+1)^2}{\omega(S(a))(1-|a|)^2} (\frac{1-|a|}{2})^2 N_{\varphi,\omega}(a) \\
&\succeq \frac{N_{\varphi,\omega}(a)}{\omega(S(a))} \\
&\succeq \frac{N_{\varphi,\omega}(a)}{\omega^*(a)}.
\end{aligned} \tag{2.13}$$

It follows that

$$\sup_{a\in\mathbb{D}} \frac{N_{\varphi,\omega}(a)}{\omega^*(a)} \preceq \sup_{a\in\mathbb{D}} \|C_\varphi F_{a,p}\|_{\mathcal{A}^p_\omega}^p \preceq \|C_\varphi\|^p \sup_{a\in\mathbb{D}} \|F_{a,p}\|_{\mathcal{A}^p_\omega}^p \asymp 1.$$

□

Recall that a holomorphic function f in $\mathbb{D}$ has the Taylor expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

For the Taylor expansion of f and any integer $n \geqslant 1$, let

$$R_n f(z) = \sum_{k=n}^{\infty} a_k z^k,$$

and $K_n = I - R_n$, where $If = f$ is the identity operator. It is an important quaestion that, when does the partial sums of the Taylor expansion of f converge to f in the norm topology of the function space? Zhu [18] have answered the question as follows.

Lemma 2.3. *[18] Suppose X is a Banach space of holomorphic functions in $\mathbb{D}$ with the property that the polynomials are dense in X. Then $||K_n f - f||_X \to 0$ as $n \to \infty$ if and only if* $\sup\{||K_n|| : n \geqslant 1\} < \infty$.

We establish that the partial sums of the Taylor expansion of f converge to f in the norm topology of $\mathcal{A}^p_\omega$ as follows.

Lemma 2.4. *If* $1 < p < \infty$ *and* $\omega \in \mathcal{R}$, *then* $\sup\{||K_n|| : n \geqslant 1\} < \infty$. *Moreover,* $||K_n f - f||_{\mathcal{A}^p_\omega} \to 0$ *as* $n \to \infty$ *for each* $f \in \mathcal{A}^p_\omega$.

Proof. We begin with recalling that there exist $\beta > -1$ such that the inclusion $\mathcal{A}^p_\omega \subset \mathcal{A}^p_\beta$ is holds. Thus for each $f \in \mathcal{A}^p_\omega$ and $r \in (0,1)$, [17, Proposition3] implies that there is a positive constant C depend only on p such that

$$\int_{\mathbb{T}} |K_n f(r\xi)|^p d\sigma(\xi) \leqslant C \int_{\mathbb{T}} |f(r\xi)|^p d\sigma(\xi),$$

where $\mathbb{T}$ denots the boundary of $\mathbb{D}$ and $d\sigma$ denots the normalized Lebesgue measure on $\mathbb{T}$. Using polar coordinates, the above inequality gives

$$\begin{aligned} \|K_n f\|^p_{\mathcal{A}^p_\omega} &= \int_{\mathbb{D}} |(K_n f)(z)|^p \omega(z) dA(z) \\ &= \int_0^1 \omega(r) r (\int_0^{2\pi} |K_n f(re^{i\theta})|^p d\theta) dr \\ &\leqslant C \int_0^1 \omega(r) r (\int_0^{2\pi} |f(re^{i\theta})|^p d\theta) dr \\ &= C\|f\|^p_{\mathcal{A}^p_\omega}. \end{aligned}$$

Therefore, $\{K_n\}$ is a bounded sequence of operators on $\mathcal{A}^p_\omega$. Since the polynomials are dense in $\mathcal{A}^p_\omega$, whenever ω is a radial weight, see [3], the Lemma 2.3 ensures that $||K_n f - f||_{\mathcal{A}^p_\omega} \to 0$ as $n \to \infty$ for each $f \in \mathcal{A}^p_\omega$. □

According to our notations, we see $R_n f = f - K_n f$. Therefore, the following result is an immediate consequence of Lemma 2.4.

Corollary 2.1. *If* $1 < p < \infty$ *and* $\omega \in \mathcal{R}$, *then* $\|R_n f\|_{\mathcal{A}^p_\omega} \to 0$ *as* $n \to \infty$ *and* $\sup\{||R_n|| : n \geqslant 1\} < \infty$.

Theorem 2.3. *Let* ω *be a regular weight and* C_φ *be a bounded operator on* $\mathcal{A}^p_\omega$ *such that* $\varphi(0) = 0$. *Then*

$$||C_\varphi||^p_e \asymp \limsup_{|z|\to 1} \frac{N_{\varphi,\omega}(z)}{\omega^*(z)}. \tag{2.14}$$

Proof. We use the same technique used in the [12, Theorem 3.2]. Suppose that

$$L := \limsup_{|z|\to 1} \frac{N_{\varphi,\omega}(z)}{\omega^*(z)}.$$

Since K_n is compact and C_φ is bounded, we have

$$||C_\varphi||_e = ||C_\varphi(K_n + R_n)||_e \leqslant ||C_\varphi K_n||_e + ||C_\varphi R_n||_e \leqslant ||C_\varphi R_n||, \tag{2.15}$$

for each $n \in \mathbb{N}$. By Corollary 2.1 and (2.15) we have,

$$\begin{aligned}
\|C_\varphi\|_e^p \leqslant & \liminf_{n\to\infty} \|C_\varphi R_n\|^p \\
\leqslant & \liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{A}^p_\omega}\leqslant 1} \|(C_\varphi R_n)(f)\|^p_{\mathcal{A}^p_\omega} \\
= & p^2 \liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{A}^p_\omega}\leqslant 1} \int_{\varphi(\mathbb{D})} |(R_n f)(\varphi(z))|^{p-2} |(R_n f)'(\varphi(z))|^2 |\varphi'(z)|^2 \omega^*(z) dA(z) \\
& + \omega(\mathbb{D}) |(R_n f)(\varphi(0))| \\
= & p^2 \liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{A}^p_\omega}\leqslant 1} \int_{\mathbb{D}} |(R_n f)(z)|^{p-2} |(R_n f)'(z)|^2 N_{\varphi,\omega}(z) dA(z) \\
\preceq & L \liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{A}^p_\omega}\leqslant 1} \int_{\mathbb{D}} |(R_n f)(z)|^{p-2} |(R_n f)'(z)|^2 \omega^*(z) dA(z) \\
\preceq & L \liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{A}^p_\omega}\leqslant 1} \|R_n f\|^p_{\mathcal{A}^p_\omega} \\
\preceq & L
\end{aligned}$$

Now consider the functions defind in the proof Theorem 2.2. If γ is fixed appropriately, by simple calculation

$$\lim_{|a|\to 1} F_{a,p}(z) = \lim_{|a|\to 1} \left(\frac{(1-|a|)^\gamma \pi}{(1-\bar{a}z)^{\gamma+1} \int_{|a|}^1 \omega(s) s ds} \right) = 0$$

uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. Therefore for every compact operator K on $\mathcal{A}^p_\omega$, we have $\lim_{|a|\to 1} \|KF_{a,p}\|_{\mathcal{A}^p_\omega}$
0. It follows that

$$\begin{aligned}
\|C_\varphi - K\| \geqslant & \limsup_{|a|\to 1} \|C_\varphi F_{a,p} - KF_{a,p}\|_{\mathcal{A}^p_\omega} \\
\geqslant & \limsup_{|a|\to 1} \|C_\varphi F_{a,p}\|_{\mathcal{A}^p_\omega} - \limsup_{|a|\to 1} \|KF_{a,p}\|_{\mathcal{A}^p_\omega} \\
= & \limsup_{|a|\to 1} \|C_\varphi F_{a,p}\|_{\mathcal{A}^p_\omega}
\end{aligned}$$

Hence, applying (2.13), we give

$$\|C_\varphi\|_e^p \succeq \limsup_{|z|\to 1} \frac{N_{\varphi,\omega}(z)}{\omega^*(z)}.$$

□

Now we present a unified approach for the boundedness and essential norm of composition operators on the weighted bergman spaces with regular weights.

Theorem 2.4. *Let ω be a regular weight and let φ be an analytic self-map of $\mathbb{D}$ such that $\varphi(0) = 0$. Then the following statements are equivalent:*

(i) $C_\varphi : \mathcal{A}^p_\omega \to \mathcal{A}^p_\omega$ is bounded,

(ii) $\sup_{z\in\mathbb{D}} \frac{N_{\varphi,\omega}(z)}{\omega^*(z)} < \infty$,

(iii) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\sigma_a'(z)|^2}{\omega(S(z))} N_{\varphi,\omega}(z)dA(z) < \infty$,

(iv) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\sigma_a'(\varphi(z))|^2}{\omega(S(\varphi(z)))} |\varphi'(z)|^2\omega^*(z)dA(z) < \infty$,

(v) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\sigma_a'(\varphi(z))|^2}{\omega(S(\varphi(z)))} |\varphi'(z)|^2(1-|z|^2)^2\omega(z)dA(z) < \infty$.

Theorem 2.5. *Let ω be a regular weight and let φ be an analytic self-map of $\mathbb{D}$ such that $\varphi(0) = 0$. Then the following quantities are comparable:*

(i) $\|C_\varphi\|_e^p$,

(ii) $\limsup_{|z|\to 1} \frac{N_{\varphi,\omega}(z)}{\omega^*(z)}$,

(iii) $\limsup_{|z|\to 1} \int_{\mathbb{D}} \frac{|\sigma_z'(w)|^2}{\omega(S(w))} N_{\varphi,\omega}(w)dA(w)$,

(iv) $\limsup_{|z|\to 1} \int_{\mathbb{D}} \frac{|\sigma_z'(\varphi(w))|^2}{\omega(S(\varphi(w)))} |\varphi'(w)|^2\omega^*(w)dA(w)$,

(v) $\limsup_{|z|\to 1} \int_{\mathbb{D}} \frac{|\sigma_z'(\varphi(w))|^2}{\omega(S(\varphi(w)))} |\varphi'(w)|^2(1-|w|^2)^2\omega(w)dA(w)$.

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