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Homogeneous fuzzy wave equation on the half-line under generalized Hukuhara differentiability

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Abstract. An analytical fuzzy solution is achieved by means of the fuzzy d'Alembert formula for the fuzzy one-dimensional homogeneous wave equation in a half-line considering the generalized Hukuhara partial differentiability of the solution. In the current article, the exclusive solution and the stability of the homogeneous fuzzy wave equations are brought into existence. Eventually, given the various instances represented, the efficacy and accuracy of the method are scrutinized.

Keywords: Generalized Hukuhara differentiability, fuzzy partial differential equation, fuzzy wave equation, Leibniz rule.

2010 AMS Subject Classification: 26E50, 35R13, 35L05.

1. Introduction

It is well known that there are many phenomena in including fluid flow, electrical networks, fractals theory, control theory, optics, biology, chemistry and other sciences can be described by models that use mathematical tools of partial differential equations. The concept of the fuzzy partial differential equations (FPDEs) was extended in [6], and according to this definition, different method are obtained an approximation fuzzy solution for partial differential equations [1, 3]. In addition, Oberguggenberger described fuzzy weak solutions for the FPDEs [15], and Chen et al. presented a new inference method with applications to the FPDEs [7]. A substantial amount of research work has to find an approximation solution for the FPDEs, convert the FPDEs to two crisp corresponding

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problems, then find the approximate solutions of crisp problems and hence obtain an approximation of fuzzy solutions, see [1, 3, 5]. To overcome this shortcoming, generalized Hukuhara partial derivative based on gH-difference is introduced and the existence and uniqueness of the solution of the fuzzy heat equation based on this concept are investigated by Allahviranloo et al. [2] and after that Gouyandeh et al. [8] introduced the fuzzy Fourier transform and obtained an analytical solution for the fuzzy heat equation. The fuzzy Poisson and fuzzy Laplace equations are solved in [12], and also some useful research papers for solving fuzzy partial differential equations are [11, 13, 14]. Recently, Rahimi Chermahini et al. [16] studied the existence and uniqueness of the solution of the fuzzy wave equation based on generalized Hukuhara partial derivative and obtained their d'Alembert solutions on infinite interval.

The aim of the present paper is to obtain the d'Alembert solution of a fuzzy wave equation on a half-line. Using the properties of generalized Hukuhara partial derivative, the d'Alembert's formulas for fuzzy Dirichlet problem on the half-line is obtained without embedding them to crisp equations.

The remainder of this paper is organized as follows. In Section 2, we briefly introduce the basic notations and preliminaries and prove some new theorems and lemmas to be used in the main part of the paper. In Section 3, the canonical form of a solution of a fuzzy wave equation is showed and the fuzzy d'Alembert's formulas for a fuzzy homogeneous wave equation based on the type of $[gH-p]-$differentiability are showed. In Section 4, the fuzzy d'Alembert's formulas for the fuzzy Dirichlet problem on the half-line based on the type of $[gH-p]-$differentiability are showed. Moreover, some examples are given to clarify the details and efficiency of the method in Section 5 and conclusions are given in Section 6.

2. Basic preliminaries

In this section, we present some definitions and introduce the necessary notations, which will be used throughout the paper. Denote

$$\mathbb{R}_{\mathcal{F}} = \big\{u : \mathbb{R}^n \to [0,\ 1] \mid u \text{ satisfies (i)-(iv) below}\big\},$$

where

(i) u is fuzzy convex;
(ii) u is normal, i.e., there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$;
(iii) u is upper semi-continuous;
(iv) closure of $\{x \in \mathbb{R}^n \mid u(x) > 0\}$ is compact.

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers. For $0 < \alpha \leqslant 1$ denote

$$[u]^\alpha = \left\{x \in \mathbb{R}^n \middle| u(x) \geqslant \alpha\right\} = [u^-(\alpha),\ u^+(\alpha)].$$

Then from (i) to (iv), it follows that the α-level set $[u]^\alpha$ is a closed interval for all $\alpha \in [0,\ 1]$. A triangular fuzzy number is defined as a fuzzy set in $\mathbb{R}_{\mathcal{F}}$, that is specified by an ordered triple $u = (a, b, c) \in \mathbb{R}^3$ with $a \leqslant b \leqslant c$ such that $u^-(\alpha) = a + (b - a)\alpha$ and $u^+(\alpha) = c - (c - b)\alpha$ are the endpoints of α-level sets for all $\alpha \in [0, 1]$.

The Hausdorff distance between fuzzy numbers is given by $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}^+ \cup \{0\}$

as in [10],

$$D(u,\ v) = \sup_{\alpha\in[0,\ 1]} d\Big([u]^{\alpha}\ \ ,\ \ [v]^{\alpha}\Big) = \sup_{\alpha\in[0,\ 1]} \max\Big\{|u^{-}(\alpha)-v^{-}(\alpha)|,\ |u^{+}(\alpha)-v^{+}(\alpha)|\Big\},$$

where d is the Hausdorff metric. The metric space $(\mathbb{R}_{\mathcal{F}},\ D)$ is complete, separable and locally compact and the following properties from [10] for metric D are valid:

1. $D(u\oplus w,\ v\oplus w) = D(u,\ v),\qquad \forall u,\ v,\ w\in\mathbb{R}_{\mathcal{F}};$
2. $D(\lambda u,\ \lambda v) = |\lambda| D(u,\ v),\qquad \forall\lambda\in\mathbb{R},\ u,\ v\in\mathbb{R}_{\mathcal{F}};$
3. $D(u\oplus v,\ w\oplus z)\leqslant D(u,\ w)+D(v,\ z),\qquad \forall u,\ v,\ w,\ z\in\mathbb{R}_{\mathcal{F}};$
4. $D(u\ominus v,\ w\ominus z)\leqslant D(u,\ w)+D(v,\ z)$, as long as $u\ominus v$ and $w\ominus z$ exist, where $u,\ v,\ w,\ z\in\mathbb{R}_{\mathcal{F}}$,

where $\ominus$ is the Hukuhara difference(H-difference), it means that $w\ominus v = u$ if and only if $u\oplus v = w$.

Definition 2.1 [4] The generalized Hukuhara difference of two fuzzy numbers $u,\ v\in\mathbb{R}_{\mathcal{F}}$ is defined as follows:

$$u\ominus_{gH} v = w \quad\Leftrightarrow\quad u = v+w, \quad\text{or}\quad v = u+(-1)w.$$

In terms of α-levels we have

$$[u\ominus_{gH} v]^{\alpha} = [\min\{u^{-}(\alpha)-v^{-}(\alpha), u^{+}(\alpha)-v^{+}(\alpha)\}\ \ ,\ \ \max\{u^{-}(\alpha)-v^{-}(\alpha), u^{+}(\alpha)-v^{+}(\alpha)\}],$$

and if the H-difference exists, then $u\ominus v = u\ominus_{gH} v$; the conditions for the existence of $w = u\ominus_{gH} v\in\mathbb{R}_{\mathcal{F}}$ are

$$case(i)\begin{cases} w^{-}(\alpha) = u^{-}(\alpha)-v^{-}(\alpha) \text{ and } w^{+}(\alpha) = u^{+}(\alpha)-v^{+}(\alpha), & \forall\alpha\in[0,1],\\ \text{with } w^{-}(\alpha) \text{ increasing, } w^{+}(\alpha) \text{ decreasing, } w^{-}(\alpha)\leqslant w^{+}(\alpha), & \end{cases}$$

$$case(ii)\begin{cases} w^{-}(\alpha) = u^{+}(\alpha)-v^{+}(\alpha) \text{ and } w^{+}(\alpha) = u^{-}(\alpha)-v^{-}(\alpha), & \forall\alpha\in[0,1].\\ \text{with } w^{-}(\alpha) \text{ increasing, } w^{+}(\alpha) \text{ decreasing, } w^{-}(\alpha)\leqslant w^{+}(\alpha). & \end{cases}$$

It is easy to show that (i) and (ii) are both valid if and only if w is a crisp number.

Remark 1 *Throughout the rest of this paper, we assume that* $u\ominus_{gH} v\in\mathbb{R}_{\mathcal{F}}$.

2.1 *Generalized Hukuhara Derivative*

In this section, we present some definitions and theorems for a fuzzy-valued function $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}_{\mathcal{F}}$. The $\alpha-$level representation of fuzzy-valued function f given by $f(x;\alpha) = [f^{-}(x;\alpha)\ \ ,\ \ f^{+}(x;\alpha)]$, for all $x\in[a,b]$ and $\alpha\in[0,1]$.

Definition 2.2 [9] A fuzzy valued function $f:[a,\ b]\to\mathbb{R}_{\mathcal{F}}$ is said to be continuous at $t_0\in[a,\ b]$ if for each $\epsilon>0$ there is $\delta>0$ such that $D(f(t), f(t_0))<\epsilon$, whenever $t\in[a,b]$ and $|t-t_0|<\delta$. We say that f is fuzzy continuous on $[a,b]$ if f is continuous at each $t_0\in[a,\ b]$.

Definition 2.3 [4] The generalized Hukuhara derivative of a fuzzy-valued function $f : (a,\ b) \to \mathbb{R}_{\mathcal{F}}$ at $x_0 \in (a,b)$ is defined as

$$f'_{gH}(x_0) = \lim_{h\to 0} \frac{f(x_0+h) \ominus_{gH} f(x_0)}{h}. \tag{1}$$

If $f'_{gH}(x_0) \in \mathbb{R}_{\mathcal{F}}$ satisfying (1) exists, we say that f is generalized Hukuhara differentiable (gH-differentiable for short) at x_0.

Definition 2.4 [4] Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in (a,b)$ with $f^-(x;\alpha)$ and $f^+(x;\alpha)$ both differentiable at x_0. Also, we say that

- f is $[(i) - gH]$-differentiable at x_0 if

$$f'_{i.gH}(x_0;\alpha) = [(f^-)'(x_0;\alpha)\ \ ,\ \ (f^+)'(x_0;\alpha)], \qquad 0 \leqslant \alpha \leqslant 1, \tag{2}$$

- f is $[(ii) - gH]$-differentiable at x_0 if

$$f'_{ii.gH}(x_0;\alpha) = [(f^+)'(x_0;\alpha)\ \ ,\ \ (f^-)'(x_0;\alpha)], \qquad 0 \leqslant \alpha \leqslant 1. \tag{3}$$

Definition 2.5 [17] We say that a point $x_0 \in (a,b)$ is a switching point for the differentiability of f if in any neighborhood V of x_0 there exist points $x_1 < x_0 < x_2$ such that

(1) **type(I)** at x_1 (2) holds while (3) does not hold and at x_2 (3) holds and (2) does not hold, or
(2) **type(II)** at x_1 (3) holds while (2) does not hold and at x_2 (2) holds and (3) does not hold.

Definition 2.6 [2] Let $f : (a,\ b) \to \mathbb{R}_{\mathcal{F}}$. We say that $f(x)$ is gH-differentiable of the $2^{th}-$ order at $x_0 \in (a,b)$ whenever the function $f(t)$ is gH-differentiable of the order $i,\ i = 0,1$, at x_0, $((f^{(i)}(x_0))_{gH} \in \mathbb{R}_{\mathcal{F}})$, moreover there isn't any switching point on (a,b). Then there exist $f''_{gH}(x_0) \in \mathbb{R}_{\mathcal{F}}$ such that

$$f''_{gH}(x_0) = \lim_{h\to 0} \frac{f'_{gH}(x_0+h) \ominus_{gH} f'_{gH}(x_0)}{h},$$

if $f'_{gH}(x_0+h) \ominus_{gH} f'_{gH}(x_0) \in \mathbb{R}_{\mathcal{F}}$.

Definition 2.7 [2] Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ and $f'_{gH}(x)$ be gH-differentiable at $x_0 \in (a,b)$, moreover there isn't any switching point on (a,b) and $(f^-)'(x;\alpha)$ and $(f^+)'(x;\alpha)$ both differentiable at x_0. We say that

- $f'_{gH}(x)$ is $[(i) - gH]$-differentiable whenever the type of gH-differentiability $f(x)$ and $f'_{gH}(x)$ are the same:

$$f''_{i.gH}(x_0;\alpha) = [(f^-)''(x_0;\alpha)\ \ ,\ \ (f^+)''(x_0;\alpha)], \qquad 0 \leqslant \alpha \leqslant 1,$$

- $f'_{gH}(x)$ is $[(ii) - gH]$-differentiable if the type of gH-differentiability $f(x)$ and $f'_{gH}(x)$ are the different :

$$f''_{ii.gH}(x_0;\alpha) = [(f^+)''(x_0;\alpha)\ \ ,\ \ (f^-)''(x_0;\alpha)], \qquad 0 \leqslant \alpha \leqslant 1.$$

In this paper, we assume that the notations $\mathcal{C}^k([a,b],\mathbb{R}_{\mathcal{F}}),\ k=0,1,2$ for the set of fuzzy valued function f which are defined on $[a,b]$ and it's first k gH-derivative are fuzzy continuous.

Theorem 2.8 [4] If f is gH-differentiable with no switching point in the interval $[a,b]$, then we have

$$\int_a^b f'_{gH}(x)\mathrm{d}x = f(b)\ominus_{gH} f(a).$$

Theorem 2.9 [16] Consider $g:[a,b]\to J$ is real and differentiable function at x and $f:J\to\mathbb{R}_{\mathcal{F}}$ is gH-differentiable at the point $g(x)$. Then we observe that

1. If $f(x)$ is a $[(i)-gH]-$differentiable fuzzy function at the point $g(x)$, then

$$\Big(f(g(x))\Big)'_{i.gH} = \begin{cases} g'(x)\odot f'_{i.gH}(g(x)), & \text{If } g'(x)>0 \\ \ominus(-1)g'(x)\odot f'_{i.gH}(g(x)), & \text{If } g'(x)<0. \end{cases} \tag{4}$$

2. If $f(x)$ is a $[(ii)-gH]-$differentiable fuzzy function at the point $g(x)$, then

$$\Big(f(g(x))\Big)'_{ii.gH} = \begin{cases} g'(x)\odot f'_{ii.gH}(g(x)), & \text{If } g'(x)>0. \\ \ominus(-1)g'(x)\odot f'_{ii.gH}(g(x)), & \text{If } g'(x)<0. \end{cases} \tag{5}$$

Remark 2 *[16] By attention to Theorem 2.9, we conclude that*

$$\Big(f(g(x))\Big)'_{gH} = \begin{cases} g'(x)\odot f'_{gH}(g(x)), & \textit{If } g'(x)>0; \\ \ominus(-1)g'(x)\odot f'_{gH}(g(x)), & \textit{If } g'(x)<0. \end{cases} \tag{6}$$

2.2 *Generalized Hukuhara partial differentiation*

In this section, some preliminary results related to the fuzzy generalized Hukuhara partial derivatives are presented. The parametric representation of the fuzzy value function $f:\mathbb{D}\to\mathbb{R}_{\mathcal{F}}$ is expressed by $f(x,t;\alpha)=[f^-(x,t;\alpha),\ f^+(x,t;\alpha)]$, for all $(x,t)\in\mathbb{D}$ and $\alpha\in[0,\ 1]$.

Definition 2.10 [2] Let $(x_0,t_0)\in\mathbb{D}$, then the first generalized Hukuhara partial derivative ([gH-p]-derivative for short) of a fuzzy value function $f(x,t):\mathbb{D}\to\mathbb{R}_{\mathcal{F}}$ at (x_0,t_0) with respect to variables x,t are the functions $f_{x_{gH}}(x_0,t_0)$ and $f_{t_{gH}}(x_0,t_0)$ given by

$$f_{x_{gH}}(x_0,t_0)=\lim_{h\to 0}\frac{f(x_0+h,t_0)\ominus_{gH} f(x_0,t_0)}{h},$$

$$f_{t_{gH}}(x_0,t_0)=\lim_{k\to 0}\frac{f(x_0,t_0+k)\ominus_{gH} f(x_0,t_0)}{k},$$

provided that $f_{x_{gH}}(x_0,t_0)$ and $f_{t_{gH}}(x_0,t_0)\in\mathbb{R}_{\mathcal{F}}$.

Definition 2.11 [2] Let $f(x,t) : \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$, $(x_0, t_0) \in \mathbb{D}$. Also, assume $f^-(x,t;\alpha)$ and $f^+(x,t;\alpha)$ are two real valued differentiable w.r.t. x. We say that

- $f(x,t)$ is $[(i)-p]$-differentiable w.r.t. x at (x_0,t_0) if

$$f_{x_{i.gH}}(x_0,t_0;\alpha) = [f_x^-(x_0,t_0;\alpha) \quad , \quad f_x^+(x_0,t_0;\alpha)]. \tag{7}$$

- $f(x,t)$ is $[(ii)-p]$-differentiable w.r.t. x at (x_0,t_0) if

$$f_{x_{ii.gH}}(x_0,t_0;\alpha) = [f_x^+(x_0,t_0;\alpha) \quad , \quad f_x^-(x_0,t_0;\alpha)]. \tag{8}$$

Definition 2.12 [2] For any fixed ξ_0, we say that $(\xi_0,t) \in \mathbb{D}$ is a switching points for the differentiability of $f(x,t)$ with respect to x if in any neighborhood V of (ξ_0,t) there exist points $(x_1,t) < (\xi_0,t) < (x_2,t)$ such that

type I. at (x_1,t) (7) holds while (8) does not hold and at (x_2,t) (8) holds and (7) does not hold for all t, or

type II. at (x_1,t) (8) holds while (7) does not hold and at (x_2,t) (7) holds and (8) does not hold for all t.

Definition 2.13 [2] Let $f(x,t) : \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$, and $\partial_x f(x,t)$ is a [gH-p]-differentiable at $(x_0,t_0) \in \mathbb{D}$ with respect to x. Moreover, there is not any switching point on $\mathbb{D}$. We say that

- $f_{x_{gH}}(x,t)$ is $[(i)-p]$-differentiable w.r.t x if the type of [gH-p]-differentiability of both $f(x,t)$ and $f_{x_{gH}}(x,t)$ are the same:

$$f_{xx_{i.gH}}(x_0,t_0;\alpha) = [f_{xx}^-(x_0,t_0;\alpha) \quad f_{xx}^+(x_0,t_0;\alpha)].$$

- $f_{x_{gH}}(x,t)$ is $[(ii)-p]$-differentiable w.r.t x if the type of [gH-p]-differentiability $f(x,t)$ and $f_{x_{gH}}(x,t)$ are different:

$$f_{xx_{ii.gH}}(x_0,t_0;\alpha) = [f_{xx}^+(x_0,t_0;\alpha) \quad f_{xx}^-(x_0,t_0;\alpha)].$$

Lemma 2.14 [2] Consider $f : \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$ as a fuzzy continuous function. Assume that f is [gH-p]-differentiable with respect to t, with no switching point in the interval $[a,t]$ and fuzzy continuous, then we have

$$\int_a^t f_{s_{gH}}(x,s)ds = f(x,t) \ominus_{gH} f(x,a).$$

3. The fuzzy homogeneous wave equation in infinite interval

Consider the following fuzzy Cauchy one-dimensional homogeneous wave equation in infinite interval

$$\begin{cases} u_{tt_{gH}}(x,t) \ominus_{gH} c^2 \odot u_{xx_{gH}}(x,t) = 0, \, (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x), \\ u_{t_{gH}}(x,0) = g(x), \end{cases} \tag{9}$$

where 0 denotes the crisp set $\{0\}$ and $c \in \mathbb{R}$ is called the wave speed. The functions $f(x)$ and $g(x)$ are fuzzy functions and $f \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$, $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$.

The canonical form of the solution of (9) is $u(x,t) = F(x+ct) \oplus G(x-ct)$, where F and G can be any two twice generalized Hukuhara differentiable functions [16].

Based on the type of $[gH-p]-$differentiability with respect to t, we have the following statements [16].

Case 1. If $u(x,t)$ is $[(i)-p]-$differentiable with respect to t, the d'Alambert's solution of (9) is equal

$$u(x,t) = \frac{1}{2}\Big(f(x+ct) \oplus f(x-ct)\Big) \oplus \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds. \tag{10}$$

Case 2. If $u(x,t)$ is $[(ii)-p]-$differentiable with respect to t, therefore the d'Alambert fuzzy solution of (9) is

$$u(x,t) = \frac{1}{2}\Big(f(x+ct) \oplus f(x-ct)\Big) \ominus \frac{(-1)}{2c}\int_{x-ct}^{x+ct} g(s)ds. \tag{11}$$

Theorem 3.1 [16] Let $T > 0$ be a fixed constant. The homogeneous wave equation (9) in the domain $-\infty < x < \infty$, $0 \leqslant t \leqslant T$ is well-posed for $f \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$, $g \in\in \mathcal{C}^1(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$.

4. The fuzzy wave equation on the half-line

Consider the following fuzzy Dirichlet problem on the half-line

$$\begin{cases} u_{tt_{gH}}(x,t) \ominus_{gH} c^2 \odot u_{xx_{gH}}(x,t) = 0, & x > 0,\ t > 0, \\ u(x,0) = f(x), \quad u_{t_{gH}}(x,0) = g(x), & x \geqslant 0, \\ u(0,t) = 0, & t > 0. \end{cases} \tag{12}$$

For the vibrating string, the boundary condition of (12) means that the end of the string at $x = 0$ is held fixed. We reduce the Dirichlet problem (12) to the whole line $\mathbb{R}$ by odd reflection. Namely, we set

$$\tilde{f}(x) = \begin{cases} f(x), & \text{for } x > 0, \\ 0, & \text{for } x = 0, \\ (-1)f(-x), & \text{for } x < 0. \end{cases} \qquad \tilde{g}(x) = \begin{cases} g(x), & \text{for } x > 0, \\ 0, & \text{for } x = 0, \\ (-1)g(-x), & \text{for } x < 0. \end{cases} \tag{13}$$

Then $\tilde{f}$ and $\tilde{g}$ are odd fuzzy function. Now, we have the following fuzzy wave equation on the whole line with the extended initial data

$$\begin{cases} \tilde{u}_{tt_{gH}}(x,t) \ominus_{gH} c^2 \odot \tilde{u}_{xx_{gH}}(x,t) = 0, & x \in \mathbb{R},\ t > 0, \\ \tilde{u}(x,0) = \tilde{f}(x), & \tilde{u}_{t_{gH}}(x,0) = \tilde{g}(x), \quad x \geqslant 0, \end{cases} \tag{14}$$

For $x > 0$, we have

$$\tilde{u}(x,0) = \tilde{f}(x) = f(x), \qquad \tilde{u}_t(x,0) = \tilde{g}(x) = g(x).$$

It remains to show that $\tilde{u}(0,t) = 0$. For this it suffices to show that $u(x,t)$ is an odd fuzzy function for all $t > 0$. Indeed, set $v(x,t) = (-1)\tilde{u}(-x,t)$, hence by Remark *2* we have

$$v_x(x,t) = (-1) \ominus \tilde{u}_x(-x,t), \qquad v_{xx}(x,t) = (-1)\tilde{u}_{xx}(-x,t),$$
$$v_t(x,t) = (-1)\tilde{u}_t(-x,t), \qquad v_{tt}(x,t) = (-1)\tilde{u}_{tt}(-x,t).$$

Hence,

$$v_{tt}(x,t) \ominus_{gH} c^2 v_{xx}(x,t) = (-1)\Big(\tilde{u}_{tt}(-x,t) \ominus_{gH} c^2 \tilde{u}_{xx}(-x,t)\Big) = 0$$

and

$$v(x,0) = (-1)\tilde{u}(-x,0) = (-1)\tilde{f}(-x) = \tilde{f}(x).$$

Also,

$$v_t(x,0) = (-1)\tilde{u}_t(-x,0) = (-1)\tilde{g}(-x) = \tilde{g}(x).$$

So by (14), we can write $v(x,t) = \tilde{u}(x,t)$, so $\tilde{u}(x,t) = (-1)\tilde{u}(-x,t)$. We show that $\tilde{u}(x,t)$ is an odd fuzzy function in the variable x, and hence $\tilde{u}(0,t) = 0$ for $t > 0$.

Then defining the restriction of $\tilde{u}(x,t)$ to the positive half-line $x \geqslant 0$,

$$u(x,t) = \tilde{u}(x,t)|_{x \geqslant 0}. \tag{15}$$

We automatically have that $u(0,t) = \tilde{u}(0,t) = 0$. So, the boundary condition of the Dirichlet problem (12) is satisfied for u. Obviously the initial conditions are satisfied as well, since the restrictions of $\tilde{f}(x)$ and $\tilde{g}(x)$ to the positive half-line are $f(x)$ and $g(x)$ respectively. Finally, $u(x,t)$ solves the fuzzy wave equation for $x > 0$, since $\tilde{u}(x,t)$ satisfies the fuzzy wave equation for all $x \in \mathbb{R}$, and in particular for $x > 0$. Thus, $\tilde{u}(x,t)$ defined by (15) is a solution of the Dirichlet problem (12).

Now we investigate the solution for different type of $[gH - p]$-differentiability.

Case 1. Consider $u(x,t)$ is $[(i) - p]-$differentiable with respect to t so the d'Alambert's solution of (14) by (10) is equal

$$\tilde{u}(x,t) = \frac{1}{2}\Big(\tilde{f}(x+ct) \oplus \tilde{f}(x-ct)\Big) \oplus \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{g}(s)ds.$$

Then, for $x \geqslant 0$ and $t > 0$, we have $x + ct > 0$, and $\tilde{f}(x+ct) = f(x+ct)$.

a) If $t > 0$ and $x - ct > 0$, then $\tilde{f}(x-ct) = f(x-ct)$, and over the interval $s \in [x-ct, x+ct]$, $\tilde{g}(s) = g(s)$. Thus, for $x > ct$, we have

$$u(x,t) = \frac{1}{2}\Big(f(x+ct) \oplus f(x-ct)\Big) \oplus \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

b) If $t > 0$ and $x - ct < 0$, then by (13) we observe that

$$\tilde{f}(x-ct) = (-1)f(-(x-ct)) = (-1)f(ct-x)$$

and $\tilde{g}(s) = (-1)g(-s)$ for $s < 0$. So,

$$u(x,t) = \frac{1}{2}\Big(f(x+ct) \oplus (-1)f(ct-x)\Big) \oplus \frac{1}{2c}\Big[\int_{x-ct}^{0} -g(-s)ds \oplus \int_{0}^{x+ct} g(s)ds\Big].$$

Making the change of variables $s \mapsto -s$ in the first integral on the right, we get

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big(f(x+ct) \oplus (-1)f(ct-x)\Big) \oplus \frac{1}{2c}\Big[\int_{ct-x}^{0} g(s)ds \oplus \int_{0}^{x+ct} g(s)ds\Big] \\ &= \frac{1}{2}\Big(f(x+ct) \oplus (-1)f(ct-x)\Big) \oplus \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)ds. \end{aligned}$$

Combining the two expression for $u(x,t)$ over the two regions, we arrive at the $[(i)-p]-$differentiable solution

$$u(x,t) = \begin{cases} \frac{1}{2}\Big(f(x+ct) \oplus f(x-ct)\Big) \oplus \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds, & \text{for } x > ct, \\ \\ \frac{1}{2}\Big(f(x+ct) \oplus (-1)f(ct-x)\Big) \oplus \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)ds, & \text{for } 0 < x < ct. \end{cases} \tag{16}$$

Case 2. If $u(x,t)$ is a $[(ii)-p]-$differentiable with respect to t. According to the process described in Case 1, we obtain the following $[(ii)-p]-$differentiable solution

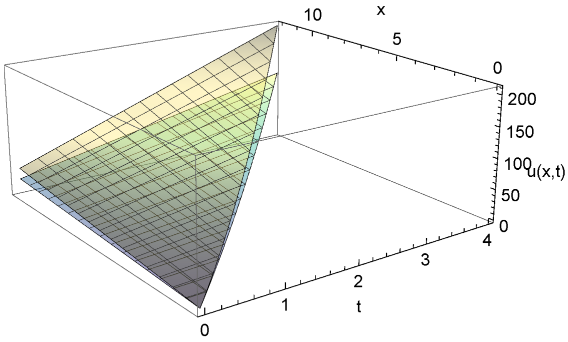
$$u(x,t) = \begin{cases} \frac{1}{2}\Big(f(x+ct) \oplus f(x-ct)\Big) \ominus (-1)\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds, & \text{for } x > ct, \\ \\ \frac{1}{2}\Big(f(x+ct) \oplus (-1)f(ct-x)\Big) \ominus (-1)\frac{1}{2c}\int_{ct-x}^{x+ct} g(s)ds, & \text{for } 0 < x < ct. \end{cases} \tag{17}$$

5. Examples

In this section, we will use the above proposed method to solve different examples. The computations associated with the examples are performed using Mathematica software.

***Example* 5.1** Consider the following fuzzy wave equation

$$\begin{cases} u_{tt_{gH}}(x,t) \ominus_{gH} 9 \odot u_{xx_{gH}}(x,t) = 0, & x > 0,\, t > 0, \\ u(x,0) = (1,3,4)x, \quad u_{t_{gH}}(x,0) = (1,3,4)x, & x \geqslant 0, \\ u(0,t) = 0, & t > 0. \end{cases}$$



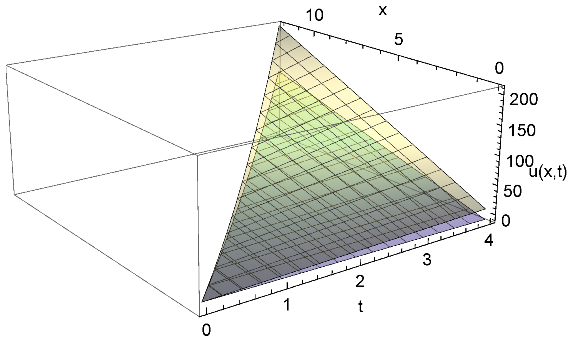


Figure 1. Graph of $u(x,t)$ of Example 5.1 for $x > 3t$ (left) and for $0 < x < 3t$ (right) in $\alpha = \frac{1}{2}$

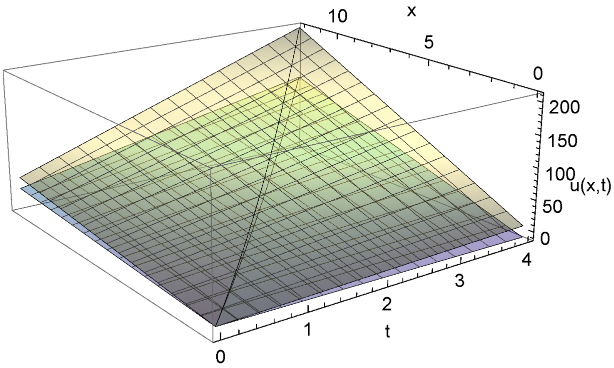


Figure 2. Graph of $u(x,t)$ of Example 5.1 in $\alpha = \frac{1}{2}$ for $x \in \mathbb{R}$

The $[(i)-p]-$differentiable solution of this equation for $x > 3t$

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big([1+2\alpha, 4-\alpha](x+3t) \oplus [1+2\alpha, 4-\alpha](x-3t)\Big) \\ &\quad \oplus \frac{1}{6}\Big[\int_{x-3t}^{x+3t}(1+2\alpha)sds, \int_{x-3t}^{x+3t}(4-\alpha)sds\Big] \\ &= [(1+2\alpha)(t+1)x, (4-\alpha)(t+1)x] \end{aligned}$$

and for $0 < x < 3t$, we have

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big([1+2\alpha, 4-\alpha](x+3t) \oplus (-1)[1+2\alpha, 4-\alpha](3t-x)\Big) \\ &\quad \oplus \frac{1}{6}\Big[\int_{3t-x}^{x+3t}(1+2\alpha)sds, \int_{3t-x}^{x+3t}(4-\alpha)sds\Big] \\ &= \frac{1}{2}\Big[9t(\alpha-1)+x(5+\alpha)+(1+2\alpha)tx, -9t(\alpha-1)+x(5+\alpha)+(4-\alpha)tx\Big]. \end{aligned}$$

These solutions are illustrated in Figures 1 and 2.

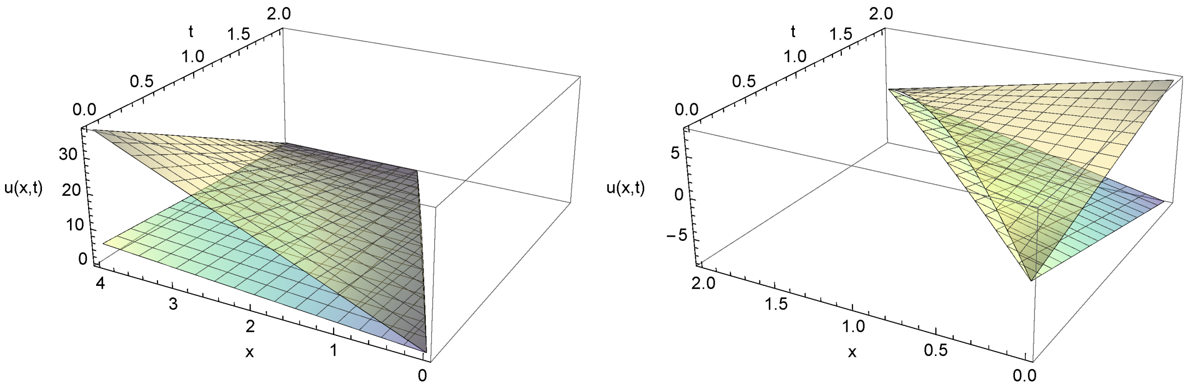


Figure 3. Graph of $u(x,t)$ of Example 5.2 for $x > t$ (left) and for $0 < x < t$ (right) in $\alpha = \frac{1}{3}$

***Example* 5.2** Consider the following fuzzy Dirichlet problem on the half-line

$$\begin{cases} u_{tt_{gH}}(x,t) \ominus_{gH} \odot u_{xx_{gH}}(x,t) = 0, & x > 0,\, t > 0, \\ u(x,0) = (0,4,6)x, \quad u_{t_{gH}}(x,0) = (-6,-2,0)x, & x \geqslant 0, \\ u(0,t) = 0, & t > 0. \end{cases}$$

The $[(ii) - p]-$differentiable solution of this equation for $x > t$

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big([4\alpha, 12-8\alpha](x+t) \oplus [4\alpha, 12-8\alpha](x-t)\Big) \\ &\qquad \ominus (-1)\frac{1}{2}\Big[\int_{x-t}^{x+t}(-6+4\alpha)sds, \int_{x-t}^{x+t}(-2\alpha)sds\Big] \\ &= [(2\alpha)(2-t)x, (6-4\alpha)(2-t)x] \end{aligned}$$

and for $0 < x < t$ by (17), we have

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big([4\alpha, 12-8\alpha](x+t) \oplus (-1)[4\alpha, 12-8\alpha](t-x)\Big) \\ &\qquad \ominus (-1)\frac{1}{2}\Big[\int_{t-x}^{x+t}(-6+4\alpha)sds, \int_{t-x}^{x+t}(-2\alpha)sds\Big] \\ &= \Big[-2x(\alpha-3) + 6t(\alpha-1) - 2tx\alpha, -2x(\alpha-3) - 6t(\alpha-1) - tx(6-4\alpha)\Big]. \end{aligned}$$

The solutions are depicted in Figures 3 and 4.

6. Conclusion

In this paper a fuzzy solution for the fuzzy wave equation on the Half-line is introduced under different type of generalized Hukuhara partial differentiability. The d'Alembert's formulas for fuzzy Dirichlet problem on the half-line obtained without embedding them to crisp equations. To illustrate the technique, some examples are solved by this method and analytical solution for the fuzzy wave equation is obtained.

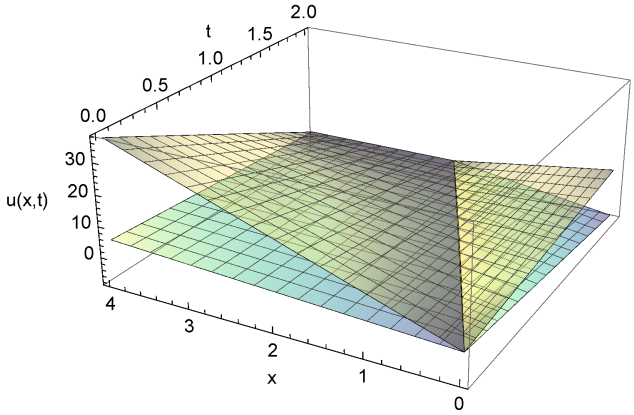


Figure 4. Graph of $u(x,t)$ of Example 5.2 in $\alpha = \frac{1}{3}$ for $x \in \mathbb{R}$

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