Beyond half synchronized systems

Dawoud Ahmadi Dastjerdi[1], Manouchehr Shahamat[2]
1- Faculty of Mathematical Sciences, Department of Pure Mathematics
Email: dahmadi1387@gmail.com
2- Faculty of Mathematical Sciences, Department of Pure Mathematics
Email: rezashahamat80@yahoo.com (Corresponding author)

Received: XXX Revised: XXX Accepted: XXX

April 9, 2019

Abstract

In irreducible subshifts, a word m is synchronizing if whenever vm and mw are admissible words, then vmw is admissible as well. A word m is (left) half (resp. weak) synchronizing, when there is a left transitive ray (resp. a left ray) x_- such that if x_-m and mw are admissible, then x_-mw is also admissible. The respective subshifts are called half (resp. weak) synchronized. K. Thomsen in [On the structure of a sofic shift space, American Mathematical Society, 356, Number 9, 3557-3619] considers a synchronized component of a general subshift and investigates the approximation of entropy from inside of this component by some certain SFTs. We, using a rather different approach, show how this result extends to weak synchronized systems.

Keywords and phrases. synchronized, half synchronized, Kreiger graph, Fischer cover, entropy.

1 Introduction

One of the most studied dynamical systems is a subshift of finite type (SFT). An SFT is a system whose set of forbidden blocks is finite [1]; or equivalently, X is SFT iff there is $M \in \mathbb{N}$ such that any block of length greater than M is synchronizing. A block m is *synchronizing* if whenever v_1m and mv_2 are both blocks of X, then v_1mv_2 is a block of X as well. If an irreducible system has at least one synchronizing block, then it is called a *synchronized system* and examples are *sofics*: factors of SFT's. Synchronized systems, has attracted much attention and extension of them has been of interest since that notion was introduced [2]. One was via *half synchronized systems*; that is, systems having *half synchronizing* blocks. In fact, if for a left transitive point such as rm and mv any block in X one has again $rmv \in X^- = \{x_- := \cdots x_{-1}x_0 : x = \cdots x_{-1}x_0x_1 \cdots \in X\}$, then m is called half synchronizing [2]. Clearly any synchronized system is half synchronized. Dyke (or Dyck!) subshifts and certain β-shifts are non-synchronized but half synchronized systems [3].

Synchronized entropy of a synchronized system X denoted by $h_{\mathrm{syn}}(X)$ was considered in [4] as a value of exponential rate of change of orbits having a synchronized block. In section (4), we extend this notion to weak synchronized entropy $h_{\mathrm{wsyn}}(X)$ and will show there are some certain SFT's X_k such that $X_k \subseteq X_{k+1}$ and $h_{\mathrm{wsyn}}(X) = \lim_{k\to\infty} h(X_k)$.

2 Background and definitions

This section is devoted to the very basic definitions in symbolic dynamics. The notations has been taken from [1] and [2] for the relevant concepts.

First we present some elementary concept from [1]. Let $\mathcal{A}$ be an alphabet, that is a non-empty finite set. The full shift $\mathcal{A}$-shift denoted by $\mathcal{A}^{\mathbb{Z}}$, is the collection of all bi-infinite sequences of symbols in $\mathcal{A}$. Equip $\mathcal{A}$ with discrete topology and $\mathcal{A}^{\mathbb{Z}}$ with product topology. A *block* or *word* over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. It is convenient to include the sequence of no symbols, called the *empty block* which is denoted by ε. If x is a point in $\mathcal{A}^{\mathbb{Z}}$ and $i \leq j$, then we will denote a block of length $j - i + 1$ by $x_{[i,j]} = x_ix_{i+1}\dots x_j$. If $n \geq 1$, then u^n denotes the

[0]**2010 Mathematics Subject Classification:** 37B10.

concatenation of n copies of u, and put $u^0 = \varepsilon$. The *shift map* σ on the full shift $\mathcal{A}^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose i-th coordinate is $y_i = x_{i+1}$. By our topology, σ is a homeomorphism. Let $\mathcal{F}$ be the collection of all forbidden blocks over $\mathcal{A}$. For a full shift $\mathcal{A}^{\mathbb{Z}}$, define $X_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{Z}}$ not containing any block from $\mathcal{F}$. A *shift space* or a *subshift* is a subset X of a full shift $\mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden blocks.

Let $\mathcal{B}_n(X)$ denote the set of all admissible n-blocks. The *language* of X is the collection $\mathcal{B}(X) = \cup_n \mathcal{B}_n(X)$. A shift space X is *irreducible* if for every ordered pair of blocks $u, v \in \mathcal{B}(X)$ there is a block $w \in \mathcal{B}(X)$ so that $uwv \in \mathcal{B}(X)$. A shift space X is called a *shift of finite type* (SFT) if there is a finite set $\mathcal{F}$ of forbidden blocks such that $X = X_{\mathcal{F}}$. A shift of *sofic* is the image of an SFT by a factor code (an onto sliding block code). Every SFT is sofic [1, Theorem 3.1.5], but the converse is not true [1, Page 67].

Let G be a graph with edge set $\mathcal{E} = \mathcal{E}(G)$ and the set of vertices $\mathcal{V} = \mathcal{V}(G)$. The *edge shift* X_G is the shift space over the alphabet $\mathcal{A} = \mathcal{E}$ defined by

$$X_G = \{\xi = (\xi_i)_{i\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} : t(\xi_i) = i(\xi_{i+1})\}.$$

Each edge e initiates at a vertex denoted by $i(e)$ and terminates at a vertex $t(e)$.

A labeled graph is a pair $\mathcal{G} = (G, \mathcal{L})$, where G is a graph with edge set $\mathcal{E}$, and the labeling $\mathcal{L} : \mathcal{E}(G) \to \mathcal{A}$ assigns to each edge e of G a label $\mathcal{L}(e)$ from the finite alphabet $\mathcal{A}$. For a path $\pi = \pi_0 \dots \pi_k$, $\mathcal{L}(\pi) = \mathcal{L}(\pi_0) \dots \mathcal{L}(\pi_k)$ is the label of π. By π_u we mean a path labeled u.

Let $\mathcal{L}_\infty(\xi)$ be the sequence of bi-infinite labels of a bi-infinite path ξ in G and set

$$X_{\mathcal{G}} := \{\mathcal{L}_\infty(\xi) : \xi \in X_G\} = \mathcal{L}_\infty(X_G).$$

We say $\mathcal{G}$ is a *presentation* or *cover* of $X = \overline{X_{\mathcal{G}}}$. In particular, X is sofic if and only if $X = X_{\mathcal{G}}$ for a finite graph G [1, Proposition 3.2.10].

In this part we collect some information from [2]. Let X be a subshift and $x \in X$. Then, $x_+ = (x_i)_{i\in Z^+}$ (resp. $x_- = (x_i)_{i\le 0}$) is called right (resp. left) infinite X-ray. Let $X^+ = \{x_+ : x \in X\}$ and $X^- = \{x_- : x \in X\}$. For a left infinite X-ray, say x_-, its follower set is $w_+(x_-) = \{x_+ \in X^+ : x_- x_+ \in X\}$ and for $m \in \mathcal{B}(X)$ its follower set is $w_+(m) = \{x_+ \in X^+ : m x_+ \in X^+\}$. Analogously, we define predecessor sets $\omega_-(x_+) = \{x_- \in X^- : x_- x_+ \in X\}$ and $\omega_-(m) = \{x_- \in X^- : x_- m \in X^-\}$.

Consider the collection of all follower sets $\omega_+(x_-)$ as the set of vertices of a graph. There is an edge from I_1 to I_2 labeled a if and only if there is an X-ray x_- such that $x_- a$ is an X-ray and $I_1 = \omega_+(x_-), I_2 = \omega_+(x_- a)$. This labeled graph is called the *Krieger graph* for X. A block $m \in \mathcal{B}(X)$ is *synchronizing* if whenever um and mv are in $\mathcal{B}(X)$, we have $umv \in \mathcal{B}(X)$. An irreducible shift space X is *synchronized system* if it has a synchronizing block. A block $m \in \mathcal{B}(X)$ is *half synchronizing* if there is a left transitive point $x \in X$ such that $x_{[-|m|+1,\,0]} = m$ and $\omega_+(x_{(-\infty,\,0]}) = \omega_+(m)$. If X is a half synchronized system with half synchronizing m, the irreducible component of the Krieger graph containing the vertex $\omega_+(m)$ is denoted by X_0^+ and is called the *right Fischer cover* of X. A shift space that is the closure of the set of sequences obtained by freely concatenating the blocks in a list of countable blocks, called the set of generators, is a *coded system* [1].

3 Weak synchronized systems

Definition 3.1 *A shift space X is called* right *(resp.* left*)* weak synchronized system *if there is a block m of X and a point $x \in X$ such that $x_{[-|m|+1,\,0]} = m$ (resp. $x_{[0,\,|m|-1]} = m$) and $\omega_+(x_{(-\infty,\,0]}) = \omega_+(m)$ (resp. $\omega_-(x_{[0,\,\infty)}) = \omega_-(m)$) that we call m a* right weak synchronizing *(resp. left) block of X. Then, $\omega_+(m)$ is called a* weak synchronized vertex.

Note that if x was left (resp. right) transitive, then by definition, X would be right (resp. left) half synchronized system and so any half synchronized system is a weak synchronized system.

Here, whenever we say "weak synchronizing", we mean the right weak synchronizing.

Example 3.2 *Now we present an example of coded weak synchronized system which are not half synchronized and whose any of their blocks are weak synchronizing.*

Let X_β denote the beta-shift corresponding to $\beta > 1$. We first choose a $1 < \beta \in \mathbb{R}$ such that X_β is not synchronized. Let m^{-1} be an arbitrary block in $W(X_\beta^{-1})$. First we show that $0^\infty m^{-1} \in (X_\beta^{-1})^-$ and $m^{-1}0^\infty \in (X_\beta^{-1})^+$.

Since X_β is not synchronized and $m \in \mathcal{B}(X_\beta)$, m is not a synchronizing block for X_β where then by [3, Proposition 2.23], $m \subseteq 1_\beta = a_1 a_2 a_3 \cdots$. Assume $m = a_{j_m} a_{j_m+1} \dots a_{j_m+|m|-1}$ (Figure 1) and set $k := \min\{i > j_m + |m| - 1 : a_i > 0\}$. Then, there is a finite path labeled $m 0^{k_m - j_m - |m| + 1}$ with initial vertex I_{j_m-1} and terminal vertex I_0. Hence $m0^\infty$ is a right infinite X_β-ray and so $0^\infty m^{-1}$ is a left infinite $(X_\beta)^{-1}$-ray. Similar reasoning works for $m^{-1}0^\infty \in (X_\beta^{-1})^+$ and so we have $w_+(0^\infty m^{-1}) = w_+(m^{-1})$ and $\omega_-(m^{-1}0^\infty) = \omega_-(m^{-1})$ which that in turn shows *that m^{-1} is a right and left weak synchronizing block for X_β^{-1}. But m was arbitrary and so we are done.*

4 Weak synchronized entropy

Let $H = (\mathcal{V}, \mathcal{E})$ be a connected graph. For each pair of vertices $I, J \in \mathcal{V}$, let $r_n(I, J)$ denote the number of paths of length n starting at I and ending at J. Then,

$$h(H) = \limsup_{n\to\infty} \frac{1}{n} \log r_n(I,\, J)$$

is independent of I, J, and it is called the *Gurevic entropy* of H [5]. For any synchronized system X, the *synchronized entropy* $h_{\text{syn}}(X)$ is defined by

$$h_{\text{syn}}(X) = \limsup_n \frac{1}{n} \log(|\{a \in \mathcal{B}_n(X) :\ mam \in \mathcal{B}(X)\}|),$$

where $m \in \mathcal{B}(X)$ is an arbitrary synchronizing block [6]. In 2004, Thomsen in [6] proves that it is equal to the topological entropy of the system.

Now let X be a weak synchronized system and let $\mathrm{WH}(X)$ denote the set of weak synchronizing blocks for X. For $m \in \mathrm{WH}(X)$, denote by $(X_m)_0^+$ the maximal irreducible component of the Krieger graph X containing the vertex $\omega_+(m)$. Note that irreducible components are countable labeled graphs and so $\overline{\mathcal{L}((X_m)_0^+)}$ is a coded system.

Let m be a weak synchronizing block for X. Fix m and x provided by the definition of weak synchronizing. Notice that x_- terminates at m and set

$$h(m,\, X) := \limsup_{n\to\infty} \frac{1}{n} \log |\{a \in \mathcal{B}_n(X) :\ \omega_+(x_- am) = \omega_+(m)\}|\,.$$

A block $m \in \mathrm{WH}(X)$ is called *residual weak synchronizing* if there is a finite path π_m in $(X_m)_0^+$ labeled m such that $\omega_+(m) = t(\pi_m)$. For example in Example 3.2 $m := 0$ is a residual weak synchronizing block for X_β^{-1} such that it is not a half synchronizing of X_β^{-1}.

Proposition 4.1 *Let m be an arbitrary residual weak synchronizing block for X Then,*

$$h(m,\, X) = h((X_m)_0^+).$$

Proof. Let x be a point in X provided by the definition of weak synchronizing. Pick a finite path π_m in $(X_m)_0^+$ labeled m such that $\theta := \omega_+(m) = t(\pi_m)$ (Figure 2). Set $\delta := i(\pi_m)$). Let π_u be a finite path in $(X_m)_0^+$ labeled u such that $i(\pi) = \theta$, $t(\pi) = \delta$. Suppose that τ is an arbitrary cycle from θ to θ in $(X_m)_0^+$ labeled v. Then, $\omega_+(x_- vum) = \omega_+(m)$. Thus the number of cycles of length $|v|$ from θ to θ is at most

$$|\{a \in \mathcal{B}_{|v|+|u|}(X) : \omega_+(x_- am) = \omega_+(m)\}|.$$

This means that $h(X_0^+) \le h(m,\, X)$. Conversely, let $a \in \mathcal{B}_n(X)$ such that $\omega_+(x_- am) = \omega_+(m)$. Then, there is a cycle C_a in $(X_m)_0^+$ labeled am and initialing at $\theta = \omega_+(m)$ and so

$$|\{a \in \mathcal{B}_n(X) : \omega_+(x_- am) = \omega_+(m)\}|$$

is at most as large as the number of cycles of length $n + |m|$ based at $\theta = \omega_+(m)$. Thus $h(m,\, X) \le h((X_m)_0^+)$.

□

Let $\mathrm{RW}(X)$ denote the set of residual weak synchronizing blocks for X and set

$$\mathcal{H}_X := \{h((X_m)_0^+) : m \in \mathrm{RW}(X)\}.$$

Then, it is natural to define the *weak synchronized entropy* $h_{\text{wsyn}}(X)$ to be

$$h_{\text{wsyn}}(X) = \sup \mathcal{H}_X.$$

If m is a synchronizing block of X, then

$$\begin{aligned}\{a \in \mathcal{B}_n(X) : \omega_+(x_- am) = \omega_+(m)\}\\ =\{a \in \mathcal{B}_n(X) :\ mam \in \mathcal{B}(X)\}\end{aligned}$$

and so

$$h(m,\, X) = h_{\text{wsyn}}(X) = h_{\text{syn}}(X) = h(X_0^+).$$

Thus

$$\sup\{h((X_m)_0^+) : m \in \mathrm{RW}(X)\} = h(X_0^+).$$

Thomsen in [6] considers a synchronized component X of a general subshift and proves that

$$\begin{aligned}\sup\{h(A) : &A \subseteq X \ \text{ is an irreducible SFT}\}\\ &= h(X_0^+) = h_{\text{syn}}(X). \qquad (1)\end{aligned}$$

Now we will extend this notion to weak synchronized with a new and shorter proof which naturally will imply (1) as well.

Proposition 4.2 *Let X be a weak synchronized system and $RW(X) \neq \emptyset$. Then,*

$$\begin{aligned}t_0 := \sup\{h(A) \colon \exists\, m \in RW(X) \text{ such that}\\ A \subseteq \mathcal{L}((X_m)_0^+) \text{ is an irreducible SFT}\} = h_{wsyn}(X).\end{aligned}$$

Proof. Let $A \subseteq \mathcal{L}\left((X_m)_0^+\right)$ be an irreducible SFT for some $m \in \mathrm{RW}(X)$. Then, $h(A) = h_{\mathrm{syn}}(A) \leq h((X_m)_0^+)$ and this implies $t_0 \leq h_{\mathrm{wsyn}}(X)$.

It suffices to show that $h_{\mathrm{wsyn}}(X) \leq t_0$. Fix $\epsilon' > 0$ and choose $m \in \mathrm{RW}(X)$ and $0 < \epsilon < \epsilon'$ such that

$$h_{\mathrm{wsyn}}(X) - \epsilon' \leq h\left((X_m)_0^+\right) - \epsilon. \tag{2}$$

Set

$$C_n := \{C : \ C \text{ is a cycle in } (X_m)_0^+ \\ \text{starting at } \omega_+(m), |C| = n\}.$$

Let $\{n_k\}$ be an increasing sequence of natural numbers such that

$$h((X_m)_0^+) - \epsilon < \lim_k \frac{1}{n_k} \log |C_{n_k}| \leq h((X_m)_0^+).$$

Thus by (2),

$$h_{\mathrm{wsyn}}(X) - \epsilon' < \lim_k \frac{1}{n_k} \log |C_{n_k}| \leq h((X_m)_0^+). \tag{3}$$

Now set $C_{n_k} := \left\{C_1^k, \ldots, C_{j_k}^k\right\}$ and

$$\begin{aligned} H_1 &:= C_1^1 \cup \cdots \cup C_{j_1}^1, \\ H_2 &:= H_1 \cup C_1^2 \cup \cdots \cup C_{j_2}^2, \cdots, \\ H_k &:= H_{k-1} \cup C_1^k \cup \cdots \cup C_{j_k}^k. \end{aligned}$$

Then, for all $k \in \mathbb{N}$,

$$|C_{n_k}| \leq |\{C : \ C \text{ is a cycle in } H_k \\ \text{starting at } \omega_+(m), |C| = n_k\}|.$$

We shall need the following lemma.

Lemma 4.3 $\lim_k \frac{1}{n_k} \log |C_{n_k}| \leq \lim_k h(X_{H_k})$.

Proof. All $(X_{H_k})'s$ are irreducible sofic and $h(X_{H_k}) = h(H_k)$ by [6, Lemma 3.1]. So it suffices to show that $\lim_k \frac{1}{n_k} \log |C_{n_k}| \leq \lim_k h(H_k)$.

Let $\lim_k h(H_k) < \lim_k \frac{1}{n_k} \log |C_{n_k}|$. Set $r := \lim_k \frac{1}{n_k} \log |C_{n_k}| - \lim_k h(H_k)$. Thus

$$\lim_k h(H_k) < \lim_k \frac{1}{n_k} \log |C_{n_k}| - \frac{r}{3}. \tag{4}$$

Set

$$C_{i,\, n_k} := \{C : C \text{ is a cycle in } H_i \\ \text{starting at } \omega_+(m), |C| = n_k\}$$

and so $\lim_k \frac{1}{n_k} \log |C_{i,\, n_k}| \leq h(H_i)$. But $h(H_i) \leq \lim_k h(H_k)$. Hence

$$\lim_k \frac{1}{n_k} \log |C_{i,\, n_k}| \leq \lim_k h(H_k)$$

and so by (4), $\lim_k \frac{1}{n_k} \log |C_{i,\, n_k}| < \lim_k \frac{1}{n_k} \log |C_{n_k}| - \frac{r}{3}$. Thus for each $i > 0$, there is k_i such that $k_i < k_{i+1}$ and

$$\frac{1}{n_{k_i}} \log |C_{i,\, n_{k_i}}| < \lim_k \frac{1}{n_k} \log |C_{n_k}| - \frac{r}{3}. \tag{5}$$

Since $\frac{1}{n_{k_i}} \log |C_{n_{k_i}}| \leq \frac{1}{n_{k_i}} \log |C_{i,\, n_{k_i}}|$ for all i, by (5),

$$\frac{1}{n_{k_i}} \log |C_{n_{k_i}}| < \lim_k \frac{1}{n_k} \log |C_{n_k}| - \frac{r}{3}.$$

Hence

$$\lim_i \frac{1}{n_{k_i}} \log |C_{n_{k_i}}| \leq \lim_k \frac{1}{n_k} \log |C_{n_k}| - \frac{r}{3}. \tag{6}$$

But

$$\lim_i \frac{1}{n_{k_i}} \log |C_{n_{k_i}}| = \lim_k \frac{1}{n_k} \log |C_{n_k}|$$

and so by (6),

$$\lim_k \frac{1}{n_k} \log |C_{n_k}| \leq \lim_k \frac{1}{n_k} \log |C_{n_k}| - \frac{r}{3}$$

that is absurd. □

Completing the proof of Proposition 4.2. By Lemma 4.3 and by (3),

$$h_{\mathrm{wsyn}}(X) - \epsilon' < \lim_k h(X_{H_k})$$

and so there is $k_0 \in \mathbb{N}$ such that

$$h_{\mathrm{wsyn}}(X) - \epsilon' < h(X_{H_{k_0}}) \leq h((X_m)_0^+).$$

Where the last equality is satisfied because $h(H_{k_0}) = h(X_{H_{k_0}})$ and H_{k_0} is a subgraph of $(X_m)_0^+$. But all C_j^i meet at $\omega_+(m)$ and so $X_{H_{k_0}}$ is an irreducible sofic. Thus by [6, Theorem 3.2], there is an irreducible SFT $A \subseteq X_{H_{k_0}} \subseteq \mathcal{L}((X_m)_0^+)$ such that

$$h_{\mathrm{wsyn}}(X) - \epsilon' < h(A) < h(X_{H_{k_0}}). \tag{7}$$

But by definition of t_0, $h(A) \leq t_0$ and so by (7), $h_{\mathrm{wsyn}}(X) - \epsilon' \leq t_0$ and we are done. □

An immediate consequence of the above proposition is

Corollary 4.4 *Suppose X is an irreducible subshift. If X is weak synchronized and $h_{wsyn}(X) = h(X)$, then X is almost sofic.*

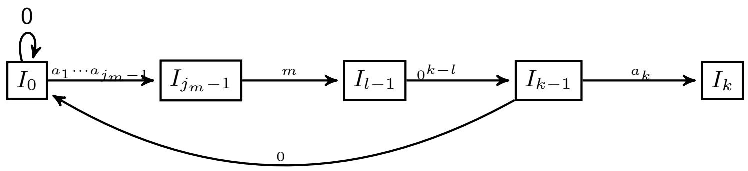

Figure 1: The subgraph H of $\mathcal{G}_\beta$ with $1_\beta = a_1a_2a_3\ldots$, where $l := j_m + |m|$.

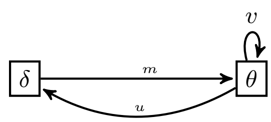

Figure 2: The subgraph of $(X_m)_0^+$.

References

[1] D. Lind and B. Marcus, ***An introduction to symbolic dynamics and coding***, Cambridge Univ. Press. 1995.

[2] D. Fiebig and U. Fiebig, ***Covers for coded systems, Contemporary Mathematics***, 135, 1992, 139-179.

[3] T. Meyerovitvh, ***Tail invariant measures of the Dyke-shift and non-sofic systems***, M.Sc. Thesis, Tel-Aviv university, 2004.

[4] U. Jung, ***On the existence of open and bi-continuous codes***, Trans. Amer. Math. Soc. 363 (2011), 1399-1417.

[5] K. Thomsen, ***On the ergodic theory of synchronized systems***, Ergod. Th. Dynam. Sys. 356 (2006) 1235-1256.

[6] K. Thomsen, ***On the structure of a sofic shift space***, American Mathematical Society, 356, Number 9, 3557-3619.