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On Semitopological De Morgan Residuated Lattices

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Abstract. The class of De Morgan residuated lattices was introduced by L. C. Holdon (Kybernetika 54(3):443-475, 2018), recently, many mathematicians have studied the theory of ideals or filters in De Morgan residuated lattices and some of them investigated the properties of De Morgan residuated lattices endowed with a topology. In this paper, we introduce the notion of semitopological De Morgan residuated lattice, we present some examples and by considering the notion of upsets, for any element a of a De Morgan residuated lattice L, there is a topology τ_a on L and we show that L endowed with the topology τ_a is semitopological with respect to $\vee, \wedge$ and $\odot$, and right topological with respect to $\rightarrow$. Moreover, in the general case of residuated lattices we prove that L endowed with the topology τ_a is semitopological with respect to $\odot$ and right topological with respect to $\rightarrow$. Finally, we obtain some of the topological aspects of this structure such as L endowed with the topology τ_a is a $\mathbf{T_0}$-space, but it is not a $\mathbf{T_1}$-space or Hausdorff space.

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1 Introduction

Residuation is a fundamental concept of ordered structures and categories. The origin of residuated lattices is in Mathematical Logic without contraction. It is known to all that algebraic research on logical systems has considerable applications. In particular, it plays a meaningful role in artificial intelligence, which make computer simulate human being in dealing with fuzzy and uncertain information. Residuated structures were introduced by Ward and Dilworth in [18] as a generalization of ideal lattices of rings. The general definition of a residuated lattice was given by Galatos et al. (2007) [4]. They first developed the structural theory of this kind of algebra.

Hájek (1998) [5] introduced the notion of BL-algebras and the concepts of filters and prime filters in BL-algebras in order to provide an algebraic proof of the completeness theorem of *basic logic* (*BL*, for short), arising from the continuous triangular norms, familiar in the fuzzy logic frame-work. Using prime filters in BL-algebras, he proved the completeness of Basic Logic. Soon after, Turunen (1999) [17] published a study on BL-algebras and their deductive systems.

A weaker logic than *BL* called *Monoidal t-norm based logic* (*MTL*, for short) was defined by Esteva and Godo (2001) [10] and proved by Jenei and Montagna (2002) [12] to be the logic of left continuous t-norms and their residua. The algebraic counterpart of this logic is *MTL-algebra*, also introduced by Esteva and Godo (2001) [10]. In Esteva and Godo (2001) [10] a residuated lattice L is called *MTL-algebra* if the prelinearity property holds in L.

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In 2018, L. C. Holdon introduced a new class of residuated lattices called De Morgan residuated lattices, which comprises salient subclasses of residuated lattices such as Boolean algebras, BL-algebras, MTL-algebras, Stonean residuated lattices, and regular residuated lattices (MV-algebras, IMTL-algebras), the author investigated it from the view of ideal theory ([6]). Recently, a lot of work has been done using De Morgan residuated lattices. For example, in 2020, L. C. Holdon [8] studied the prime and maximal spectra and the reticulation of residuated lattices with applications to De Morgan residuated lattices. In 2020, D. Piciu [16] found new characterizations for prime and maximal ideals in De Morgan residuated lattices with interesting applications. In 2021, D. Buşneag et al. [3] and L. C. Holdon and A. Borumand Saeid [9] developed a theory of ideals in residuated lattices with interesting applications in De Morgan residuated lattices. In 2022, F. Woumfo et al. [19] developed a study on state ideals and state relative annihilators in De Morgan state residuated lattices. In conclusion, the class of De Morgan residuated lattices played an important role in the theory of ideals.

In 2018, L. C. Holdon ([7]) studied a new topology based on upsets (filters) in residuated lattices and proved that the class of divisible residuated lattices with respect to that topology form semitopological divisible residuated lattices.

In this paper, motivated by the previous research on upsets (filters) in residuated lattices and their generated topology on residuated lattices and by the importance of the class of De Morgan residuated lattices in the theory of ideals, we want to answer the following question: Are De Morgan residuated lattices semitopological algebras? Moreover, we investigate the topological properties of De Morgan residuated lattices.

This paper is organized as follows: in Section 2, we recall from the literature some preliminaries including the basic definitions, some examples of residuated lattices, rules of calculus and theorems that are needed in the sequel.

In Section 3, using examples we show that the class of divisible and De Morgan residuated lattices are different, we notice that the class of BL-algebras is a subclass of divisible and De Morgan residuated lattices. In order to give an answer to the question that "Are De Morgan residuated lattices semitopological algebras?", it is necessary to find a nontrivial topology to work with, and the idea comes from the paper [7] where was proved that any divisible residuated lattice is a semitopological algebra. Using examples we show that the classes of divisible residuated lattices and De Morgan residuated lattices are different, and by considering the notion of upsets, for any element a of a De Morgan residuated lattice L, there is a topology τ_a on L and in Corollary 3.7 we show that $(L, \{\vee, \wedge, \odot\}, \tau_a)$ is a semitopological De Morgan residuated lattice and $(L, \{\rightarrow\}, \tau_a)$ is a right topological De Morgan residuated lattice. Moreover, in the general case of residuated lattices, in Theorem 3.6 we prove that $(L, \{\odot\}, \tau_a)$ is a semitopological residuated lattice and $(L, \{\rightarrow\}, \tau_a)$ is a right topological residuated lattice. Also, we investigate the topological properties of residuated lattices, in Theorem 3.9 we prove that (L, τ_a) is a $\mathbf{T_0}$-space, and in Corollary 3.10 we obtain that (L, τ_a) is not a $\mathbf{T_1}$-space, hence it is not a Hausdorff space.

2 Preliminaries

In this section, we recall some basic notions relevant to residuated lattices which we will need in the sequel.

Definition 2.1. ([4]) A *residuated lattice* is an algebra $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

(Lr_1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice;
(Lr_2) $(L, \odot, 1)$ is a commutative monoid;
(Lr_3) $\odot$ and $\rightarrow$ form an adjoint pair, i.e., $a \odot x \leq b$ iff $x \leq a \rightarrow b$.

Examples of residuated lattices can be found in [11, 13, 15].

We denote by L a residuated lattice (unless otherwise specified). If L is a totally ordered residuated lattice, then L is called a *chain*. For $x \in L$ and $n \geq 1$ we define $x^* = x \rightarrow 0$, $x^{**} = (x^*)^*$, $x^0 = 1$ and

$x^n = x^{n-1} \odot x$.

We refer to [10, 11, 12, 13, 15, 17, 18, 20] for detailed proofs of these well-known rules of calculus:

If L is a residuated lattice, then for every $x, y, z \in L$, we have:

(r_1) $x \to x = 1,\ x \to 1 = 1,\ 1 \to x = x$;
(r_2) $x \leq y$ iff $x \to y = 1$;
(r_3) If $x \vee y = 1$, then $x \odot y = x \wedge y$; $x \odot y \leq x \odot (x \to y) \leq x \wedge y$;
(r_4) If $x \leq y$, then $z \odot x \leq z \odot y,\ z \to x \leq z \to y,\ y \to z \leq x \to z$;
(r_5) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;
(r_6) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$, and $x \odot (y \wedge z) \leq (x \odot y) \wedge (x \odot z)$;
(r_7) $(x \to z) \wedge (y \to z) = (x \vee y) \to z$;
(r_8) $(x \to z) \vee (y \to z) \leq (x \wedge y) \to z$;
(r_9) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$;
(r_{10}) $(x \to y) \vee (x \to z) \leq x \to (y \vee z)$;
(r_{11}) $x \vee y \leq ((x \to y) \to y) \wedge ((y \to x) \to x)$;
(r_{12}) $(x \vee y)^* = x^* \wedge y^*,\ (x \wedge y)^* \geq x^* \vee y^*$;
(r_{13}) $(x \to y^{**})^{**} = x \to y^{**}$;
(r_{14}) $x^{**} \to y^{**} = y^* \to x^* = x \to y^{**} = (x \to y^{**})^{**}$;
(r_{15}) $x \odot x^* = 0,\ 1^* = 0,\ 0^* = 1,\ x^{***} = x^*$;
(r_{16}) $x \leq x^{**},\ x^{**} \leq x^* \to x,\ x \to y \leq y^* \to x^*$;
(r_{17}) $x \to y \leq (x \to y)^{**} \leq x^{**} \to y^{**}$;
(r_{18}) $x^{**} \odot y^{**} \leq (x \odot y)^{**}$, so $(x^{**})^n \leq (x^n)^{**}$
for every natural number n;
(r_{19}) $x^* \odot y^* \leq (x \odot y)^*$;
(r_{20}) $(x \wedge y)^{**} \leq x^{**} \vee y^{**} \leq (x \vee y)^{**}$.
(r_{21}) $z \to y \leq (x \to z) \to (x \to y)$ and $z \to y \leq (y \to x) \to (z \to x)$;

Following the above mentioned literature, we consider the identities:

(i_1) $x \wedge y = x \odot (x \to y)$ $\quad (divisibility)$;
(i_2) $(x^* \wedge y^*)^* = [x^* \odot (x^* \to y^*)]^*$ $\quad (semi-divisibility)$;
(i_3) $(x \to y) \vee (y \to x) = 1$ $\quad (pre-linearity)$;
(i_4) $x^* \vee x^{**} = 1$;
(i_5) $x^2 = x$;
(i_6) $x = x^{**}$;
(i_7) $x \vee x^* = 1$;

Then the *residuated lattice* L is called:

(i) *Divisible* if L verifies (i_1);
(ii) *Semi-divisible* if L verifies (i_2);
(iii) *MTL-algebra* if L verifies (i_3);
(iv) *BL-algebra* if L verifies (i_1) and (i_3);
(v) *Stonean* if L verifies (i_4);
(vi) *G-algebra* if L verifies (i_5);
(vii) *Involutive* if L verifies (i_6);
$(viii)$ *Boolean* if L verifies (i_7);
(ix) *MV-algebra* if L is a BL-algebra with (i_6).

Apart from their logical interest, filters have important algebraic properties and they have been intensively studied from an algebraic point of view. Filter theory plays an important role in studying logical algebras.

Definition 2.2. ([5]) An *implicative filter* (*filter*, for short) is a nonempty subset F of L such that

(F_1) If $x \leq y$ and $x \in F$, then $y \in F$;

(F_2) If $x, y \in F$, then $x \odot y \in F$.

We refer to [1, 2, 7, 14, 20] for detailed aspects of these well-known topological properties and concepts. In general, the concept of topology represents the study of topological spaces. Important topological properties include *connectedness* and *compactness*.

A topology tells how elements of a set relate spatially to each other. The same set can have different topologies. For instance, the real line, the complex plane, and the Cantor set can be thought of as the same set with different topologies.

Let X be a set and let τ be a family of subsets of X. We denote by $\mathcal{P}(X)$ the family of all subsets of X. Then τ is called a topology on X if:

$\tau_{\mathbf{1}}$. Both the empty set $\emptyset$ and X are elements of τ;

$\tau_{\mathbf{2}}$. Any union of elements of τ is an element of τ;

$\tau_{\mathbf{3}}$. Any intersection of finitely many elements of τ is an element of τ.

If τ is a topology on X, then the pair (X, τ) is called a *topological space*. The notation X_τ is used to denote a set X endowed with the particular topology τ.

The members of τ are called *open sets* in X. A subset of X is said to be *closed* if its complement is in τ (i.e., its complement is open). A subset of X may be open, closed, both (clopen set), or neither. The empty set $\emptyset$ and X itself are always both closed and open. An open set containing a point x is called a *neighborhood* of x.

A set with a topology is called a *topological space*. A topological space (X, τ) is called *connected* if $\{\emptyset, X\}$ is the set of all closed and open subsets of X.

A *base* (or *basis*) β for a topological space X with topology τ is a collection of open sets in τ such that every open set in τ can be written as a union of elements of β. We say that the base generates the topology τ. Let L be a residuated lattice and (L, τ_a) a topological space. We have in the literature the following well known separation axioms in (L, τ_a) :

$\mathbf{T_0}$: For each $x, y \in L$ and $x \neq y$, there is at least one in an open neighborhood excluding the other.

$\mathbf{T_1}$: For each $x, y \in L$ and $x \neq y$, each has an open neighborhood not containing the other.

$\mathbf{T_2}$: For each $x, y \in L$ and $x \neq y$, both have disjoint open neighborhoods U, V such that $x \in U$ and $y \in V$.

$\mathbf{T_3}$: If C is any closed subset of (L, τ_a) and $x \in L$ such that $x \notin C$, then there exist disjoint open sets U, V such that $x \in U$ and $C \subseteq V$.

$\mathbf{T_4}$: If C and x are as in $\mathbf{T_3}$, then there exists a real valued function $f : A \to [0, 1]$ such that $f(x) = 0$ and $f(C) = 1$.

$\mathbf{T_5}$: If C and D are two disjoint closed subsets of L, then there exist two disjoint open subsets U and V such that $C \subseteq U$ and $D \subseteq V$.

A topological space satisfying $\mathbf{T_i}$ is called a $\mathbf{T_i}$-space. A $\mathbf{T_2}$-space is also known as a *Hausdorff* space. A $\mathbf{T_1} + \mathbf{T_3}$-space will be called *regular*; A $\mathbf{T_1} + \mathbf{T_4}$-space will be called *completely regular*; A $\mathbf{T_1} + \mathbf{T_5}$-space will be called *normal*, respectively. A topological space (L, τ_a) is said to be *compact*, if each open covering of L is reducible to a finite open covering, *locally compact*, if for each $x \in L$ there exist an open neighborhood U of x and a compact subset K such that $x \in U \subseteq K$.

Definition 2.3. ([7, 20]) Let $(X, \leq)$ be an ordered set. Then we define $\uparrow : \mathcal{P}(X) \to \mathcal{P}(X)$, by $\uparrow S = \{x \in X | a \leq x,$ for some $a \in S\}$, for any subset S of X. A subset F of X is called an *upset* if $\uparrow F = F$. We denote by $U(X)$ the set of all upsets of X. An upset F is called *finitely generated* if there exists $n \in N$ such that $F = \uparrow \{x_1, x_2, ..., x_n\}$, for some $x_1, x_2, ..., x_n \in X$.

Examples of upsets in residuated lattices can be found in [7].

Definition 2.4. ([14, 20]) Let τ and $\tau^{'}$ be two topologies on a given set X. If $\tau^{'}$ is a subset of τ, then we say that τ is *finer* than $\tau^{'}$. Let (X,τ) and $(Y,\tau^{'})$ be two topological spaces. A map $f: X \to Y$ is called *continuous* if the inverse image of each open set of Y is open in X. A *homeomorphism* is a continuous function, bijective and which has a continuous inverse.

Definition 2.5. ([1, 20]) Let $(A,*)$ be an algebra of type 2 and τ be a topology on A. Then $(A,*,\tau)$ is called:

(i) Right (left) topological algebra, if for all $a \in A$ the map $*: A \to A$ defined by $x \mapsto a * x$ $(x \mapsto x * a)$ is continuous;

(ii) Semitopological algebra if A is a right and left topological algebra.

If $(A,*)$ is a commutative algebra, then right and left topological algebras are equivalent.

Definition 2.6. ([1, 20]) Let A be a nonempty set, $\{*_i\}_{i\in I}$ be a family of binary operations on A and τ be a topology on A. Then,

(i) $(A,\{*_i\}_{i\in I},\tau)$ is a right (left) topological algebra, if for every $i \in I$, $(A,*_i,\tau)$ is a right (left) topological algebra;

(ii) $(A,\{*_i\}_{i\in I},\tau)$ is a semitopological algebra, if for every $i \in I$, $(A,*_i,\tau)$ is a semitopological algebra.

On any residuated lattice L ([6, 7]) we may define two operators by setting for all $x, y \in L$,

$$x \oplus y = (x^* \odot y^*)^* \text{ and } x\Delta y = x^* \oplus y. \tag{1}$$

Definition 2.7. ([7]) Consider L a residuated lattice and $a \in L$. For any nonempty upset X of L we define the set

$$D_a(X) = \{x \in L \mid a^n\Delta x \in X, \text{ for some } n \in \mathbb{N}\},$$

where $a^n\Delta x = a\Delta(a^{n-1}\Delta x)$, for any $n \in \{2,3,4,...\}$.

By Proposition 3.9[7] we get that $D_a(X) = \{x \in L \mid (a^{**})^n \to x^{**} \in X, \text{ for some } n \in \mathbb{N}\}$.

We note that if $a = 1$, then $D_a(X) = D_1(X) = \{x \in L \mid x^{**} \in X\}$ is the set of double complemented elements of X, hence $D_a(X)$ represents a generalization of the set of double complemented elements of X.

Theorem 2.8. *([7], Theorem 3.12) Suppose L is a residuated lattice and $a, x \in L$. Consider X, Y two nonempty upsets of L. Then*

(i) *$D_a(X)$ is an upset of L;*

(ii) *$1 \in D_a(X)$, $a \in D_a(X)$ and $X \subseteq D_a(X)$;*

(iii) *$a^m\Delta(a^n\Delta x) = a^{m+n}\Delta x$, for any $m, n \in \mathbb{N}$;*

(iv) *if $X \subseteq Y$, then $D_a(X) \subseteq D_a(Y)$;*

(v) *$D_a(D_a(X)) = D_a(X)$;*

(vi) *if F is a filter of L, then $D_a(F)$ is a filter of L;*

(v) *$D_a(D_x(X)) = D_x(D_a(X))$.*

Proposition 2.9. *([7], Corollary 3.23, Proposition 3.24, Proposition 3.26) For a residuated lattice L and $x, a \in L$, we have:*

(i) *The set $\tau_a = \{D_a(X) \mid X \in U(L)\}$ is a topology on L;*

(ii) *The set $\beta_a = \{D_a(\uparrow x) \mid x \in L)\}$ is a base for the topology τ_a on L;*

(iii) *If $u, v \in L$ such that $u \leq v$, then the topology τ_v is finer than topology τ_u.*

3 Semitopological De Morgan residuated lattices

In this section, using examples we show that the classes of divisible and De Morgan residuated lattices are different. We introduce and study semitopological De Morgan residuated lattices.

We recall ([6]) that a residuated lattice L will be called *De Morgan* if it satisfies the identity $(x \wedge y)^* = x^* \vee y^*$, for all $x, y \in L$.

Examples of De Morgan residuated lattices are Boolean algebras, MV-algebras, BL-algebras, MTL-algebras, involutive and Stonean residuated lattices (see [6]).

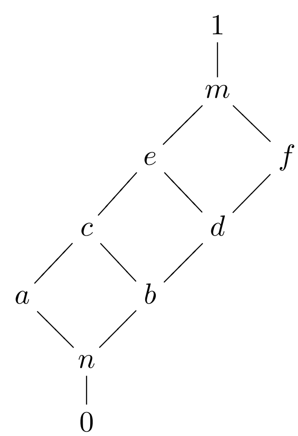
In any De Morgan residuated lattice L, for every $x, y \in L$, we have the following rules of calculus (see Lemma 2 [6]):

(r_{22}) $(x \vee y)^{**} = x^{**} \vee y^{**}$;

(r_{23}) $(x \wedge y)^{**} = x^{**} \wedge y^{**}$.

In the following examples we show that the classes of divisible and De Morgan residuated lattices are different. However, we notice that the class of BL-algebras is a subclass of divisible and De Morgan residuated lattices.

Example 3.1. Let L=$\{0, n, a, b, c, d, e, f, m, 1\}$ with $0 < n < a < c < e < m < 1$, $0 < n < b < d < f < m < 1$ and the elements $\{a, b\}$, $\{c, d\}$, $\{e, f\}$ are pairwise incomparable.



Then ([11]) L is a residuated lattice with respect to the following operations:

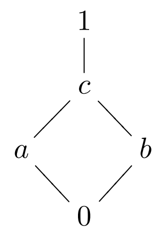
$\rightarrow$	0	n	a	b	c	d	e	f	m	1
0	1	1	1	1	1	1	1	1	1	1
n	m	1	1	1	1	1	1	1	1	1
a	f	f	1	f	1	f	1	f	1	1
b	e	e	e	1	1	1	1	1	1	1
c	d	d	e	f	1	f	1	f	1	1
d	c	c	c	e	e	1	1	1	1	1
e	b	b	c	d	e	f	1	f	1	1
f	a	a	a	c	c	e	e	1	1	1
m	n	n	a	b	c	d	e	f	1	1
1	0	n	a	b	c	d	e	f	m	1

$\odot$	0	n	a	b	c	d	e	f	m	1
0	0	0	0	0	0	0	0	0	0	0
n	0	0	0	0	0	0	0	0	0	n
a	0	0	a	0	a	0	a	0	a	a
b	0	0	0	0	0	0	0	b	b	b
c	0	0	a	0	a	0	a	b	c	c
d	0	0	0	0	0	b	b	d	d	d
e	0	0	a	0	a	b	c	d	e	e
f	0	0	0	b	b	d	d	f	f	f
m	0	0	a	b	c	d	e	f	m	m
1	0	n	a	b	c	d	e	f	m	1

It is easy to verify that L is a De Morgan residuated lattice. Since $a \wedge b = n$ and $a \odot (a \rightarrow b) = a \odot f = 0$, it follows that $a \wedge b \neq a \odot (a \rightarrow b)$, consequently, L is not a divisible residuated lattice.

Divisible residuated lattices are not always De Morgan as we can see in the following example.

Example 3.2. Let $L = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$, and a and b are incomparable.



Then ([11], page 187) L is a residuated lattice with respect to the following operations:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

It easy to see that L is a divisible residuated lattice. Since $(a \wedge b)^* = 0^* = 1$ and $a^* \vee b^* = b \vee a = c$, it follows that $(a \wedge b)^* \neq a^* \vee b^*$, hence L is not De Morgan.

We recall (Lemma 4 and Corollary 3 [7]) that for any residuated lattice L and $a, x \in L$, we have the following rules of calculus:
(r_{24}) $a^n \Delta x = (a^{**})^n \to x^{**}$;
(r_{25}) $(a^n \Delta x)^{**} = a^n \Delta x$, for any $n \in \mathbb{N}$.
(r_{26}) $a \odot x \leq a \wedge x \leq a \vee x \leq a^{**} \vee x^{**} \leq a \oplus x$;
(r_{27}) $x^{**} \leq a \Delta x \leq a^n \Delta x$, for any $n \in \mathbb{N}$.
By Lemma 11 [7], in any residuated lattice L, for every $a, p, q \in L$ and $m \geq 1$ we have the following rules of calculus:
(ε_1) $(a^{**})^m \to (q \vee p)^{**} = [(a^{**})^m \to (q \vee p)^{**}]^{**}$;
(ε_2) $[q \odot ((a^{**})^m \wedge p^{**})] \to (q \odot p)^{**} = 1$.

Now, we introduce the notion of a *semitopological De Morgan residuated lattice.*

Definition 3.3. Let τ be a topology on the De Morgan residuated lattice L. If $(L, \{*_i\}, \tau)$, where $\{*_i\} \subseteq \{\vee, \wedge, \odot, \to\}$ is a semitopological algebra, then $(L, \{*_i\}, \tau)$ is a semitopological De Morgan residuated lattice. For simplicity, if $\{*_i\} \subseteq \{\vee, \wedge, \odot, \to\}$, we consider (L, τ) instead of $(L, \{\vee, \wedge, \odot, \to\}, \tau)$.

Example 3.4. We consider the De Morgan residuated lattice L from Examples 3.1, we notice that $0^{**} = 0$, $n^{**} = n$, $a^{**} = a$, $b^{**} = b$, $c^{**} = c$, $d^{**} = d$, $e^{**} = e$, $f^{**} = f$, $m^{**} = m$, $1^{**} = 1$, so L is an involutive residuated lattice ($x^{**} = x$, for all $x \in L$), by Corollary 5.4[7] we get that $(L, \{\vee, \wedge, \odot\}, \tau_a)$ is a semitopological De Morgan residuated lattice, and $(L, \{\to\}, \tau_a)$ is a right semitopological De Morgan residuated lattice, for any element $a \in L$.

Theorem 3.5. *Let L be a De Morgan residuated lattice and $a \in L$. Then*
(i) $(L, \{\vee\}, \tau_a)$ is a semitopological De Morgan residuated lattice;
(ii) $(L, \{\wedge\}, \tau_a)$ is a semitopological De Morgan residuated lattice.

Proof. (i). We prove that $(L, \{\vee\}, \tau_a)$ is a semitopological De Morgan residuated lattice.

Consider q an arbitrary element of L and the map $\varphi_q : L \to L$, defined by $\varphi_q(x) = q \vee x$, for any $x \in L$. Following Proposition 2.9 the set $\beta_a = \{D_a(\uparrow x) \mid\ x \in L)\}$ is a base for the topology τ_a on L. Then it suffices to prove that $\varphi_q^{-1}(D_a(\uparrow x)) \in \tau_a$, for any $x \in L$.

Let $x \in L$, then we obtain successively $\varphi_q^{-1}(D_a(\uparrow x)) = \{p \in L \mid \varphi_q(p) \in D_a(\uparrow x)\} = \{p \in L \mid (a^{**})^n \to (q \vee p)^{**} \in \uparrow x,\ \text{for some } n \in \mathbb{N}\} = \{p \in L \mid x \leq (a^{**})^n \to (q \vee p)^{**},\ \text{for some } n \in \mathbb{N}\}$.

Consider the set $A = \{p \in L \mid x \leq (a^{**})^n \to (q \vee p)^{**},\ \text{for some } n \in \mathbb{N}\}$ and we show that the set A is an upset of L. Let $p \in A$ and $p \leq s$, for some $s \in L$. Then there is $n \in \mathbb{N}$ such that $x \leq (a^{**})^n \to (q \vee p)^{**}$. By (r_4), $(a^{**})^n \to (q \vee p)^{**} \leq (a^{**})^n \to (q \vee s)^{**}$, and so $s \in A$. Hence A is an upset. Now, we prove that $D_a(A) = A$, then we deduce that $A \in \tau_a$, that is $\varphi_q^{-1}(D_a(\uparrow x)) \in \tau_a$, hence φ_q is a continuous map.

Let $p \in D_a(A)$. Then there is $m \in \mathbb{N}$ such that $(a^{**})^m \to p^{**} \in A$, that is $x \leq (a^{**})^n \to (q \vee ((a^{**})^m \to p^{**}))^{**}$, for some $n \in \mathbb{N}$.

We obtain successively

$[(a^{**})^n \to (q \vee ((a^{**})^m \to p^{**}))^{**}] \to [(a^{**})^{m+n} \to (q \vee p)^{**}] \overset{(r_5)}{=}$

$[(a^{**})^n \to (q \vee ((a^{**})^m \to p^{**}))^{**}] \to [(a^{**})^n \to ((a^{**})^m \to (q \vee p)^{**})] \overset{(r_{21})}{\geq}$

$[q \vee ((a^{**})^m \to p^{**})]^{**} \to [(a^{**})^m \to (q \vee p)^{**}] \overset{(r_{22})}{=}$

$[q^{**} \vee ((a^{**})^m \to p^{**})^{**}] \to [(a^{**})^m \to (q^{**} \vee p^{**})] \overset{(\varepsilon_1)}{=}$

$[q^{**} \vee ((a^{**})^m \to p^{**})] \to [(a^{**})^m \to (q^{**} \vee p^{**})] \overset{(r_{10})}{\geq}$

$[q^{**} \vee ((a^{**})^m \to p^{**})] \to [((a^{**})^m \to q^{**}) \vee ((a^{**})^m \to p^{**})] \overset{(r_{16}),(r_4)}{\geq}$

$[q^{**} \vee ((a^{**})^m \to p^{**})] \to [q^{**} \vee ((a^{**})^m \to p^{**})] \overset{(r_1)}{=} 1.$

By (r_2), $(a^{**})^n \to (q \vee ((a^{**})^m \to p^{**}))^{**} \leq (a^{**})^{m+n} \to (q \vee p)^{**}$.

We deduce that $x \leq (a^{**})^n \to (q \vee ((a^{**})^m \to p^{**}))^{**} \leq (a^{**})^{m+n} \to (q \vee p)^{**}$, hence $x \leq (a^{**})^{m+n} \to (q \vee p)^{**}$, that is $p \in A$. Following Theorem 2.8 (ii), we deduce that $A = D_a(A)$, that is φ_q is a continuous map. Since $\vee$ is commutative we deduce that $(L, \{\vee\}, \tau_a)$ is a semitopological De Morgan residuated lattice.

(ii). We prove that $(L, \{\wedge\}, \tau_a)$ is a semitopological De Morgan residuated lattice.

Consider q an arbitrary element of L and the map $\phi_q : L \to L$, defined by $\phi_q(x) = q \wedge x$, for any $x \in L$. Following Proposition 2.9 the set $\beta_a = \{D_a(\uparrow x) \mid\ x \in L)\}$ is a base for the topology τ_a on L. Then it suffices to prove that $\phi_q^{-1}(D_a(\uparrow x)) \in \tau_a$, for any $x \in L$.

Let $x \in L$, then we obtain successively $\phi_q^{-1}(D_a(\uparrow x)) = \{p \in L \mid \phi_q(p) \in D_a(\uparrow x)\} = \{p \in L \mid (a^{**})^n \to (q \wedge p)^{**} \in \uparrow x,\ \text{for some } n \in \mathbb{N}\} = \{p \in L \mid x \leq (a^{**})^n \to (q \wedge p)^{**},\ \text{for some } n \in \mathbb{N}\}$.

Consider the set $B = \{p \in L \mid x \leq (a^{**})^n \to (q \wedge p)^{**},\ \text{for some } n \in \mathbb{N}\}$ and we show that the set B is an upset of L. Let $p \in B$ and $p \leq s$, for some $s \in L$. Then there is $n \in \mathbb{N}$ such that $x \leq (a^{**})^n \to (q \wedge p)^{**}$. By (r_4), $(a^{**})^n \to (q \wedge p)^{**} \leq (a^{**})^n \to (q \wedge s)^{**}$, and so $s \in B$. Hence B is an upset. Now, we prove that $D_a(B) = B$, then we deduce that $B \in \tau_a$, that is $\phi_q^{-1}(D_a(\uparrow x)) \in \tau_a$, hence ϕ_q is a continuous map.

Let $p \in D_a(B)$. Then there is $m \in \mathbb{N}$ such that $(a^{**})^m \to p^{**} \in B$, that is $x \leq (a^{**})^n \to (q \wedge ((a^{**})^m \to p^{**}))^{**}$, for some $n \in \mathbb{N}$.

We obtain successively

$[(a^{**})^n \to (q \wedge ((a^{**})^m \to p^{**}))^{**}] \to [(a^{**})^{m+n} \to (q \wedge p)^{**}] \overset{(r_5)}{=}$

$[(a^{**})^n \to (q \wedge ((a^{**})^m \to p^{**}))^{**}] \to [(a^{**})^n \to ((a^{**})^m \to (q \wedge p)^{**})] \overset{(r_{21})}{\geq}$

$[q \wedge ((a^{**})^m \to p^{**})]^{**} \to [(a^{**})^m \to (q \wedge p)^{**}] \overset{(r_{22})}{=}$

$[q^{**} \wedge ((a^{**})^m \to p^{**})^{**}] \to [(a^{**})^m \to (q^{**} \wedge p^{**})] \overset{(\varepsilon_1)}{=}$

$[q^{**} \wedge ((a^{**})^m \to p^{**})] \to [(a^{**})^m \to (q^{**} \wedge p^{**})] \overset{(r_9)}{=}$

$[q^{**} \wedge ((a^{**})^m \to p^{**})] \to [((a^{**})^m \to q^{**}) \wedge ((a^{**})^m \to p^{**})] \overset{(r_{16}),(r_4)}{\geq}$

$[q^{**} \wedge ((a^{**})^m \to p^{**})] \to [q^{**} \wedge ((a^{**})^m \to p^{**})] \overset{(r_1)}{=} 1.$

By (r_2), $(a^{**})^n \to (q \wedge ((a^{**})^m \to p^{**}))^{**} \leq (a^{**})^{m+n} \to (q \wedge p)^{**}$.

We deduce that $x \leq (a^{**})^n \to (q \wedge ((a^{**})^m \to p^{**}))^{**} \leq (a^{**})^{m+n} \to (q \wedge p)^{**}$, hence $x \leq (a^{**})^{m+n} \to (q \wedge p)^{**}$, that is $p \in B$. Following Theorem 2.8 (ii), we deduce that $B = D_a(B)$, that is ϕ_q is a continuous map. Since $\wedge$ is commutative we deduce that $(L, \{\wedge\}, \tau_a)$ is a semitopological De Morgan residuated lattice. $\square$

Theorem 3.6. *Let L be a residuated lattice and $a \in L$. Then*
(i) $(L, \{\odot\}, \tau_a)$ is a semitopological residuated lattice;
(ii) $(L, \{\to\}, \tau_a)$ is a right topological residuated lattice.

Proof. (i). We prove that $(L, \{\odot\}, \tau_a)$ is a semitopological residuated lattice.

Consider q an arbitrary element of L and the map $\psi_q : L \to L$, defined by $\psi_q(x) = q \odot x$, for any $x \in L$. Following Proposition 2.9 the set $\beta_a = \{D_a(\uparrow x) \mid x \in L)\}$ is a base for the topology τ_a on L. Then it suffices to prove that $\psi_q^{-1}(D_a(\uparrow x)) \in \tau_a$, for any $x \in L$.

Let $x \in L$, then we obtain successively $\psi_q^{-1}(D_a(\uparrow x)) = \{p \in L \mid \psi_q(p) \in D_a(\uparrow x)\} = \{p \in L \mid (a^{**})^n \to (q \odot p)^{**} \in \uparrow x,\ \text{for some } n \in \mathbb{N}\} =$

$\{p \in L \mid x \leq (a^{**})^n \to (q \odot p)^{**},\ \text{for some } n \in \mathbb{N}\}$.

Consider the set $C = \{p \in L \mid x \leq (a^{**})^n \to (q \odot p)^{**},\ \text{for some } n \in \mathbb{N}\}$ and we show that the set C is an upset of L. Let $p \in C$ and $p \leq s$, for some $s \in L$. Then there is $n \in \mathbb{N}$ such that $x \leq (a^{**})^n \to (q \odot p)^{**}$. By (r_4), $(a^{**})^n \to (q \odot p)^{**} \leq (a^{**})^n \to (q \odot s)^{**}$, and so $s \in C$. Hence C is an upset. Now, we prove that $D_a(C) = C$, then we deduce that $C \in \tau_a$, that is $\psi_q^{-1}(D_a(\uparrow x)) \in \tau_a$, hence ψ_q is a continuous map.

Let $p \in D_a(C)$. Then there is $m \in \mathbb{N}$ such that $(a^{**})^m \to p^{**} \in C$, that is $x \leq (a^{**})^n \to (q \odot ((a^{**})^m \to p^{**}))^{**}$, for some $n \in \mathbb{N}$.

We obtain successively

$[(a^{**})^n \to (q \odot ((a^{**})^m \to p^{**}))^{**}] \to [(a^{**})^{m+n} \to (q \odot p)^{**}] \overset{(r_5)}{=}$

$[(a^{**})^n \to (q \odot ((a^{**})^m \to p^{**}))^{**}] \to [(a^{**})^n \to ((a^{**})^m \to (q \odot p)^{**})] \overset{(r_{21})}{\geq}$

$[q \odot ((a^{**})^m \to p^{**})]^{**} \to [(a^{**})^m \to (q \odot p)^{**}] \overset{(\varepsilon_1)}{=}$

$[q \odot ((a^{**})^m \to p^{**})]^{**} \to [(a^{**})^m \to (q \odot p)^{**}]^{**} \overset{(r_{17})}{\geq}$

$[q \odot ((a^{**})^m \to p^{**})] \to [(a^{**})^m \to (q \odot p)^{**}] \overset{(r_5)}{=}$

$[q \odot (a^{**})^m \odot ((a^{**})^m \to p^{**})] \to (q \odot p)^{**} \overset{(r_3),(r_4)}{\geq}$

$[q \odot ((a^{**})^m \wedge p^{**})] \to (q \odot p)^{**} \overset{(\varepsilon_2)}{=} 1.$

By (r_2), $(a^{**})^n \to (q \odot ((a^{**})^m \to p^{**}))^{**} \leq (a^{**})^{m+n} \to (q \odot p)^{**}$.

We deduce that $x \leq (a^{**})^n \to (q \odot ((a^{**})^m \to p^{**}))^{**} \leq (a^{**})^{m+n} \to (q \odot p)^{**}$, hence $x \leq (a^{**})^{m+n} \to (q \odot p)^{**}$, that is $p \in C$. Following Theorem 2.8 (ii), we deduce that $C = D_a(C)$, that is ψ_q is a continuous map. Since $\odot$ is commutative we deduce that $(L, \{\odot\}, \tau_a)$ is a semitopological residuated lattice.

(ii). We prove that $(L, \{\to\}, \tau_a)$ is a right topological residuated lattice.

Consider q an arbitrary element of L and the map $\omega_q : L \to L$, defined by $\omega_q(x) = q \to x$, for any $x \in L$. Following Proposition 2.9 the set $\beta_a = \{D_a(\uparrow x) \mid x \in L)\}$ is a base for the topology τ_a on L. Then it suffices to prove that $\omega_q^{-1}(D_a(\uparrow x)) \in \tau_a$, for any $x \in L$.

Let $x \in L$, then we obtain successively $\omega_q^{-1}(D_a(\uparrow x)) = \{p \in L \mid \omega_q(p) \in D_a(\uparrow x)\} = \{p \in L \mid (a^{**})^n \to (q \to p)^{**} \in \uparrow x,\ \text{for some } n \in \mathbb{N}\} = \{p \in L \mid x \leq (a^{**})^n \to (q \to p)^{**},\ \text{for some } n \in \mathbb{N}\}$.

Consider the set $D = \{p \in L \mid x \leq (a^{**})^n \to (q \to p)^{**},\ \text{for some } n \in \mathbb{N}\}$ and we show that the set D is an upset of L. Let $p \in D$ and $p \leq s$, for some $s \in L$. Then there is $n \in \mathbb{N}$ such that $x \leq (a^{**})^n \to (q \to p)^{**}$. By (r_4), $(a^{**})^n \to (q \to p)^{**} \leq (a^{**})^n \to (q \to s)^{**}$, and so $s \in D$. Hence D is an upset. Now, we prove that $D_a(D) = D$, then we deduce that $D \in \tau_a$, that is $\omega_q^{-1}(D_a(\uparrow x)) \in \tau_a$, hence ω_q is a continuous map.

Let $p \in D_a(D)$. Then there exists $m, n \in \mathbb{N}$ such that $(a^{**})^n \to p^{**} \in D$ and $x \leq (a^{**})^m \to ((a^{**})^n \to p^{**}))^{**}$.

We obtain successively

$x \leq (a^{**})^m \to ((a^{**})^n \to p^{**})^{**} \overset{(\varepsilon_1)}{=}$

$(a^{**})^m \to ((a^{**})^n \to p^{**}) \overset{(r_5)}{=}$

$(a^{**})^m \odot (a^{**})^n \to p^{**} =$

$(a^{**})^{m+n} \to p^{**}$.

Since $p \leq q \to p$, $p^{**} \leq (q \to p)^{**}$, by (r_4) we deduce that $x \leq (a^{**})^m \to ((a^{**})^n \to p^{**}))^{**} \leq (a^{**})^{m+n} \to p^{**} \leq (a^{**})^{m+n} \to (q \to p)^{**}$, hence $x \leq (a^{**})^{m+n} \to (q \to p)^{**}$, that is $p \in D$. Following Theorem 2.8 (ii), we deduce that $D = D_a(D)$, that is ω_q is a continuous map. Since $\to$ is not commutative we deduce that $(L, \{\to\}, \tau_a)$ is a right topological residuated lattice. $\square$

The next result follows from Theorems 3.5 and 3.6:

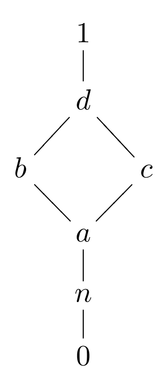
Corollary 3.7. *If L is a De Morgan residuated lattice and $a \in L$, then*

(i) $(L, \{\vee, \wedge, \odot\}, \tau_a)$ is a semitopological De Morgan residuated lattice;

(ii) $(L, \{\to\}, \tau_a)$ is a right topological De Morgan residuated lattice.

Now, we give an example for Corollary 3.7:

Example 3.8. We present a De Morgan residuated lattice L without the involutive property, we construct the topology τ_a, for $a \in L$, and we show that $(L, \{\vee, \wedge, \odot\}, \tau_a)$ is a semitopological De Morgan residuated lattice, and $(L, \{\to\}, \tau_a)$ is a right topological De Morgan residuated lattice. We consider $L = \{0, n, a, b, c, d, 1\}$ with $0 < n < a < b, c < d < 1$, and b and c are incomparable.



Then ([11], page 247) L is a residuated lattice with respect to the following operations:

$\rightarrow$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1
n	0	1	1	1	1	1	1
a	0	d	1	1	1	1	1
b	0	c	c	1	c	1	1
c	0	b	b	b	1	1	1
d	0	a	a	b	c	1	1
1	0	n	a	b	c	d	1

$\odot$	0	n	a	b	c	d	1
0	0	0	0	0	0	0	0
n	0	n	n	n	n	n	n
a	0	n	n	n	n	n	a
b	0	n	n	b	n	b	b
c	0	n	n	n	c	c	c
d	0	n	n	b	c	d	d
1	0	n	a	b	c	d	1

It is easy to ascertain that L is a De Morgan residuated lattice. Since $n^{**} = 1 \neq n$, it follows that L is not an involutive residuated lattice. The upsets of L are $U(L) = \{\uparrow 0, \uparrow n, \uparrow a, \uparrow b, \uparrow c, \uparrow d, \uparrow 1\}$, where $\uparrow 0 = L$, $\uparrow n = \{n, a, b, c, d, 1\}$, $\uparrow a = \{a, b, c, d, 1\}$, $\uparrow b = \{b, d, 1\}$, $\uparrow c = \{c, d, 1\}$, $\uparrow d = \{d, 1\}$, $\uparrow 1 = \{1\}$. We notice that $0^{**} = 0$, and $n^{**} = a^{**} = b^{**} = c^{**} = d^{**} = 1^{**} = 1$.

Now, we identify the elements of the topology τ_a, for a fixed element $a \in L : D_a(\uparrow 0) = L$; $D_a(\uparrow n) = \{x \in L \mid (a^{**})^n \rightarrow x^{**} \in \uparrow n, \text{ for some } n \in \mathbb{N}\} = \{x \in L \mid 1 \rightarrow x^{**} \in \uparrow n\} \overset{(r_1)}{=} \{x \in L \mid x^{**} \in \uparrow n\} = \{x \in L \mid x^{**} \in \{n, a, b, c, d, 1\}\} = \{x \in L \mid x^{**} \in \{1\}\} = \uparrow n$. Similarly, we deduce that $D_a(\uparrow a) = D_a(\uparrow b) = D_a(\uparrow c) = D_a(\uparrow d) = D_a(\uparrow 1) = \uparrow n$. Hence $\tau_a = \{\{\emptyset\}, \{\uparrow n\}, L\}$.

Now, following the proof of Theorem 3.5, we show that $(L, \{\vee\}, \tau_a)$ is a semitopological De Morgan residuated lattice. For that we consider q an arbitrary element of L and the map $\varphi_q : L \rightarrow L$, defined by $\varphi_q(x) = q \vee x$, for any $x \in L$. Since $\varphi_q^{-1}(D_a(\{\emptyset\})) = \{\emptyset\}$ and $\varphi_q^{-1}(D_a(L)) = L$, it suffices to prove that $\varphi_q^{-1}(D_a(\uparrow n)) \in \tau_a$. We obtain successively $\varphi_q^{-1}(D_a(\uparrow n)) = \{p \in L \mid \varphi_q(p) \in D_a(\uparrow n)\} = \{p \in L \mid (a^{**})^n \rightarrow (q \vee p)^{**} \in \uparrow n, \text{ for some } n \in \mathbb{N}\} = \{p \in L \mid 1 \rightarrow (q \vee p)^{**} \in \uparrow n\} \overset{(r_1)}{=} \{p \in L \mid (q \vee p)^{**} \in \uparrow n\} = \{p \in L \mid (q \vee p)^{**} \in \{n, a, b, c, d, 1\}\} = \{p \in L \mid (q \vee p)^{**} \in \{1\}\} = \{p \in L \mid q \vee p \in \{n, a, b, c, d, 1\}\} = L$ if $q \in \{n, a, b, c, d, 1\}$, otherwise $\varphi_q^{-1}(D_a(\uparrow n)) = \uparrow n$ if $q = 0$. Hence $\varphi_q^{-1}(D_a(\uparrow n)) \in \tau_a$.

Similarly, we show that $(L, \{\wedge\}, \tau_a)$ is a semitopological De Morgan residuated lattice. For that we consider q an arbitrary element of L and the map $\phi_q : L \rightarrow L$, defined by $\phi_q(x) = q \wedge x$, for any $x \in L$. Then it suffices to prove that $\phi_q^{-1}(D_a(\uparrow n)) \in \tau_a$. We obtain successively $\phi_q^{-1}(D_a(\uparrow n)) = \{p \in L \mid \phi_q(p) \in D_a(\uparrow n)\} = \{p \in L \mid (a^{**})^n \rightarrow (q \wedge p)^{**} \in \uparrow n, \text{ for some } n \in \mathbb{N}\} = \{p \in L \mid (q \wedge p)^{**} \in \uparrow n\} = \{p \in L \mid (q \wedge p)^{**} \in \{n, a, b, c, d, 1\}\} = \{p \in L \mid (q \wedge p)^{**} \in \{1\}\} = \{p \in L \mid q \wedge p \in \{n, a, b, c, d, 1\}\} = \uparrow n$ if $q \in \{n, a, b, c, d, 1\}$, otherwise $\phi_q^{-1}(D_a(\uparrow n)) = \{\emptyset\}$ if $q = 0$. Hence $\phi_q^{-1}(D_a(\uparrow n)) \in \tau_a$.

Now, following the proof of Theorem 3.6, we show that $(L, \{\odot\}, \tau_a)$ is a semitopological De Morgan residuated lattice. For that we consider q an arbitrary element of L and the map $\psi_q : L \rightarrow L$, defined by $\psi_q(x) = q \odot x$, for any $x \in L$. Then it suffices to prove that $\psi_q^{-1}(D_a(\uparrow n)) \in \tau_a$. We obtain successively $\psi_q^{-1}(D_a(\uparrow n)) = \{p \in L \mid \psi_q(p) \in D_a(\uparrow n)\} = \{p \in L \mid (a^{**})^n \rightarrow (q \odot p)^{**} \in \uparrow n, \text{ for some } n \in \mathbb{N}\} = \{p \in L \mid (q \odot p)^{**} \in \uparrow n\} = \{p \in L \mid (q \odot p)^{**} \in \{n, a, b, c, d, 1\}\} = \{p \in L \mid (q \odot p)^{**} \in \{1\}\} = \{p \in L \mid q \odot p \in \{n, a, b, c, d, 1\}\} = \uparrow n$ if $q \in \{n, a, b, c, d, 1\}$, (because $\uparrow n$ is a filter of L); otherwise $\psi_q^{-1}(D_a(\uparrow n)) = \{\emptyset\}$ if $q = 0$. Hence $\psi_q^{-1}(D_a(\uparrow n)) \in \tau_a$.

Similarly, we show that $(L, \{\rightarrow\}, \tau_a)$ is a right topological De Morgan residuated lattice. For that we consider q an arbitrary element of L and the map $\omega_q : L \rightarrow L$, defined by $\omega_q(x) = q \rightarrow x$, for any $x \in L$. Then it suffices to prove that $\omega_q^{-1}(D_a(\uparrow n)) \in \tau_a$. We notice that $\uparrow n$ is a filter of L. We obtain successively $\omega_q^{-1}(D_a(\uparrow n)) = \{p \in L \mid \omega_q(p) \in D_a(\uparrow n)\} = \{p \in L \mid (a^{**})^n \rightarrow (q \rightarrow p)^{**} \in \uparrow n, \text{ for some } n \in \mathbb{N}\} = \{p \in L \mid (q \rightarrow p)^{**} \in \uparrow n\} = \{p \in L \mid (q \rightarrow p)^{**} \in \{n, a, b, c, d, 1\}\} = \{p \in L \mid (q \rightarrow p)^{**} \in \{1\}\} = \{p \in L \mid q \rightarrow p \in \{n, a, b, c, d, 1\}\} = \uparrow n$ if $q \in \{n, a, b, c, d, 1\}$, (because $\uparrow n$ is a filter of L); otherwise, if $q = 0$ we get that $\omega_q^{-1}(D_a(\uparrow n)) = \{p \in L \mid 0 \rightarrow p \in \{n, a, b, c, d, 1\}\} \overset{(r_1)}{=} \{p \in L \mid 1 \in \{n, a, b, c, d, 1\}\} = L$. Hence $\omega_q^{-1}(D_a(\uparrow n)) \in \tau_a$.

Theorem 3.9. *Let L be a residuated lattice and $a \in L$. Then (L, τ_a) is a $\mathbf{T_0}$-space.*

Proof. Let L be a residuated lattice, $a, b \in L$ and X a nonempty upset of L. We show that the set $D_a(X) \to a := \{x \to a \mid x \in D_a(X)\}$ is an upset of L. By Theorem 2.8(ii), we have that $a \in D_a(X)$, since $D_a(X)$ is an upset of L and $a \leq x \to a$ we get that $x \to a \in D_a(X)$, hence $D_a(X) \to a \subseteq D_a(X)$. Let $u \in D_a(X) \to a$ and let $L \ni v \geq u$. Since $D_a(X) \to a \subseteq D_a(X)$ and $D_a(X)$ is an upset we get that $v \in D_a(X)$. Then by (r_7) we obtain that $(v \vee a) \to a = (v \to a) \wedge (a \to a) \overset{(r_1)}{=} v \to a$ belongs to $D_a(X) \to a$. Similarly, we obtain that $D_b(X) \to b$ is an upset of L.

Now, we prove that $D_a(X) \to a$ is a neighborhood of a. Since $D_a(X) \to a$ is an upset of L and following Theorem 2.8(v) we get that $D_a(D_a(X) \to a) = D_a(X) \to a$, hence $D_a(X) \to a \in \tau_a$. We conclude that $D_a(X) \to a$ is an open set of the topology τ_a, and so it is a neighborhood of a. Similarly, we get that $D_b(X) \to b$ is a neighborhood of b.

We claim that $a \notin D_b(X) \to b$ or $b \notin D_a(X) \to a$. If $a \in D_b(X) \to b$ and $b \in D_a(X) \to a$, then there are $x_1 \in D_b(X)$ and $x_2 \in D_a(X)$ such that $a = x_1 \to b$ and $b = x_2 \to a$. Since $a = x_1 \to b \geq b$ and $b = x_2 \to a \geq a$, we deduce that $a = b$, a contradiction. □

Corollary 3.10. *Let L be a residuated lattice. Then (L, τ_a) is not a $\mathbf{T_1}$-space, so it is not a $\mathbf{T_2}$-space, that is (L, τ_a) is not a Hausdorff space;*

Proof. Let L be a residuated lattice and a an element of L. We suppose that (L, τ_a) is a $\mathbf{T_1}$-space. By Theorem 3.6(ii), we have that $(L, \{\to\}, \tau_a)$ is a right topological residuated lattice. By Proposition 4.9[2] we have that $(L, \to, \tau_a)$ is a $\mathbf{T_1}$-space iff for any $x \neq 1$ there are neighborhoods U and V of x and 1, respectively, such that $1 \notin U$ and $x \notin V$. Since U and V are two neighborhoods of x and 1, we deduce that U and V are open sets in τ_a, hence $U = D_a(X)$ and $V = D_a(Y)$, for some nonempty upsets $X, Y \in U(L)$. By Theorem 2.8(ii), we have that $1 \in D_a(X) = U$, a contradiction. Since every $\mathbf{T_2}$-space is a $\mathbf{T_1}$-space, we deduce that (L, τ_a) is not a $\mathbf{T_2}$-space, so it is not a Hausdorff space;

□

4 Conclusions

In the last five years many mathematicians have studied properties of De Morgan residuated lattices endowed with a topology. For example L. C. Holdon (Kybernetika 54(3):443-475, 2018 [6]) introduced the class of De Morgan residuated lattices and studied the theory of ideals and annihilators, also, some important results on prime and maximal spectrum of a De Morgan residuated lattice have been proven. L. C. Holdon (Open Math 18:12061226, 2020 [8]) investigated the Zariski topology and stable topology in residuated lattices with applications in De Morgan residuated lattices. D. Piciu (Fuzzy Sets Syst 405:47-64, 2021 [16]) studied some aspects of prime, minimal prime and maximal ideals spaces in residuated lattices. D. Buşneag et al.(Carpathian J. Math 37:53-63, 2021 [3]) and L. C. Holdon et al.(Stud. Sci. Math. Hung 58(2):182205, 2021 [9]) investigated congruences based on ideals in residuated lattices and the relationships between different types of ideals in residuated lattices with applications in De Morgan residuated lattices. All these works represent open gates for further studies. In this paper, using examples we showed that the classes of divisible and De Morgan residuated lattices are different. Based on the topology generated by upsets (filters) in residuated lattices we introduced semitopological De Morgan residuated lattices and studied separation axioms $\mathbf{T_0}$, $\mathbf{T_1}$ and $\mathbf{T_2}$. In Corollary 3.7 we showed that $(L, \{\vee, \wedge, \odot\}, \tau_a)$ is a semitopological De Morgan residuated lattice and $(L, \{\to\}, \tau_a)$ is a right topological De Morgan residuated lattice, we presented some examples of semitopological De Morgan residuated lattices in Examples 3.4 and 3.8. Moreover, in the general case of residuated lattices, in Theorem 3.6 we proved that $(L, \{\odot\}, \tau_a)$ is a semitopological residuated lattice and $(L, \{\to\}, \tau_a)$ is a right topological residuated lattice. Also, we investigated the topological properties of residuated lattices, in Theorem 3.9 we prove that (L, τ_a) is a $\mathbf{T_0}$-space, and in Corollary 3.10 we obtained that (L, τ_a) is not a $\mathbf{T_1}$-space, hence it is not a Hausdorff space. The present paper represents an open gate for searching new classes of semitopological residuated lattices endowed with non trivial topologies, it will be interesting to find such a topology τ such that (L, τ) to be a Hausdorff space.

Next researchers can study normality, regularity, metrizability and uniformity on semitopological residuated lattices.

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