Optimal Control of Nonlinear Systems Using the Shifted Legendre Polynomials

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ABSTRACT
A numerical technique based on Legendre Polynomials for finding the optimal control of nonlinear systems with quadratic performance index is presented. An operational matrix of integration and product matrix are introduced and are used to reduce the nonlinear differential equations for the solution of nonlinear algebraic equations. The optimal solution from two classes of first and second order nonlinear systems is considered. In the case of second-order nonlinear systems, a new approach is introduced to find the optimal solution. In both cases, numerical examples are given and compared with the Taylor polynomials to confirm the accuracy of the proposed method.

KEYWORDS: Non-linear systems; Legendre Polynomials; Optimal Control; Numerical Methods

1. INTRODUCTION

Most of the systems have nonlinear dynamics. So, the study of these nonlinear systems are very important. Hence, many researchers and designers have showed an active interest in the development and applications of nonlinear systems [1-3].

Orthogonal functions and polynomial series have received considerable attention in dealing with various problems of dynamical systems. Examples are the use of the Walsh functions (Chen Shih 1978)[4], the block-pulse functions (Maleknejad and Shahrezaee 2005, wang and Li 2009) [5,6], the Chebyshev polynomials (H. Jaddu and E. Shimemura 1973) [7], the Taylor series (Mouroutsos and Sparis 1985, Gulsu and Sezer 2006, S. Yalcinbas 2002) [8-10], the Fourier series (M.L. Nagurka, V. Yen 1990 ,Ardekani and Keyhani 1989, Ardekani , Samavat and Rahmani 1991, Samavat and Rashidi 1995, Ebrahimi , Samavat , Vali and Gharavisi 2007) [11-15]. The main characteristic of the technique is that it reduces these problems to those of solving a system of algebraic equations; thus it greatly simplifies the problem.

In this paper, we use the Legendre Polynomials to find the optimal control of nonlinear systems. For the first time in this paper, the optimal control of a particular class of second-order nonlinear systems using a new approach has been proposed. By this numerical technique, a difficult problem is reduced to the straightforward nonlinear algebraic equations which can be solved by using a digital computer. Numerical examples are given to show the accuracy of the technique.

2. PROPERTIES OF LEGENDER POLYNOMIALS

2.1. Legendre Polynomials

The shifted Legendre polynomials, $P_n(t)$, where $0 \le t \le t_f$ are obtained from[16],

$$P_{n+1}(t) = \left(\frac{2n+1}{n+1}\right)(2\frac{t}{t_f} - 1)P_n(t) - \left(\frac{n}{n+1}\right)P_{n-1}(t)$$

$$n \quad = 1,2,3, \dots \tag{1}$$

Where

$$P_0(t) = 1 \quad , P_1(t) = 2\frac{t}{t_f} - 1 \tag{2}$$

The orthogonal property is given by

$$\int_0^{t_f} P_i(t)\, P_j(t) = \begin{cases} 0, & i \ne j \\ \left(\frac{t_f}{2i+1}\right), & i = j \end{cases} \tag{3}$$

2.2. Function Approximation

A function $f(t)\, \epsilon\, L^2[0, t_f]$ can be approximated as:

$f(t) = \sum_{i=0}^{\infty} f_i P_i(t)$, (4)

In practice we only consider a finite number of terms, that is

$f(t) = \sum_{i=0}^{m-1} f_i P_i(t)$, (5)

The shifted Legendre polynomial f_i can be obtained by using

$$f_i = \frac{2i+1}{t_f}\int_0^{t_f} f(\tau)\, P_i(\tau)d\tau$$

Equation (5) can be written in a matrix form as:

$f(t) = F^T P(t)$ (6)

Or

$f(t) = P(t)^T F$ (7)

Where F and $P(t)$ are $m\times 1$ matrices which are given by:

$F^T = [f_0 \quad f_1 \quad f_2 \qquad f_{m-1}]$ (8)

$P(t) = [P_0 \quad P_1 \quad P_2 \qquad P_{m-1}]^T$ (9)

2.3 The Operational Matrix of Integration

Integration of the vector $P(t)$ defined in Eq. (9) can be written as:

$\int_0^t P(s)\, ds = H_{m\times m} P(t)_{m\times 1}$, (10)

By using Eqs.(3) and (7) we have :

$\int_0^t f(s)ds = \int_0^t F^T P(s)\, ds = F^T\, HP(t)$ (11)

By using Eqs.(7) and (10) we have :

$\int_0^t f(s)ds = \int_0^t P(s)^T F\, ds = P(t)^T H^T F$ (12)

Where the matrix H is obtained as follows[16]:

$$H = t_f \begin{bmatrix} 1/2 & 1/2 & 0 & \dots\dots\dots\dots & 0 \\ -1/6 & 0 & 1/6 & \dots\dots\dots\dots & 0 \\ 0 & -1/10 & 0 & \dots\dots\dots\dots & 0 \\ 0 & 0 & -1/14 & \dots\dots\dots\dots & \vdots \\ \vdots & \vdots & \vdots & \dots & 1/2(2m-3) \\ 0 & 0 & 0 & -1/(2(2m-1)) & 0 \end{bmatrix}_{m\times m}$$

2.4 The Product Operational Matrix

The product operational matrix $\tilde{F}$ can be defined as follows[17]:

$P(t)P_i^T(t)F_i = \tilde{F}_i P_i(t)$, (13)

for i = 0,1,2,. . .,m − 1. In fact

$PP^T F \approx \tilde{F}P$, (14)

To illustrate the calculation procedures, we choose m=3. Thus, we have:

$F = [f_0 \quad f_1 \quad f_2]^T$

$P(t) = [P_0 \quad P_1 \quad P_2]^T$, (15)

Where

$P_0(t) = 1 \quad , P_1(t) = 2t - 1 \quad , P_2(t) = 6t^2 - 6t + 1$

Using equation (14), we get:

$$\begin{bmatrix} P_0{}^2 & P_0P_1 & P_0P_1 \\ P_1P_0 & P_1{}^2 & P_1P_2 \\ P_2P_0 & P_2P_1 & P_2{}^2 \end{bmatrix} [f_0 \quad f_1 \quad f_2]^T = \tilde{F}[P_0 \quad P_1 \quad P_2]^T \qquad (16)$$

Finally, we have:

$$\tilde{F} = \begin{bmatrix} f_0 & f_1 & f_2 \\ \frac{f_1}{4} & f_0 & \frac{2f_1}{3} \\ 0 & 0 & f_0 \end{bmatrix} \qquad (17)$$

3. THE OPTIMAL CONTROL PROBLEM

The aim of this section is to explain how we can use the Legendre polynomials to find the optimal solution of first and second order nonlinear systems. In both cases the results are compared with the results of the Taylor polynomials.

3.1. The first-order systems:

Example 1:

Consider the optimal control problem of the fist-order nonlinear system[12]:

$2[y^2(t)]^{\cdot} + \dot{y}(t) = u(t), \ \ y(0) = 0.2,$

$0 \le t \le 1$ (18)

With respect to a quadratic performance index:

$J = \int_0^1 [y^2(t) + u^2(t)]dt$ (19)

Integrating Eq. (18) from zero to t and using Eqs. (6), (7), (10), (11), (12) and (14) we have:

$$\int_0^t \dot{y}(s)\, ds = y(t) - y(0) = Y^T P(t) - Y_0{}^T P(t) ,$$

$$\int_0^t u(s)\, ds = \int_0^t U^T P(s)\, ds = U^T \int_0^t P(s)\, ds = U^T HP(t) ,$$

$$\int_0^t y(s) ds = \int_0^t Y^T P(s)\, ds = Y^T \int_0^t P(s)\, ds = Y^T H\, P(t) ,$$

$$\int_0^t \left(y^2(S)\right)^{\cdot} ds = y^2(t) - y^2\,(0)$$

$$= Y^T P(t) P^T(t) Y - {Y_{20}}^T P(t)\,,$$

Finally

$$2[Y^T\tilde{Y}P(t) - (Y_0^2)^T P(t)] + [Y^T P(t) - {Y_0}^T P(t)] = U^T H P(t), \qquad (20)$$

Eliminating $P(t)$ in equation (20) gives:

$$2Y^T\tilde{Y} + Y^T - U^T H - 2(Y_0^2)^T - Y_0^T = 0 \qquad (21)$$

Where Y_0^T and $(Y_0^2)^T$ are 1×m matrices which given by:

$$Y_0^T = [0.2\ 0\ 0\ 0\ \dots\dots\dots\dots.]$$

$$(Y_0^2)^T = [0.04\ 0\ 0\ \dots\dots\dots\dots]$$

For the performance index we have:

$$J = 10\int_0^1 [Y^T P(t) P^T(t) Y + U^T P(t) P^T(t) U]dt \qquad (22)$$

$$J = 10 \times Y^T L Y + 10 \times U^T L U \qquad (23)$$

Where

$$L = \int_0^1 [\ P(t) P^T(t)]dt \qquad (24)$$

We now minimize equation (23) related to the equation (21),by using the Lagrange multiplier technique we get:

$$J^* =$$

$$J + \lambda \left[2Y^T\tilde{Y} + Y^T - U^T H - 2(Y_0^2)^T - Y_0^T\right]^{\mathbf{T}} \qquad (25)$$

Where λ is a 1×m matrix as follows:

$$\lambda = [\lambda_0\ \lambda_1\ \lambda_2\ \lambda_3\ \dots\dots\dots\dots\ \lambda_{m-1}]$$

The necessary conditions for finding the minimum are:

$$\frac{\partial J^*}{\partial y_i} = 0\ , \frac{\partial J^*}{\partial u_i} = 0 \quad i = 1,2,\dots,m-1 \qquad (26)$$

$$\frac{\partial J^*}{\partial \lambda} = 0 \qquad (27)$$

By using this technique Eq. (25) turns into a set of nonlinear algebraic equations which can be solved using the Newton's iterative method to obtain J. The approximated values of J and $y(0)$ in comparison with the Taylor polynomials are given in Table 1.

Table 1. Approximated values of J *and* $y(0)$, using the proposed method in comparison with the Taylor polynomials for m= 4, m=5 and m=6 for example 1

m	Exact values of $\boldsymbol{y(0)}$	Approximated values of $\boldsymbol{y(0)}$ by the Taylor polynomials	Approximated values of $\boldsymbol{y(0)}$ by the proposed method	Approximated values of J by the Taylor polynomials	Approximated values of J by the proposed method
4	0.2	0.21312	0.21146	0.03954	0.03614
5	0.2	0.20862	0.20657	0.03910	0.03602
6	0.2	0.20345	0.20089	0.03881	0.03601

Example 2:
Consider the optimal control problem of a different fist-order nonlinear system[18]:

$$\dot{y}(t) = -y^2(t) + u(t),\ \ y(0) = 10,$$

$$0 \le t \le 1 \qquad (28)$$

With respect to a quadratic performance index:

$$J = 0.5\int_0^1 [y^2(t) + u^2(t)]dt \qquad (29)$$

Integrating Eq. (28) from zero to t and using Eqs. (6), (7), (10), (11), (12) and (14) and eliminating $P(t)$ we have:

$$Y^T - Y_0^T + Y^T\tilde{Y}H = U^T H \qquad (30)$$

Using the Newton's iterative method explained in example 1, the approximated values of J and $y(0)$ and comparison with Taylor polynomials are given in Table 2.

Table 2. Approximated values of J *and* $y(0)$, using the proposed method in comparison with the Taylor polynomials for m= 4, m=5 and m=6 for example 2

m	Exact values of $\boldsymbol{y(0)}$	Approximated values of $\boldsymbol{y(0)}$ by the Taylor polynomials	Approximated values of $\boldsymbol{y(0)}$ by the proposed method	Approximated values of J by the Taylor polynomials	Approximated values of J by the proposed method
4	10	10.1028	10.0245	4.93954	4.51236
5	10	10.0420	10.0126	4.64052	4.50828
6	10	10.0089	10.0098	4.60091	4.50726

3.2. The second order systems:
Example 3:

Consider the optimal controlling problem of the second-order nonlinear system[12]:

$y(t)\ddot{y}(t) + \dot{y}(t) + y^2(t) = u(t)$,

$y(0) = 0.3,\ \dot{y}(0) = 0$ (31)

Using the same performance index as example 1. Let us use the following assumption:

$\ddot{y}(t) = Y^T P(t)$ (32)

Integrating Eq. (32) from zero to t:

$\dot{y}(t) - \dot{y}(0) = Y^T H P(t),$ (33)

Integrating Eq. (33) from zero to t:

$y(t) - y(0) = Y^T H^2 P(t)$ (34)

Expanding $y(0)$ using the Legendre polynomials:

$y(0) = f^T P(t)$ (35)

Where f^T is a 1×m matrix given by:

$f^T = [0.3\ 0\ 0 \dots \dots]$

Substituting Eq. (35) into Eq. (34) gives

$y(t) = Y^T H^2 P(t) + f^T P(t),$ (36)

Expanding $u(t)$ within using the Legendre polynomials:

$u(t) = U^T P(t)$, (37)

Substituting Eqs. (32), (36),(37) into Eq. (31) gives:

$$Y^T H^2 P(t)P(t)^T Y + f^T P(t)P(t)^T Y + Y^T H P(t) + Y^T H^2 P(t)P(t)^T (H^2)^T Y + f^T P(t)P(t)^T f + Y^T H^2 P(t)P(t)^T f + f^T P(t)P(t)^T (H^2)^T Y = U^T P(t) \quad (38)$$

Let us define:

$Y^T H^2 = Z^T$ (39)

Substituting Eq. (39) and Eq. (14) into Eq. (38) and eliminating $P(t)$ gives:

$$Z^T\tilde{Y} + S^T\tilde{Y} + f^T\tilde{Y} + Y^T H + Z^T\tilde{Z} + f^T\tilde{f} + Z^T\tilde{f} + f^T\tilde{Z} + = U^T \quad (40)$$

Therefore Eq. (40) turns into a set of nonlinear algebraic equations. By using the Newton's iterative method explained in example 1, we get Y^T , $\boldsymbol{\lambda}$, U^T. In this case Y^T is the coefficients of $\ddot{y}(t)$. In order to find the coefficients of $y(t)$, we should solve Eq. (36) then we get the following equation:

Coefficients of $y(t) = Y^T H^2 + f^T$ (41)

Now this gives us the minimum of J. The approximated values of J and $y(0)$ and comparing them with Taylor polynomials are given in Table 3.

Table 3. Approximated values of $J\ and\ y(0)$, using the proposed method in comparison with the Taylor polynomials for m= 2, m=3 and m=4 for example 3

m	Exact values of $\boldsymbol{y(0)}$	Approximated values of $\boldsymbol{y(0)}$ by the Taylor polynomials	Approximated values of $\boldsymbol{y(0)}$ by the proposed method	Approximated values of J by the Taylor polynomials	Approximated values of J by the proposed method
2	0.3	0.34721	0.32001	0.42432	0.38982
3	0.3	0.33121	0.31456	0.40027	0.38745
4	0.3	0.31011	0.30865	0.39925	0.38599

4. CONCLUSION

In the proposed method, using the Legendre polynomials, the nonlinear differential equations are reduced into a set of nonlinear algebraic equations, which can be solved using a digital computer. Since the operational matrix of integration and the product matrix contain many zero entries, it gives computational advantages when compared with the other possible approximations. Numerical examples are given to show the accuracy and applicability of the technique. Results show that the Legendre polynomials have accurate approximate values than the Taylor polynomials. Finally, the method can be extended for the optimal control of nonlinear time within varying systems.

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