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On Characterizing Solutions of Optimization Problems with Roughness in the Objective Functions

S. A. Eldalatpanah[a], Hamiden Abd El- Wahed Khalifa[b,c], and Hashem Saberi Najafi[a,*]

[a] *Department of Applied Mathematics, Ayandegan Institute of Higher Education, Tonekabon,*

[b] *Department of Operations Research, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza 12613, Egypt*

[c] *Department of Mathematics, College of Science and Arts, Qassim University, Al- Badaya 51951, Saudi Arabia,*

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A B S T R A C T

Rough set theory expresses vagueness, not by means of membership, but employing a boundary region of a set. If the boundary region of a set is empty, it means that the set is crisp. Otherwise, the set is rough. Nonempty boundary region of a set means that our knowledge about the set is not sufficient to define the set precisely. In this paper, we introduce the concept of rough function and its convexity and differentiability based on its boundary region. The RP problem is converted into two subproblems namely, lower and upper approximation problem. The Kuhn-Tucker. Saddle point of rough programming problem (RPP) is discussed. In addition, in the case of differentiability assumption the solution of the RP problem is investigated. A numerical example is given to illustrate the methodology.

1. Introduction

Rough set theory has found in many interesting applications. The rough set approach seems to be of fundamental importance to cognitive sciences, especially in the areas of machine learning, knowledge discovery from databases, expert systems, inductive reasoning and pattern recognition. First of all, Pawlak et al. [15] and Pawlak [16] introduced the concept of a rough set. There are many applications for the rough set theory as artificial intelligence, expert systems, civil engineering [4], medical data analysis [5], data mining [5, 14, 17, 23], Pattern recognition [14, 19], and decision theory [8, 9].

* Correspondig author
E-mail address: hnajafi@aihe.ac.ir (H. Saberi Najafi)

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According to the decision maker (DM) influence in the optimization process, multiobjective optimization (MO) methods can be classified into four categories (Hwang and Masud [7]). Sasaki and Gen [20]) proposed a hybridized genetic algorithm for solving multiple- objective nonlinear programming having fuzzy multiple objective functions and constraints with generalized upper bounding structure. Wang and Chaing [23] applied user preference enabling method to solve general constrained nonlinear MO problems. Kundu and Islam [12] introduced an interactive method to design a high reliable and productivity system with minimum cost to solve multi- objective optimization problem. Waliv et al. [22] studied the effect of capital investment and warehouses space on profits as well as shortage cost through sensitivity analysis and compared the efficiency of fuzzy nonlinear programming and intuitionistic fuzzy optimization techniques to obtain the solution. Ahmed [1] proposed a method to solve MO problems with intuitionistic fuzzy parameters. Liu et al. [13] introduced a new systematic method for determining an optimal operation scheme for minimizing octane number loss and operational risks.

In this paper, the concept of rough function and its convexity and differentiability based on its boundary region are introduced. In addition, a new kind of rough programming problem and its solutions is discussed based on the notion of boundary region. Many researchers investigated the study of rough either in the objective functions or constraints or the twice (Khalifa [9]; Khalifa et al. [10]; Garg et al. [6]; Khalifa et al. [11]; Zaher et al. [25]; Zaher et al. [26]; Ammar and Emsimir [2]; and Ammar and Al- Asfar[3]).

This paper is organized as: Section 2, some preliminaries related to the rough function and its convexity based on its boundary region are introduced. Section 3 concerns with the formulation of rough programming problem, the related two problems, which one of them is called upper approximation problem (UAP) and the second is the lower approximation problem LAP and surely and possible optimal solution. In Section 4, we discuss the Kuhn-Tucker. Saddle point of rough programming (RP) problem. In section 5, we investigate the solution of (RPP) in the cases of differentiability. In Section 6, a numerical example is given in the sake of the paper for illustration. Finally, some concluding remarks are reported in Section 7.

2. Preliminaries

In this section, definition of rough function and its convexity based on its boundary region is introduced.

Definition 1. Let $\tilde{f}^R : \mathbb{R}^n \to \mathbb{R}$ and $u,\ \hat{u} \in \mathbb{R}, u < \hat{u}$. Suppose that the universal set $V(V = \{f(x) : f : \mathbb{R}^n \to \mathbb{R}\})$. The set of functions $\{f_i\} \subset V$ is the lower approximation of $\tilde{f}^R$ which is denoted by $f^{LA}(x)$ and is defined as $f^{LA}(x) = \{f(x) \in V : |f_j(x) - \tilde{f}^R| < u\}$, and the set of functions $\{f_j\} \subset V$ is the upper approximation of $\tilde{f}^R$ which is denoted by $f^{UA}(x)$ and is defined as $f^{UA}(x) = \{f(x) \in V : |f_j(x) - \tilde{f}^R| < \hat{u}\}$. The function $\tilde{f}^R$ is called rough function if $f^{LA}(x) \neq f^{UA}(x)$.

Definition 2. The boundary function of the rough function $\tilde{f}^R$ *is* $F(x) = f^{UA}(x) - f^{LA}(x)$, where $f^{LA}(x)$, and $f^{UA}(x)$ are the lower and upper approximations of $\tilde{f}^R$; respectively.

Definition 3. A rough function $\tilde{f}^R$ is said to be convex if the boundary function $F(x)$ is convex.

3. Problem statement

A rough programming (RP) problem in which the objective function is rough is formulated as

(RP) $\min \tilde{f}^R(x)$

Subject to

$$\mathrm{X} = \{x \in \mathbb{R}^{\mathrm{n}} : h_r(x) \leq 0, r = \overline{1, m}\}.$$

Where, $\tilde{f}^R(x)$ is rough function with lower and upper approximations $f^{LA}(x)$, and $f^{UA}(x)$; respectively and

$f^{LA}(x) \leq \tilde{f}^{R}(x) \leq f^{UA}(x)$, and X represents the crisp feasible region.

In order to solve the RP problem, let us solve the following boundary problem

(BP) $\quad min\, F(x) = f^{UA}(x) - f^{LA}(x)$

Subject to

$$X = \{x \in \mathbb{R}^n : h_r(x) \leq 0, r = 1, 2, \ldots, m\}.$$

Where, X is convex set and $h_r(x), r = 1, 2, \ldots$,m are convex and continuous functions.

The BP problem can be separated into the following two subproblems as:

(LA) $\quad min\, F(x) = f^{LA}(x)$

Subject to

$$\mathrm{X} = \{x \in \mathbb{R}^n : h_r(x) \leq 0, r = \overline{1,\ m}\}, \text{ and}$$

(UA) $\quad min\, F(x) = f^{UA}(x)$

Subject to

$$X = \{x \in \mathbb{R}^n : h_r(x) \leq 0, r = 1, 2, \ldots, m\}.$$

Here, we assume that $f^{UA}(x)$ is convex function and $f^{LA}(x)$ is concave function.

The optimal solution of lower problem (LA) is denoted by $f^{LA}(x^*) = \max_{x \in X} f^{LA}(x)$, and the optimal solution of upper approximation problem (UA) is denoted by

$$f^{UA}(x^*) = \min_{x \in X} f^{UA}(x).$$

Definition 4. The optimal solution of the RP problem is $\tilde{\mathrm{f}}^{\mathrm{R}}(\mathrm{x}^*)$ where $f^{LA}(x^*) \leq \tilde{f}^{R}(x^*) \leq f^{UA}(x^*)$, where S^{L}, and S^{U} are the sets of the solutions of problems (LA) and (UA); respectively.

Definition 5. A solution $x^* \in S^L \cap S^U, F(x^*) = 0$ is called surely optimal solution of the RP problem.

Definition 6. A solution $x^* \in S^L \cap S^U, F(x^*) \neq 0$n is called possibly optimal solution of the RP problem.

Definition 7. A solution $x^* \in \mathrm{S}^L \cap \mathrm{S}^U$ is called nearly possibly optimal solution of the RP problem.

Lemma 1. If x^* is the solution of the boundary problem (BP), then x^* is the solution for the lower and upper approximation problems.

Proof. Let x^* be a solution of the BP, then

$$f^{UA}(x^*) - f^{LA}(x^*) \leq f^{UA}(x) - f^{LA}(x);\ \forall x$$

Suppose that x^* is not a solution for the (LAP) and (UA), then there exists an $\overline{A} \in X$ such that $f^{UA}(\overline{x}) \leq f^{UA}(x^*)$, this implies that $f^{UA}(\overline{x}) - f^{LA}(\overline{x}) < f^{UA}(x^*) - f^{LA}(\overline{x}), f^{LA}(x^*) < f^{LA}(\overline{x})$ which leads to

$$f^{UA}(x^*) - f^{LA}(x^*) > f^{UA}(x^*) - f^{LA}(\overline{x}).$$

Thus $f^{UA}(\overline{x}) - f^{LA}(\overline{x}) < f^{UA}(x^*) - f^{LA}(x^*)$, contradicts that x^* is a solution of BP. Therefore, x^* is a solution of the two problems (LA) and (UA).

4. Rough Kuhn- Tucker Saddle point

Consider the rough problem

$$min\, \tilde{f}^R(x)$$

Subject to (1)

$$X = \{x \in \mathbb{R}^n : h_r(x) \le 0, r = \overline{1,\ m}\},$$

$$f^{LA}(x) \le \tilde{f}^R(x) \le f^{UA}(x).$$

The rough Kuhn- Tucker saddle point for problem (1) takes the form

$$\tilde{f}^R(x^*) + \sum_{r=1}^m \gamma_r h_r(x^*) + \gamma_{m+1}\left(f^{LA}(x^*) - \tilde{f}^R(x^*)\right) + \gamma_{m+2}\left(\tilde{f}^R(x^*) - f^{UA}(x^*)\right)$$

$$\le \quad \tilde{f}^R(x^*) + \sum_{r=1}^m \gamma_r^*\, h_r(x^*) + \gamma_{m+1}^*\left(f^{LA}(x^*) - \tilde{f}^R(x^*)\right) + \gamma_{m+2}^*\left(\tilde{f}^R(x^*) - f^{UA}(x^*)\right)$$

$$\le \quad \tilde{f}^R(x) + \sum_{r=1}^m \gamma_r^*\, h_r(x) + \gamma_{m+1}^*\left(f^{LA}(x) - \tilde{f}^R(x)\right) + \gamma_{m+2}^*\left(\tilde{f}^R(x) - f^{UA}(x)\right), or$$

$$(1 - \gamma_{m+1} + \gamma_{m+2})\, \tilde{f}^R(x^*) + \sum_{r=1}^m \gamma_r h_r(x^*) + \gamma_{m+1} f^{LA}(x^*) - \gamma_{m+2} f^{UA}(x^*)$$

$$\le (1 - \gamma_{m+1}^* + \gamma_{m+2}^*)\, \tilde{f}^R(x) + \sum_{r=1}^m \gamma_r^* h_r(x^*) + \gamma_{m+1}^* f^{LA}(x^*) - \gamma_{m+2} f^{UA}(x^*)$$

$$\le (1 - \gamma_{m+1}^* + \gamma_{m+2}^*)\, \tilde{f}^R(x) + \sum_{r=1}^m \gamma_r^* h_r(x) + \gamma_{m+1}^* f^{LA}(x) - \gamma_{m+2} f^{UA}(x).$$

Theorem 1. If (x^*, γ_r^*), where $\gamma_r^* \ge 0, r = \overline{1, m+2}$, and $\sum_{r=1}^{m+1} \gamma_r^*$ is a rough Kuhn- Tucker saddle point, then x^* is a solution of the RP problem.

Proof. Assume that $(x^*, \gamma_r^*), r = \overline{1, m+2}$ is a rough Kuhn- Tucker saddle point**,** then for $\gamma_r \ge 0, \gamma_r \in \mathbb{R}^{m+2}$, we get

$$(1 - \gamma_{m+1} + \gamma_{m+2})\, \tilde{f}^R(x^*) + \sum_{r=1}^m \gamma_r h_r(x^*) + \gamma_{m+1} f^{LA}(x^*) - \gamma_{m+2} f^{UA}(x^*)$$

$$\le (1 - \gamma_{m+1}^* + \gamma_{m+2}^*)\, \tilde{f}^R(x^*) + \sum_{r=1}^m \gamma_r^* h_r(x^*) + \gamma_{m+1}^* f^{LA}(x^*) - \gamma_{m+2}^* f^{UA}(x^*)$$

$$\le (1 - \gamma_{m+1}^* + \gamma_{m+2}^*)\, \tilde{f}^R(x) + \sum_{r=1}^m \gamma_r^* h_r(x) + \gamma_{m+1}^* f^{LA}(x) - \gamma_{m+2}^* f^{UA}(x).$$

From the first inequality, we have

$$(1 - \gamma_{m+1} + \gamma_{m+2})\, \tilde{f}^R(x^*) + \sum_{r=1}^m \gamma_r h_r(x^*) + \gamma_{m+1} f^{LA}(x^*) - \gamma_{m+2} f^{UA}(x^*)$$

$$\le (1 - \gamma_{m+1}^* + \gamma_{m+2}^*)\, \tilde{f}^R(x) + \sum_{r=1}^m \gamma_r^* h_r(x^*) + \gamma_{m+1}^* f^{LA}(x^*) - \gamma_{m+2} f^{UA}(x^*),$$

Or

$$(1 - \gamma_{m+1} + \gamma_{m+2} + 1 - \gamma_{m+1}^* + \gamma_{m+2}^*)\, \tilde{f}^R(x^*) + \sum_{r=1}^m (\gamma_r - \gamma_r^*) h_r(x^*)$$

$$+(\gamma_{m+1} - \gamma_{m+1}^*) f^{LA}(x^*) - (\gamma_{m+2} - \gamma_{m+2}^*) f^{UA}(x^*) \le 0,$$

which implies to

$$(\gamma_{m+1}-\gamma_{m+1}^*)\left(f^{LA}(x^*)-\tilde{f}^R(x^*)\right)+(\gamma_{m+2}-\gamma_{m+2}^*)\left(\tilde{f}^R(x^*)-f^{UA}(x^*)\right)+\sum_{r=1}^{m}(\gamma_r-\gamma_r^*)\,h_r(x^*)\le 0.$$

This inequality is true for all $\gamma_r,\gamma_r^*,\gamma_{m+1},\gamma_{m+1}^*,\gamma_{m+2},\gamma_{m+2}^*.$ In the case, $\gamma_{m+1}=\gamma_{m+1}^*$ and $\gamma_{m+2}=\gamma_{m+2}^*$, we have $\sum_{r=1}^{m}(\gamma_r-\gamma_r^*)\,\mathrm{h_r}(\mathrm{x}^*)\le 0$. Assume that $\gamma_r=\gamma_r^*, r=1,2,\dots,i-1,i+1,\dots,m$ and $\gamma_i^*=\gamma_i-1$. Then, $h_r(x^*)\le 0$. By repeating this for all i, we have $h_r(x^*)\le 0$ and hence x^* is feasible point. Since $\gamma_r^*\ge 0$ and $h_r(x^*)\le 0$, we get $\sum_{r=1}^{m}\gamma_r^*\,h_r(x^*)\le 0$. Again and from the first inequality, where $\gamma_{m+1}=\gamma_{m+1}^*$ and $\gamma_{m+2}=\gamma_{m+2}^*$, and by setting γ_r we obtain $\sum_{r=1}^{m}\gamma_r^*\,h_r(x^*)\ge 0$. Hence, $\sum_{r=1}^{m}\gamma_r^*\,h_r(x^*)=0$. Thus,

$$(\gamma_{m+1}-\gamma_{m+1}^*)\left(f^{LA}(x^*)-\tilde{f}^R(x^*)\right)+(\gamma_{m+2}-\gamma_{m+2}^*)\left(\tilde{f}^R(x^*)-f^{UA}(x^*)\right)+\sum_{r=1}^{m}(\gamma_r-\gamma_r^*)\,h_r(x^*)\le 0.$$

By taking, $\gamma_{m+1}=\gamma_{m+1}^*-1$, and $\gamma_{m+2}=\gamma_{m+2}^*-1$, we have

$$(\gamma_{m+1}-1-\gamma_{m+1}^*)\left(f^{LA}(x^*)-\tilde{f}^R(x^*)\right)+(\gamma_{m+2}-1-\gamma_{m+2}^*)\left(\tilde{f}^R(x^*)-f^{UA}(x^*)\right)+\textstyle\sum_{r=1}^{m}\gamma_r\,h_r(x^*)\le 0.$$

This leads to

$$\left(f^{LA}(x^*)-\tilde{f}^R(x^*)\right)+\left(\tilde{f}^R(x^*)-f^{UA}(x^*)\right)+\textstyle\sum_{r=1}^{m}\gamma_r\,h_r(x^*)\le 0.$$

Since the inequality is valid for each $\gamma_r\ge 0$, then for $\gamma_r=0$, we get $\left(f^{LA}(x^*)-\tilde{f}^R(x^*)\right)+\left(\tilde{f}^R(x^*)-f^{UA}(x^*)\right)\le 0$, and

$$f^{UA}(x^*)-f^{LA}(x^*)\le 0. \tag{2}$$

Taking $\gamma_{m+1}=\gamma_{m+1}^*+1$, and $\gamma_{m+2}=\gamma_{m+2}^*+1$, we have

$$(\gamma_{m+1}+1-\gamma_{m+1}^*)\left(f^{LA}(x^*)-\tilde{f}^R(x^*)\right)+(\gamma_{m+2}+1-\gamma_{m+2}^*)\left(\tilde{f}^R(x^*)-f^{UA}(x^*)\right)+\textstyle\sum_{r=1}^{m}\gamma_r\,h_r(x^*)\le 0.$$

Thus,

$$\left(f^{LA}(x^*)-\tilde{f}^R(x^*)\right)+\left(\tilde{f}^R(x^*)-f^{UA}(x^*)\right)+\textstyle\sum_{r=1}^{m}\gamma_r\,h_r(x^*)\le 0.$$

Since the inequality is valid for each $\gamma_r\ge 0$, then for $\gamma_r=0$, we have $\left(f^{LA}(x^*)-\tilde{f}^R(x^*)\right)+\left(\tilde{f}^R(x^*)-f^{UA}(x^*)\right)\le 0$, and

$$f^{UA}(x^*)-f^{LA}(x^*)\ge 0. \tag{3}$$

Hence from (2) and (3), we conclude that $f^{LA}(x^*)=\tilde{f}^R(x^*)=f^{UA}(x^*)$ (i. e., x^* is a surely optimal solution for the RP problem.

From the second inequality we have,

$$(1-\gamma_{m+1}^*+\gamma_{m+2}^*)\,\tilde{f}^R(x^*)+\textstyle\sum_{r=1}^{m}\gamma_r^*h_r(x^*)+\gamma_{m+1}^*f^{LA}(x^*)-\gamma_{m+2}f^{UA}(x^*)$$
$$\le(1-\gamma_{m+1}^*+\gamma_{m+2}^*)\,\tilde{f}^R(x)+\textstyle\sum_{r=1}^{m}\gamma_r^*h_r(x)+\gamma_{m+1}^*f^{LA}(x)-\gamma_{m+2}^*f^{UA}(x).$$

Since, $\sum_{r=1}^{m}\gamma_r^*h_r(x^*)=0$. Then

$$(1-\gamma_{m+1}^*+\gamma_{m+2}^*)\left(\tilde{f}^R(x^*)-\tilde{f}^R(x)\right)\le\sum_{r=1}^{m}\gamma_r^*h_r(x)+\gamma_{m+1}^*\left(f^{LA}(x)-f^{LA}(x^*)\right)+\gamma_{m+2}^*\left(f^{UA}(x)-f^{UA}(x^*)\right),$$

$$\tilde{f}^R(x^*) - \tilde{f}^R(x) \leq \frac{\sum_{r=1}^{m}\gamma_r^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)} h_r(x) + \frac{\gamma_{m+1}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}\left(f^{LA}(x) - f^{LA}(x^*)\right) + \frac{\gamma_{m+2}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}\left(f^{UA}(x) - f^{UA}(x^*)\right).$$

For $x^* \in S^L \cap S^U$, we have $f^{LA}(x) \leq f^{LA}(x^*)$ and $f^{UA}(x) \geq f^{UA}(x^*)$. Since $\sum_{r=1}^{m+1}\gamma_r = 1$, and $\gamma_{m+1}^* = \gamma_1^* + \gamma_1^* + \cdots + \gamma_m^*$, thus $1 - \gamma_{m+1}^* + \gamma_{m+2}^* \leq 0$ which implies to $\tilde{f}^R(x^*) \leq \tilde{f}^R(x)$, $x \in X$. Hence, x* is a possible optimal solution of rough problem. For $x^* \in S^L, x^* \notin S^U$, we obtain $f^{LA}(x^*) \geq f^{LA}(x)$ and

$$\tilde{f}^R(x^*) - \tilde{f}^R(x) \leq \frac{\gamma_{m+2}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}\left(f^{UA}(x) - f^{UA}(x^*)\right).$$

Now, there are two cases:

Case 1: $f^{UA}(x^*) - f^{UA}(x) \leq 0$; ; $\forall x \in X$, this implies that x* is a nearly possibly optimal solution.

Case 2: $f^{UA}(x^*) - f^{UA}(x) > 0$.

Let x* be not nearly possible optimal solution of rough problem, then there is $\overline{x} \in X$: $\tilde{f}^R(\overline{x}) < \tilde{f}^R(x^*)$. Since $x^* \in S^L, x^* \notin S^U$, so x^* is not a solution for boundary problem (BP), i.e., there is $\overline{x} \in X$:

$$f^{UA}(\overline{x}) - f^{LA}(\overline{x}) < f^{UA}(x^*) - f^{LA}(x^*), f^{LA}(x^*) - f^{LA}(\overline{x}) < f^{UA}(x^*) - f^{UA}(\overline{x}).$$

(i) If $f^{UA}(x^*) < f^{UA}(\overline{x})$, then $f^{LA}(x^*) < f^{LA}(\overline{x})$. This contradicts that $x^* \in S^L$, and hence x^* must be a nearly possible optimal solution for the RP problem.

(ii) If $f^{UA}(x^*) > f^{UA}(\overline{x})$, then we may write $f^{UA}(x^*) = f^{UA}(\overline{x}) + \theta, \theta > 0$. which implies to $f^{LA}(x^*) - f^{LA}(\overline{x}) < \theta, \theta > 0$. Then, we have two cases:

(a) $f^{LA}(x^*) > f^{LA}(\overline{x})$ which is not considered, where x* ∈ S^L,

(b) $f^{LA}(x^*) < f^{LA}(\overline{x})$, which contradicts that $x^* \in S^L$, and hence x^* must be a nearly possible optimal solution for the RP problem.

For $x^* \in S^U, x^* \notin S^{UL}$, we obtain $f^{UA}(x^*) \leq f^{UA}(x)$ and

$$\tilde{f}^R(x^*) - \tilde{f}^R(x) \leq \frac{\gamma_{m+1}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}\left(f^{LA}(x) - f^{LA}(x^*)\right).$$

So, there are two cases:

Case 3: $f^{LA}(x^*) - f^{LA}(x) \leq 0$; ; $\forall x \in X$, this implies that x^* is a nearly possibly optimal solution.

Case 4: $f^{LA}(x^*) - f^{LA}(x) > 0$.

Let x^* be not nearly possible optimal solution of rough problem, then there is $\overline{x} \in X$: $\tilde{f}^R(\overline{x}) < \tilde{f}^R(x^*)$. Since $x^* \in S^U, x^* \notin S^L$, so x* is not a solution for boundary problem (BP), i.e., there is $\overline{x} \in X$:

$$f^{UA}(\overline{x}) - f^{LA}(\overline{x}) < f^{UA}(x^*) - f^{LA}(x^*), f^{UA}(\overline{x}) - f^{UA}(x^*) < f^{LA}(\overline{x}) - f^{UA}(x^*).$$

(iii) If $f^{LA}(\overline{x}) < f^{UA}(x^*)$, then $f^{UA}(\overline{x}) < f^{LA}(x^*)$. This contradicts that $x^* \in S^U$, and hence x^* must be a nearly possible optimal solution for the RP problem.

(iv) If $f^{LA}(\overline{x}) > f^{UA}(x^*)$, then we may write $f^{LA}(x^*) = f^{LA}(\overline{x}) + \theta, \theta > 0$. which implies to $f^{UA}(\overline{x}) - f^{UA}(\overline{x}) < \theta, \theta > 0$. Then, we have two cases:

(c) $f^{LA}(x^*) > f^{LA}(x^*)$ which is not considered, where $x^* \in S^U$,

(d) $f^{UA}(x^*) < f^{UA}(\overline{x})$, which contradicts that $x^* \in S^U$, and hence x^* must be a nearly possible optimal solution for the RP problem.

5. Rough function differentiability

A rough function $\tilde{f}^R(x)$ is said to be differentiable if its boundary $F(x) = f^{UA}(x) - f^{LA}(x)$ is differentiable. Then

$$F(x) - F(x^*) = \frac{\delta}{\delta x} F(x^*)(x - x^*) + \vartheta\big(x^*, \gamma(x - x^*)\big)\|x - x^*\|,$$

or equivalently

$$\tilde{f}^R(x) - \tilde{f}^R(x^*) = \frac{\delta}{\delta x}\tilde{f}^R(x^*)(x - x^*) + \vartheta\big(x^*, \gamma(x - x^*)\big)\|x - x^*\|,$$

where

$$\lim_{\vartheta \to 0} \vartheta\big(\mathrm{x}^*, \delta(x - \mathrm{x}^*)\big) = 0.$$

The rough Kuhn- Tucker conditions for the RP problem take the form

$$\frac{\delta}{\delta x}\tilde{f}^R(x^*) + \sum_{r=1}^{m} \gamma_r^* h_r(x^*) + \gamma_{m+1}^* \frac{\delta}{\delta x}\Big(f^{LA}(x^*) - \tilde{f}^R(x^*)\Big) + \gamma_{m+2}^* \frac{\delta}{\delta x}\Big(\tilde{f}^R(x^*) - f^{UA}(x^*)\Big), = 0$$

and

$$\gamma_r^* h_r(x^*) = 0, r = \overline{1, m};$$

$$\gamma_{m+1}^* \Big(f^{LA}(x^*) - \tilde{f}^R(x^*)\Big) = 0;$$

$$\gamma_{m+2}^* \Big(\tilde{f}^R(x^*) - f^{UA}(x^*)\Big) = 0;$$

$$\gamma_r^* \geq 0, r = \overline{1, m+2}.$$

Let $\sum_{r=1}^{m+1} \gamma_r^* = 1$. Then,

$$(1 - \gamma_{m+1}^* + \gamma_{m+2}^*)\frac{\delta}{\delta x}\tilde{f}^R(x^*) + \gamma_{m+1}^* \frac{\delta}{\delta x} f^{LA}(x^*) - \gamma_{m+2}^* \frac{\delta}{\delta x} f^{UA}(x^*) + \sum_{r=1}^{m} \gamma_r^* \frac{\delta}{\delta x} h_r(x^*) = 0, \text{ or}$$

$$\frac{\delta}{\delta x}\tilde{f}^R(x^*) + \frac{\gamma_{m+1}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}\frac{\delta}{\delta x} f^{LA}(x^*) - \frac{\gamma_{m+2}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}\frac{\delta}{\delta x} f^{UA}(x^*) + \frac{\sum_{r=1}^{m}\gamma_r^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}\frac{\delta}{\delta x} h_r(x^*) = 0;$$

$$\frac{\sum_{r=1}^{m}\gamma_r^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}\frac{\delta}{\delta x} h_r(x^*) = 0, r = \overline{1, m};$$

$$\frac{\gamma_{m+1}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)} f^{LA}(x^*) = 0;$$

$$\frac{\gamma_{m+2}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)} f^{UA}(x^*);$$

$$\gamma_r^* \geq 0, r = \overline{1, m+2}.$$

Theorem 2. Let $\tilde{f}^R(x), f^{UA}(x)$, and $h(x)$ are convex and differentiable functions at x^*, and let $f^{LA}(x)$ be concave and differentiable at $x^* \in X$. Suppose that $f^{UA}(x^*)$>0 and $f^{LA}(x^*) > 0$. If (x^*, γ_r^*), where $\gamma_r^* \geq 0, r = \overline{1, m+2}$ is a solution of the Kuhn- Ticker conditions, then x^* is a solution for the RP problem.

Proof. Let (x^*, γ_r^*)be a solution of the rough Kuhn- Tucker conditions. Since, $\tilde{f}^R(x)$ is a convex and differentiable at x^*, we get $\tilde{f}^R(x) - \tilde{f}^R(x^*) \geq \frac{\delta}{\delta x}\tilde{f}^R(x^*)(x - x^*)$. Since,

$\frac{\delta}{\delta x}\tilde{f}^R(x^*) = \frac{\gamma_{m+2}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}\frac{\delta}{\delta x}f^{UA}(x^*) - \frac{\gamma_{m+1}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}\frac{\delta}{\delta x}f^{LA}(x^*) - \frac{\sum_{r=1}^m \gamma_r^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}h_r(x^*)$ and $f^{UA}(x)$, $f^{LA}(x)$, and $h_r(x)$, are differentiable, then

$$f^{UA}(x) - f^{UA}(x^*) = \frac{\delta}{\delta x}f^{UA}(x^*)(x - x^*) + \vartheta\big(x^*, \gamma(x - x^*)\big)\|x - x^*\|,$$

$$f^{LA}(x) - f^{LA}(x^*) = \frac{\delta}{\delta x}f^{LA}(x^*)(x - x^*) + \vartheta\big(x^*, \gamma(x - x^*)\big)\|x - x^*\|,$$

$$h_r(x) - h_r(x^*) = \frac{\delta}{\delta x}h_r(x^*)(x - x^*) + \vartheta\big(x^*, \gamma(x - x^*)\big)\|x - x^*\|.$$

Then,

$$\tilde{f}^R(x) - \tilde{f}^R(x^*) \geq \frac{\gamma_{m+2}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}\Big(f^{UA}(x) - f^{UA}(x^*) - \vartheta\big(x^*, \gamma(x - x^*)\big)\|x - x^*\|\Big) - \frac{\gamma_{m+1}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}\Big(f^{LA}(x) - f^{LA}(x^*) - \vartheta\big(x^*, \gamma(x - x^*)\big)\|x - x^*\|\Big) - \frac{\sum_{r=1}^m \gamma_r^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}\Big(h_r(x) - h_r(x^*) - \vartheta\big(x^*, \gamma(x - x^*)\big)\|x - x^*\|\Big).$$

Since $\lim_{\vartheta \to 0} \vartheta\big(x^*, \delta(x - x^*)\big) = 0$. Then

$$\tilde{f}^R(x) - \tilde{f}^R(x^*) \geq \frac{\gamma_{m+2}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}\big(f^{UA}(x) - f^{UA}(x^*)\big) - \frac{\gamma_{m+1}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}\big(f^{LA}(x) - f^{LA}(x^*)\big) - - \frac{\sum_{r=1}^m \gamma_r^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}\big(h_r(x) - h_r(x^*)\big).$$

From the Kuhn- Tucker conditions

$$\frac{\sum_{r=1}^m \gamma_r^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}h_r(x^*) = 0, r = \overline{1, m};$$

$$\frac{\gamma_{m+1}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}f^{LA}(x^*) = 0;$$

$$\frac{\gamma_{m+2}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}f^{UA}(x^*) = 0;$$

Then, the inequality

$\tilde{f}^R(x) - \tilde{f}^R(x^*) \geq \frac{\gamma_{m+2}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}f^{UA}(x) - \frac{\gamma_{m+1}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}f^{LA}(x) - \frac{\sum_{r=1}^m \gamma_r^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}h_r(x)$ is valid for each $\gamma_r^* \geq 0, r = \overline{1, m+2}$, and for $\gamma_r^* = 0$, we have

$$\tilde{f}^R(x) - \tilde{f}^R(x^*) \geq \frac{\gamma_{m+2}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}f^{UA}(x) - \frac{\gamma_{m+1}^*}{(1-\gamma_{m+1}^*+\gamma_{m+2}^*)}f^{LA}(x).$$

If $\gamma^*_{m+1};\gamma^*_{m+2} > 0$, then from the Kuhn- Tucker conditions we obtain $\tilde{f}^R(x^*) = f^{LA}(x^*)$ and $\tilde{f}^R(x^*) = f^{UA}(x^*)$. Then x* is surely optimal solution of the RP problem.

If $x^* \in S^L \cap S^U$, then $f^{UA}(x^*) \le f^{UA}(x); \forall\, x \in X$ anf$^{LA}(x^*) \ge f^{LA}(x);\ \forall x \in X$d, and then we get

$$\tilde{f}^R(x) - \tilde{f}^R(x^*) \ge \frac{\gamma^*_{m+2}}{(1-\gamma^*_{m+1}+\gamma^*_{m+2})} f^{UA}(x^*) - \frac{\gamma^*_{m+1}}{(1-\gamma^*_{m+1}+\gamma^*_{m+2})} f^{LA}(x^*).$$

In addition, from the Kuhn- Tucker condition$\tilde{f}^R(x) - \tilde{f}^R(x^*) \ge 0$s , this leads to $\tilde{f}^R(x^*) \le \tilde{f}^R(x)$, i. e., x^* is possibly optimal solution.

If $x^* \in S^L, x^* \notin S^U$, then $f^{LA}(x^*) \ge f^{LA}(x);\ \forall x \in X$, and we have

$$\tilde{f}^R(x) - \tilde{f}^R(x^*) \ge \frac{\gamma^*_{m+2}}{(1-\gamma^*_{m+1}+\gamma^*_{m+2})} f^{UA}(x^*) - \frac{\gamma^*_{m+1}}{(1-\gamma^*_{m+1}+\gamma^*_{m+2})} f^{LA}(x^*),$$ and

$$\tilde{f}^R(x) - \tilde{f}^R(x^*) \ge \frac{\gamma^*_{m+2}}{(1-\gamma^*_{m+1}+\gamma^*_{m+2})} f^{UA}(x).$$

From the assumption that $f^{UA}(x^*) > 0$, and x^* is not solution for the BP problem, $\frac{\gamma^*_{m+2}}{(1-\gamma^*_{m+1}+\gamma^*_{m+2})} = 0$.

Hence, $\tilde{f}^R(x) - \tilde{f}^R(x^*) \ge 0$ leads to $\tilde{f}^R(x^*) \le \tilde{f}^R(x);\ \forall x \in X$. Then $x^*$is nearly possibly optimal solution for the RP problem.

If $x^* \in S^U, x^* \notin S^L$, then $f^{UA}(x^*) \le f^{UA}(x);\ \forall x \in X$ and we have

$$\tilde{f}^R(x) - \tilde{f}^R(x^*) \ge \frac{\gamma^*_{m+2}}{(1-\gamma^*_{m+1}+\gamma^*_{m+2})} f^{UA}(x^*) - \frac{\gamma^*_{m+1}}{(1-\gamma^*_{m+1}+\gamma^*_{m+2})} f^{LA}(x^*),$$

From Kuhn-Tucker conditions, we have

$$\tilde{f}^R(x) - \tilde{f}^R(x^*) \ge \frac{\gamma^*_{m+1}}{(1-\gamma^*_{m+1}+\gamma^*_{m+2})} f^{LA}(x).$$

From the assumption that $f^{LA}(x^*) > 0$, and x^* is not solution for the BP problem, $\frac{\gamma^*_{m+1}}{(1-\gamma^*_{m+1}+\gamma^*_{m+2})} = 0$.

Thus, $\tilde{f}^R(x) - \tilde{f}^R(x^*) \ge 0$, which implies to $\tilde{f}^R(x^*) \le \tilde{f}^R(x);\ \forall x \in X$. Then x* is nearly possibly optimal solution for the RP problem.

6. Numerical example

Consider the following rough function

$\tilde{f}^R(x): X \to \mathbb{R}$ with $f^{LA}(x) = x_1 + x_2$ $f^{UA}(x) = \frac{1}{3}x_1^3 - 2x_1^2 - 10x_2 + 100$, and consider the following RP problem as

(RP) $\quad min\, \tilde{f}^R(x)$

Subject to

$$X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \le 10, 3.5 \le x_1 \le 6,\ x_2 \le 6, x_1 + x_2 \ge 1\}.$$

The lower and upper approximation problems are

(LA) $\quad min\, f^{LA}(x) = x_1 + x_2$

Subject to

$x \in X$, and

(UA) $$min\, f^{UA}(x) = \frac{1}{3}x_1^3 - 2x_1^2 - 10x_2 + 100$$

Subject to

$x \in X$.

Then, the RP problem is

(BP) $$min\, F(x) = f^{UA}(x) - f^{LA}(x)$$

Subject to
$x \in X$.

The solution of the LA problem is $S^L = \{(5,5)\}$, and the solution of the UA problem is $S^U = \{(1-\lambda)(6,4) + \lambda(4,6), 0 \leq \lambda \leq 1\}$. Then

1. There is no surely optimal; solution (Definition 3).
2. The possible optimal solution is $(5,5)$, where $(5,5) \in S^L \cap S^U$ and $F(5,5) \neq 0$ (Definition 4),
3. The nearly possibly solution is $\{(1-\lambda)(6,4) + \lambda(4,6), 0 \leq \lambda \leq 1\} \cup \{(5,5)\}$ (Definition 5).

7. Concluding Remarks

In this paper, we have introduced the concept of rough function and its convexity and differentiability based on its boundary region. Also, a new kind of rough programming problem and its solutions have discussed according to the notion of boundary region**.** The result shows the proposed method has its advantage in flexible decision-making corresponding to favorite priorities of alternatives. This study may be extended to additional fuzzy-like structures, such as Interval-valued fuzzy set, Pythagorean fuzzy set, Spherical fuzzy set, Intuitionistic fuzzy set, Picture fuzzy set, Neutrosophic set, etc., in future work.

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References

1. Ahmed, F. (2021). Robust neutrosophic programming approach for solving intuitionistic fuzzy multiobjective optimization problem. *Complex& Intelligent Systems*, https://doi.org/10.1007/s40747-021-00299-9.
2. Ammar, E. E. & Emsimir, A. A. A. (2020). A mathematical model for solving integer linear programming problems. *African Journal of Mathematics and Computer Science Research*, 13(1): 39- 50. DOI: 10.5897/AJMCSR2019.0804
3. Ammar, E. E., & Al- Al- Asfar, A. (2021). Approaches for solving fully fuzzy rough multi- objective nonlinear programming problems. *Academic Journal of Applied Mathematical Sciences*, 7(2), 113- 128
4. Arciszewski,T., & W. P. Ziarko, W. P. Adaptive Expert System for Preliminary Design Wind Bracings in Steel Skeleton Structure, Second Century of the Skyscraper, Van Nostrand Reinhold Company, New York, 1988.
5. Fibak,J., Z. Pawlak,Z., & K. Slowinski,K. (1986). Rough sets-based decision algorithms for treatment of duodenal ulcer by HSV, Bulletin of PAS, (34), 227-246.
6. Garg, H., Alodhaibi, S, S., & Khalifa, H. A. (2022). Study on multi-objective nonlinear programming problem with rough parameters. *Journal of Intelligent & Fuzzy Systems*, 42(4), 3591-3604.

7. Hwang, C. L., & Masud, A. S. M. Multiple Objective Decision Making: Methods and Applications, Springer Berlin, 1979.
8. Jin-Mao, W. (2003). Rough set-based approach to selection of node, *International Journal Computational Cognition* 1(2), 25-40.
9. Khalifa, H. A. (2018). Study on multi- objective nonlinear programming in optimization of the rough interval constraints. *International Journal of Industrial Engineering& Production Research*, 29(4), 407- 413.
10. Khalifa H.A., Dragan Pamucar, Amina Hadj Kacem, & W.A. Afifi (2022). A novel approach for characterizing solutions of rough optimization problems based on boundary region. *Computational Intelligence and Neuroscience*, Vol.2022, Article ID 8662289, 12 Pages.
11. Khalifa, H.A., Pavan Kumar, & Hassan, B.A. (2021). An inexact rough interval of normalized heptagonal fuzzy numbers for solving vendor selection problem. *Applied Mathematics and Information Sciences* (Natural Science), 15(3), 317- 324.
12. Kundu, T., & Islam, S. (2019). An interactive weighted fuzzy goal programming technique to solve multi- objective reliability optimization problem. *Journal of Industrial Engineering International*, (15), 95- 104. https://doi.org/10.1007/s40092-019-0321-y.
13. Liu, X., Liu, Y., He, X., Xiao, M., & Jiang, T. (2021). Multi- objective nonlinear programming model for reducing octane number loss in gasoline refining process based on data mining technology. Processes (MDPI), 9(4), 721. https://doi.org/10.3390/pr9040721.
14. Mitatha, S., K. Dejhan, and F. Cheevasuvit. (2003). Some experimental results of using rough sets for printed Thai characters recognition, *International Journal of Computational Cognition* 1(4), 109-121.
15. Munakata,T., Rough control: A perspective, Rough Sets and Data Mining Analysis of Imprecise Data, 1997, pp. 77-87.
16. Pawlak, Z., K. Slowinski,K., & R. Slowinski, R. (1986). Rough classification of patients after highly selective vagotomy for duodenal ulcer, *International Journal of Man-Machine Studies*, (24), 413-433
17. Pawlak,Z. Rough Sets, Kluwer Academic Publishers, Dordrecht, 1991.
18. Pawlak,Z. Rough Sets, in: T. Y. Lin, N. Cercone (Eds.), Rough Sets and Data Mining Analysis of Imprecise Data. Kluwer Academic, Boston, MA, 1997.
19. Qiang,S., & C. Alexios, C. (1999). Combining rough sets and data-driven fuzzy learning for generation of classification rules. Pattern Recognition, (32), 2073-2076.
20. Sasaki, M., & Gen, M. (2010). A Method of fuzzy multi- objective nonlinear programming with GUB structure by hybrid genetic algorithm. *International Journal of Smart Engineering System Design*, 5(4), 281- 288.
21. Slowinski,R., and R. Intelligent Decision Support-Handbook of Advances and Applications of the Rough Set Theory, Kluwer Academic Publishers, Dordrecht, 1992.
22. Waliv, R. H., Mishra, U., Garg, H., & Umap, H. P. (2020). A Nonlinear programming approach to solve the stochastic multi-objective inventory model using the uncertain information. *Arabian Journal for Science and Engineering*, (45), 6963- 6973.
23. Wang, S., and Chaing, H- D. (2018). Constrained multiobjective nonlinear optimization: A User preference enabling method. IEEE *Transaction on Cybernetic*, 49(7), 2779- 2791.
24. Yao, Y. Y., S. K. Wong,S.K., and Lin,T. Y. A review of rough sets models, in: T. Y. Lin, N. Cercone (Eds.), Rough Sets and Data Mining Analysis of Imprecise data, Kluwer Academic, Boston, MA, 1997.
25. Zaher, H., Khalifa H. A., & Mohamed, S. (2018). On Rough Interval Multi Criteria Decision Making. *International Journal of Scientific& Technology*, 7(3):44- 54.
26. Zaher, H. A. Khalifa, H. A., & Abeer Ahmed. (2020). Rough interval max plus algebra algorithm for traffic problems. International *Journal of Engineering and Advanced Technology*, 9(4), 58- 60.

Eldalatpanah, S. A., Abd El-Wahed Khalifa, H. & Saberi Najafi, H. (2022). On Characterizing Solutions of Optimization Problems with Roughness in the Objective Functions. *Fuzzy Optimization and Modelling Journal*, 3(3), 48-58.

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