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Free Vibration of Functionally Graded Cylindrical Shells Based on the First Order Shear Deformation Theory

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The free vibration analysis of simply supported functionally graded cylindrical shells based on the first order shear deformation theory is presented in this paper. Assuming that the material properties graded in the thickness direction as a volume fraction power-law distribution and using the Hamilton's principle, the governing equations of motion are established and solved. The influence of constituent volume fractions and the effects of configurations of the constituent materials on the frequencies are carefully discussed. The results are validated with the known data in the literature.

Keywords: cylindrical shells; first order shear deformation theory; free vibration; functionally graded material

1 - Introduction

Functionally graded materials (FGMs) have gained considerable importance as materials to be used in extremely high temperature environments such as nuclear reactors and high-speed spacecraft industries [1] in recent years. FGMs were first introduced by a group of scientists in Sendai, Japan in 1984 [2]. FGMs are inhomogeneous materials, in which the mechanical properties vary smoothly and continuously from one layer to the other. This is achieved by gradually varying the volume fraction of the constituent materials. This continuous change in composition results in the graded properties of FGMs [3]. This gradation in properties of the material reduces thermal stresses, residual stresses and stress concentration factors [4]. These materials are made from a mixture of ceramic and metal or from a mixture of different materials. The ceramic constituent of the material provides the high-temperature resistance due to its low thermal conductivity. The ductile metal constituent on the other hand, prevents fracture caused by stresses due to the high temperature gradient in a very short period of time. Furthermore a mixture of ceramic and metal with a continuously varying volume fraction can be easily manufactured [5]. This eliminates interface problems of composite materials, and thus, the stress distributions are smooth. Studies on FGMs have been extensive but are largely confined to

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analysis of thermal stresses and deformations. Tanigawa et al. [6] derived a one dimensional temperature solution for a nonhomogeneous plate in the transient state and optimized the material compositions by introducing a laminated composite model. The optimal composition profile problems of the FGM to decrease the thermal stresses and thermal stress intensity factor were discussed by Noda [7, 8]. He concluded that when the continuously changing composition between ceramics and metals can be selected pertinently, thermal stresses in the FGM are drastically decreased. Javaheri and Eslami [9] presented the thermal buckling of rectangular functionally graded plate based on the high order plate theory. The buckling analysis of functionally graded circular plates is given by Najafizadeh and Eslami [10, 11].

Studies on vibration of cylindrical shells are extensive. Many of these studies are for pure isotropic and composite shells; see [12-19]. But, studies on vibration of functionally graded cylindrical shells (FGCSs) are limited. Loy et al. [20] presented the Rayleigh-Ritz solution for free vibration of simply supported FGCSs. Pradhan et al. [21] discussed the effects of boundary conditions and volume fractions on the natural frequencies of FGCSs.

In the present paper, free vibration analysis of simply supported functionally graded cylindrical shells (FGCSs) is presented. Using the Hamilton's principle, the governing equations are derived based on the first order shear deformation theory. The objective is to study the frequency characteristics, the influence of the constituent volume fractions, and the affects of the configurations of the constituent materials on the natural frequencies. The results are validated the known data in the literature. The comparison shows that the present results agreed well with those in the literature.

2 – Material Properties

The functionally graded cylindrical shell (FGCS) as shown in Figure 1 is assumed to be thin and of length L and thickness h with mean radius R. The x-axis is taken along a generator, the circumferential arc length subtends an angle θ, and the z-axis is directed radially outwards. The material properties P of FGMs are function of the material properties and volume fraction of the constituent materials and are expressed as [22]

$$P = \sum_{j=1}^{K} P_j V_{f_j} \tag{1}$$

where P_j and V_{f_j} are the material property and volume fraction of the constituent material j, respectively. The volume fractions of all the constituent materials should add up to one [22].

$$\sum_{j=1}^{K} V_{f_j} = 1 \tag{2}$$

For a FGCS with thickness h and a reference surface at its middle surface, the volume fraction can be written as [22]

$$V_f = \left(\frac{z + h/2}{h}\right)^N, \quad 0 \le N \le \infty \tag{3}$$

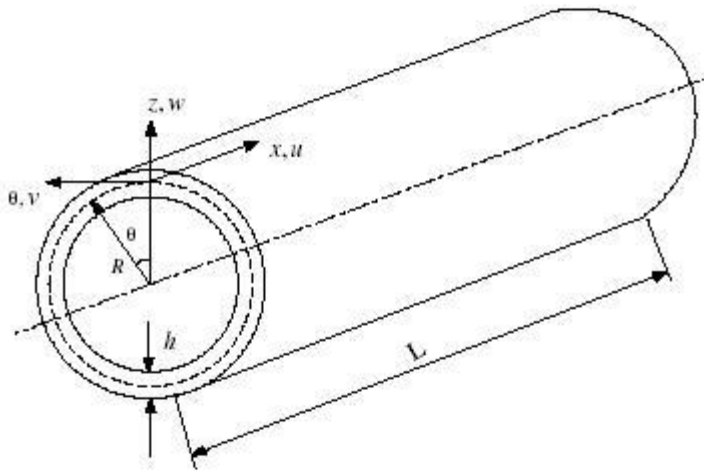

Figure 1. Coordinate system of FGCS.

where N is the power-law exponent. For a FGM with two constituent materials, the Young's modulus E , Poisson ratio υ, and the mass density ρ can be expressed as [22]

$$E=(E_1-E_2)(\frac{2z+h}{2h})^N+E_2 \tag{4}$$

$$\upsilon=(\upsilon_1-\upsilon_2)(\frac{2z+h}{2h})^N+\upsilon_2 \tag{5}$$

$$\rho=(\rho_1-\rho_2)(\frac{2z+h}{2h})^N+\rho_2 \tag{6}$$

From these equations, it is found that for $z=-h/2$, FGM properties are the same as those of material 2 and for $z=h/2$, FGM properties are the same as those of material 1. Thus, the material properties vary continuously from material 2 at the inner surface of the cylindrical shell to material 1 at the outer surface of the cylindrical shell. Figure (2) shows a geometric definition of the shell.

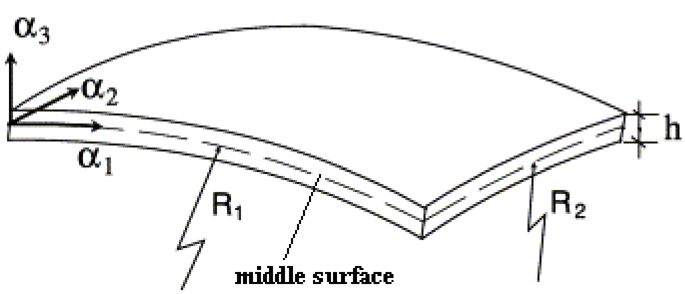

Figure 2. Configuration of a generic shell.

3 – Formulation of the problem

3.1 – The generic functionally graded shells

The strain-displacement relationships for a thin generic shell are [23]

$$\in_{11}=\frac{1}{A_1}\left[\frac{\partial U_1}{\partial \alpha_1}+\frac{U_2}{A_2}\frac{\partial A_1}{\partial \alpha_2}+U_3\frac{A_1}{R_1}\right] \tag{7a}$$

$$\in_{22}=\frac{1}{A_2}\left[\frac{\partial U_2}{\partial \alpha_2}+\frac{U_1}{A_1}\frac{\partial A_2}{\partial \alpha_1}+U_3\frac{A_2}{R_2}\right] \tag{7b}$$

$$\in_{33}=\frac{\partial U_3}{\partial \alpha_3} \tag{7c}$$

$$\in_{12}=\frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}(\frac{U_1}{A_1})+\frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}(\frac{U_2}{A_2}) \tag{7f}$$

$$\in_{13}=A_1\frac{\partial}{\partial \alpha_3}(\frac{U_1}{A_1})+\frac{1}{A_1}\frac{\partial U_3}{\partial \alpha_1} \tag{7d}$$

$$\in_{23}=A_2\frac{\partial}{\partial \alpha_3}(\frac{U_2}{A_2})+\frac{1}{A_2}\frac{\partial U_3}{\partial \alpha_2} \tag{7e}$$

where

$$A_1=\left|\frac{\partial \bar{r}}{\partial \alpha_1}\right| \tag{8a}$$

$$A_2=\left|\frac{\partial \bar{r}}{\partial \alpha_2}\right| \tag{8b}$$

Here, $\bar{r}$ is the position vector of an arbitrary point on the shell, A_1 and A_2 are the fundamental form parameters or Lame$'$ parameters, U_1, U_2 and U_3 are the displacements at any point $(\alpha_1, \alpha_2, \alpha_3)$, R_1 and R_2 are the radii of curvatures related to α_1 and α_2, respectively. The first order shear deformation theory (FSDT) is used in the present study and is based on the following displacement field [23]

$$\begin{aligned}U_1(\alpha_1,\alpha_2,\alpha_3)&=u_1(\alpha_1,\alpha_2)+\alpha_3\phi_1(\alpha_1,\alpha_2)\\ U_2(\alpha_1,\alpha_2,\alpha_3)&=u_2(\alpha_1,\alpha_2)+\alpha_3\phi_2(\alpha_1,\alpha_2)\\ U_3(\alpha_1,\alpha_2,\alpha_3)&=u_3(\alpha_1,\alpha_2)\end{aligned} \tag{9}$$

where $u_1, u_2,$ and u_3 are the middle surface displacements, and ϕ_1 and ϕ_2 are the rotations about the α_2 and α_1-directions, respectively. Substituting Eq. (9) into Eqs. (7) give [23]

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{Bmatrix} = \begin{Bmatrix} \epsilon_{11}^0 \\ \epsilon_{22}^0 \\ \gamma_{12}^0 \end{Bmatrix} + \alpha_3 \begin{Bmatrix} k_{11} \\ k_{22} \\ k_{12} \end{Bmatrix} \tag{10a}$$

$$\begin{Bmatrix} \epsilon_{13} \\ \epsilon_{23} \end{Bmatrix} = \begin{Bmatrix} \gamma_{13}^0 \\ \gamma_{23}^0 \end{Bmatrix} \tag{10b}$$

where ϵ_{ii}^0 and γ_{ij}^0 are the normal and shear strains on the middle surface, respectively, and k_{11}, k_{22}, and k_{12} are the curvatures, and defined as [23]

$$\begin{Bmatrix} \epsilon_{11}^0 \\ \epsilon_{22}^0 \\ \gamma_{12}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{u_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{u_3}{R_1} \\ \frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2} + \frac{u_1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{u_3}{R_2} \\ \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_1}{A_1}\right) \end{Bmatrix}, \quad \begin{Bmatrix} k_{11} \\ k_{22} \\ k_{12} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{A_1}\frac{\partial \phi_1}{\partial \alpha_1} + \frac{\phi_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} \\ \frac{1}{A_2}\frac{\partial \phi_2}{\partial \alpha_2} + \frac{\phi_1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} \\ \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{\phi_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{\phi_1}{A_1}\right) \end{Bmatrix} \tag{11a}$$

$$\begin{Bmatrix} \gamma_{13}^0 \\ \gamma_{23}^0 \end{Bmatrix} = \begin{Bmatrix} \phi_1 + \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha_1} \\ \phi_2 + \frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2} \end{Bmatrix} \tag{11b}$$

For an isotropic shell, the stress-strain relationship are defined as [23]

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{33} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \tag{12}$$

where

$$Q_{11} = Q_{22} = \frac{E}{1-\upsilon^2}, \qquad Q_{12} = \upsilon Q_{11}, \qquad Q_{33} = Q_{44} = Q_{55} = \frac{E}{2(1+\upsilon)} \tag{13}$$

For a thin FGCS, the forces and moments resultants are expressed as

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} d\alpha_3 \tag{14a}$$

$$\begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} \alpha_3 . d\alpha_3 \tag{14b}$$

$$\begin{Bmatrix} Q_{13} \\ Q_{23} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \tau_{13} \\ \tau_{23} \end{Bmatrix} d\alpha_3 \tag{14c}$$

Substituting Eqs. (10) into Eq. (12), and the result into Eqs. (15) give the constitutive equations as

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} \in_{11}^0 + B_{11} k_{11} + A_{12} \in_{22}^0 + B_{12} k_{22} \\ A_{12} \in_{11}^0 + B_{12} k_{11} + A_{22} \in_{22}^0 + B_{22} k_{22} \\ A_{55} \gamma_{12}^0 + B_{55} k_{12} \end{bmatrix} \tag{15a}$$

$$\begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \begin{bmatrix} B_{11} \in_{11}^0 + B_{12} \in_{22}^0 \\ B_{12} \in_{11}^0 + B_{22} \in_{22}^0 \\ B_{55} \gamma_{12}^0 \end{bmatrix} \tag{15b}$$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} A_{44} \gamma_{13}^0 \\ A_{33} \gamma_{23}^0 \end{Bmatrix} \tag{15c}$$

Here, A_{ij} and $B_{ij} (i, j = 1,2,3,4,5)$ denote the extensional, coupling, and bending stiffnesses which defined as

$$(A_{ij}, B_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij} (1, \alpha_3) d\alpha_3 \tag{16}$$

The equations of motion for vibration of shell can be derived by using Hamilton's principle which is described by

$$\delta \int_0^t (\Pi - K) dt = 0 \quad , \qquad \Pi = U - V \tag{17}$$

where $K, \Pi, U,$ and V are the total kinetic, potential, strain, external loads energies, respectively, and t is arbitrary time. The kinetic, strain, and external loads energies of a cylindrical shell can be written as

$$K = \frac{1}{2} \int_{\alpha_1} \int_{\alpha_2} \int_{\alpha_3} \rho (\dot{U}_1^2 + \dot{U}_2^2 + \dot{U}_3^2) d\forall \tag{18a}$$

$$U = \int_{\alpha_1} \int_{\alpha_2} \int_{\alpha_3} (\sigma_{11} \in_{11} + \sigma_{22} \in_{22} + \tau_{12} \gamma_{12} + \tau_{13} \gamma_{13} + \tau_{23} \gamma_{23}) d\forall \tag{18b}$$

$$V = \int_{\alpha_1} \int_{\alpha_2} (q_1 \delta U_1 + q_2 \delta U_2 + q_3 \delta U_3) A_1 A_2 d\alpha_1 d\alpha_2 \tag{18c}$$

where q_1, q_2, and q_3 are the distributed loads. The infinitesimal volume is given by

$$d\forall = A_1 A_2 d\alpha_1 d\alpha_2 d\alpha_3 \tag{19}$$

Substituting Eqs. (9-13) into Eq. (19), and the results into Eq. (18), give the equations of motion for a thin generic shell as

$$-\frac{\partial N_{11}A_2}{\partial \alpha_1} + N_{22}\frac{\partial A_2}{\partial \alpha_1} - \frac{\partial N_{12}A_1}{\partial \alpha_2} - \frac{Q_{13}}{R_1}A_1A_2 = -\ddot{u}_1 I_0 - \ddot{\phi}_1 I_1 + q_1 \tag{20a}$$

$$\frac{\partial N_{22}A_1}{\partial \alpha_2} - N_{11}\frac{\partial A_1}{\partial \alpha_2} + \frac{\partial N_{12}A_2}{\partial \alpha_1} + \frac{Q_{23}}{R_2}A_1A_2 = \ddot{u}_2 I_\circ + \ddot{\phi}_2 I_1 - q_2 \tag{20b}$$

$$-\frac{\partial Q_{13}A_2}{\partial \alpha_1} - \frac{\partial Q_{23}A_1}{\partial \alpha_2} + \frac{N_{11}}{R_1}A_1A_2 + \frac{N_{22}}{R_2}A_1A_2 = -\ddot{u}_3 I_\circ - q_3 \tag{20c}$$

$$-\frac{\partial M_{11}A_2}{\partial \alpha_1} + M_{22}\frac{\partial A_2}{\partial \alpha_1} - \frac{\partial M_{12}A_1}{\partial \alpha_2} + A_1A_2Q_{13} = -\ddot{u}_1 I_1 - \ddot{\phi}_1 I_2 \tag{20d}$$

$$-\frac{\partial M_{22}A_1}{\partial \alpha_2} M_{11}\frac{\partial A_1}{\partial \alpha_2} - \frac{\partial M_{12}A_2}{\partial \alpha_1} + A_1A_2Q_{23} = -\ddot{u}_2 I_1 - \ddot{\phi}_2 I_2 \tag{20e}$$

where

$$I_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(\alpha_3)^i \, d\alpha_3 \tag{21}$$

3.2 – The functionally graded cylindrical shells

The curvilinear coordinates and fundamental form parameters for the cylindrical shell are

$$\frac{1}{R_1} = 0,\quad R_2 = a,\quad A_1 = 1,\quad A_2 = a,\quad \alpha_1 = x,\quad \alpha_2 = \theta,\quad \alpha_3 = z \tag{22}$$

According to the Eq. (22), the strains, curvatures, and stress resultants related to the cylindrical coordinate are defined in Appendix A. Thus, the governing equations of motion for FGCS are obtained as follows

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{x\theta}}{\partial \theta} = I_0\ddot{u}_x + I_1\ddot{\phi}_1 - q_x \tag{23a}$$

$$\frac{\partial N_{\theta\theta}}{\partial \theta} + Q_{23} = I_0\ddot{u}_\theta + I_1\ddot{\phi}_2 - q_\theta \tag{23b}$$

$$N_{\theta\theta} - a\frac{\partial Q_{x3}}{\partial x} - \frac{\partial Q_{\theta 3}}{\partial \theta} = -\ddot{u}_3 I_0 - q_3 \tag{23c}$$

$$-a\frac{\partial M_{xx}}{\partial x} - \frac{\partial M_{x\theta}}{\partial \theta} + aQ_{x3} = -I_1\ddot{u}_3 - I_2\ddot{\phi}_1 \tag{23d}$$

$$-\frac{\partial M_{\theta\theta}}{\partial \theta} - a\frac{\partial M_{x\theta}}{\partial x} + aQ_{\theta 3} = -I_1\ddot{u}_2 - I_2\ddot{\phi}_2 \tag{23e}$$

3. Free Vibration Analysis

The simply supported boundary conditions for cylindrical shell are given by (24)

$$u_2 = u_3 = N_{xx} = M_{xx} = 0, \qquad x = 0, L$$

The displacement fields which satisfy these boundary conditions can be written as

$$\begin{aligned}
u_1 &= A\cos\frac{m\pi x}{l}\cos n\theta\, e^{j\omega t}\\
u_2 &= B\sin\frac{m\pi x}{l}\sin n\theta\, e^{j\omega t}\\
u_3 &= C\sin\frac{m\pi x}{l}\cos n\theta\, e^{j\omega t}\\
\phi_1 &= D\cos\frac{m\pi x}{l}\cos n\theta\, e^{j\omega t}\\
\phi_2 &= E\sin\frac{m\pi x}{l}\sin n\theta\, e^{j\omega t}
\end{aligned} \tag{25}$$

where A, B, C, D, and E are the constants denoting the amplitudes of vibration, m and n are the axial and circumferential wave numbers, respectively and ω(rad/s) is the natural angular frequency of the vibration. For free vibration case, that is

$$q_1 = q_2 = q_3 = 0 \tag{26}$$

Substituting Eqs. (11) into Eqs. (15) and then the results into (23), give the governing equations in terms of displacements. Substituting the Eq.(25) into the resulting governing equations leads to the following equations for the undetermined coefficients A, B, C, D, and E .

$$[C]\begin{Bmatrix} A\\ B\\ C\\ D\\ E\end{Bmatrix} - \omega^2[M]\begin{Bmatrix} A\\ B\\ C\\ D\\ E\end{Bmatrix} = \begin{Bmatrix} 0\\ 0\\ 0\\ 0\\ 0\end{Bmatrix} \tag{27}$$

The matrices $[C]$ and $[M]$ are listed in the Appendix B as a 5×5 matrices. Eq. (27) is solved by imposing non-trivial solutions and equating the characteristic determinant to zero, that is

$$\left|C_{ij} - M_{ij}\omega^2\right| = 0 \tag{28}$$

The smallest roots of this equation yield the natural frequencies.

4 - Results and Discussion

In this paper, the free vibration of functionally graded cylindrical shells (FGCSs) is studied based on the first order shear deformation theory. The FGCS considered here is composed of stainless steel and nickel and its properties are graded in thickness direction according to a volume fraction power-law distribution. The frequency characteristics, the influence of the constituent volume fractions and the effects of the FGM configuration are presented. The influence of constituent volume fractions is studied by varying the value of the power-law exponent N. Also, the effects of the FGM configuration are presented by studying the natural frequencies of two FGCSs. Type I which FGCS has nickel on its inner surface and stainless steel on its outer surface and Type II which FGCS has stainless steel on its inner surface and nickel on its outer surface.

The properties of stainless steel and nickel, calculated at $T = 300K$, are presented in Table 1. To validate the analysis, the results compared with the results of Loy et al. [20] which is based on the Love's shell theory. The comparisons show that the present results agreed well with those in the literature.

The variations of the natural frequencies (Hz) versus the circumferential wave numbers n and length-to-radius ratio L/R for Type I FGCS are shown in Tables 2 and 4, respectively. The frequencies for higher axial modes are higher than those for lower axial modes. Thus, the fundamental frequencies occur at $m=1$. The natural frequencies are decreased with increasing the power-law exponent N. The decrease in the natural frequencies from $N=1$ to $N=15$ is about 2.3**%** at $n=1$ and about 2.4**%** at $n=10$. For small value of N, the natural frequencies approached to the frequencies of stainless steel shell and for large value of N they approached to those for nickel. The natural frequencies for $N>0$ fell between those of stainless steel and nickel for a given circumferential wave number n. It is interesting to note that in Table 2 the fundamental frequencies for various values of N occur at circumferential wave number $n=3$.

As well as Type I FGCS, the variations of the natural frequencies (Hz) versus the circumferential wave numbers n and length-to-radius ratios L/R for Type II FGCS are shown in Tables 3 and 5, respectively. In this case, the influence of the power-law exponent N on the natural frequencies in the opposite of Type I FGCS and the natural frequencies are increased with increasing N. The increase in natural frequencies from $N=1$ to $N=15$ in about 2.3**%** at $n=1$ and about 2.4**%** at $n=10$. As can be seen, the influence of the constituent volume fractions is different for Type I and Type II FGCS. Tables 2-5 show that for $N>1$, the natural frequencies for Type II FGCS are higher than those for Type I FGCS. For example, for $N=15$ at $n=10$ and $h/R=0.002$, the natural frequency for Type II FGCS is about 4.67**%** higher than the other one. However, for $N=0.5$ at $n=10$ and $h/R=0.002$, the natural frequency for Type I FGCS is 1.66**%** higher than the other one.

Tables 6 and 7 show the variations of the fundamental natural frequencies (Hz) versus the thickness ratio h/R for Type I and Type II FGCSs. Note that, the numbers in the brackets indicate the circumferential wave numbers at which the fundamental frequencies occur. Note that, as N is increased the fundamental frequencies is decreased for Type I FGCS and increased for Type II FGCS. The difference in the fundamental frequencies between $N=1$ and $N=15$ is about 2.2**%** for Type I and Type II FGCSs. The fundamental natural frequencies for Type I and Type II FGCSs occur at the same circumferential wave numbers. For all values of N, the fundamental natural frequencies fell between the frequencies of the stainless steel and nickel.

5- Conclusion

The free vibration analysis of functionally graded cylindrical shells with simply supported boundary conditions is presented according to the first order shear deformation theory. The results show that, the constituent volume fractions and the configurations of the constituent materials affect the natural frequencies. The natural frequencies first decreased and than increased with the increasing of circumferential wave numbers. It is interesting to note that the value of the power-law exponent did not affect the value of the circumferential wave number at which the fundamental natural frequency might occur. Also, the natural frequencies of a short cylindrical shell are higher than those for a long shell, and the natural frequencies of a thick cylindrical shell are higher than those for a thin shell.

Table 1. Material properties.

Coefficients	Stainless steel			Nickel		
	E (Nm^{-2})	υ	ρ (Kgm^{-3})	E (Nm^{-2})	υ	ρ (Kgm^{-3})
P_0	201.04×10^9	0.3262	8166	223.95×10^9	0.3100	8900
P_{-1}	0	0	0	0	0	0
P_1	3.079×10^{-4}	-2.002×10^{-4}	0	-2.794×10^{-4}	0	0
P_2	-6.534×10^{-7}	3.797×10^{-7}	0	-3.998×10^{9}	0	0
P_3	0	0	0	0	0	0
	2.07788×10^{11}	0.317756	8166	2.05098×10^{11}	0.3100	8900

Table 2. Natural frequencies for Type I FGCS versus the circumferential wave number $(L/R = 20,\ h/R = 0.002)$.

n	Theory	Stainless steel	Nickel	N						
				0.5	0.7	1	2	5	15	30
1	FSDT	13.5478	12.8937	13.3209	13.268	13.2108	13.1029	12.9979	12.9328	12.9137
	Loy et al. [20]	13.548	12.894	13.321	13.269	13.211	13.103	12.998	12.933	12.914
2	FSDT	4.5919	4.3688	4.5166	4.4993	4.4800	4.4434	4.4066	4.3833	4.3764
	Loy et al. [20]	4.5920	4.3690	4.5168	4.4994	4.480	4.4435	4.4068	4.3834	4.3765
3	FSDT	4.2631	4.0487	4.1910	4.1747	4.1567	4.1233	4.0890	4.0651	4.0574
	Loy et al. [20]	4.2633	4.0489	4.1911	4.1749	4.1569	4.1235	4.0891	4.0653	4.0576
4	FSDT	7.2249	6.8574	7.0971	7.0688	7.0383	6.9818	6.9249	6.8853	6.8724
	Loy et al. [20]	7.2250	6.8577	7.0972	7.0691	7.0384	6.9820	6.9251	6.8856	6.8726
5	FSDT	11.5418	10.9547	11.3354	11.2900	11.2408	11.1509	11.0608	10.9987	10.9779
	Loy et al. [20]	11.542	10.955	11.336	11.290	11.241	11.151	11.061	10.999	10.978
6	FSDT	16.8964	16.0369	16.5938	16.5269	16.4548	16.3226	16.1918	16.1009	16.0709
	Loy et al. [20]	16.897	16.037	16.594	16.527	16.455	16.323	16.192	16.101	16.071
7	FSDT	23.2437	22.0608	22.8259	22.7347	22.6347	22.4538	22.2728	22.1479	22.1078
	Loy et al. [20]	23.244	22.061	22.826	22.735	22.635	22.454	22.273	22.148	22.108
8	FSDT	30.5728	29.0169	30.0228	29.9029	29.7708	29.5328	29.2959	29.1318	29.0777

	Loy et al. [20]	30.573	29.017	30.023	29.903	29.771	29.533	29.296	29.132	29.078
9	FSDT	38.880	36.9018	38.1808	38.0278	37.8619	37.5587	37.2568	37.0477	36.9808
	Loy et al. [20]	38.881	36.902	38.181	38.028	37.862	37.559	37.257	37.048	36.981
10	FSDT	48.1675	45.7158	47.3009	47.1110	46.9049	46.5288	46.1548	45.8968	45.8126
	Loy et al. [20]	48.168	45.716	47.301	47.111	46.905	46.529	46.155	45.897	45.813

Table 3. Natural frequencies for Type II FGCS versus the circumferential wave number $(L/R = 20,\ h/R = 0.002)$.

n	Theory	Stainless steel	Nickel	N						
				0.5	0.7	1	2	5	15	30
1	FSDT	13.5478	12.8937	13.1028	13.1538	13.2108	13.3208	13.4328	13.5049	13.5200
	Loy et al. [20]	13.548	12.894	13.103	13.154	13.211	13.321	13.433	13.505	13.526
2	FSDT	4.5919	4.3688	4.4381	4.4549	4.4741	4.5113	4.5503	4.5757	4.5834
	Loy et al. [20]	4.5920	4.3690	4.4382	4.4550	4.4742	4.5114	4.5504	4.5759	4.5836
3	FSDT	4.2631	4.0486	4.1150	4.1308	4.1485	4.1825	4.2190	4.2448	4.2535
	Loy et al. [20]	4.2633	4.0489	4.1152	4.1309	4.1486	4.1827	4.2191	4.2451	4.2536
4	FSDT	7.2249	6.8575	6.9753	7.0025	7.0328	7.0904	7.1509	7.1942	7.2084
	Loy et al. [20]	7.2250	6.8577	6.9754	7.0026	7.0330	7.0905	7.1510	7.1943	7.2085
5	FSDT	11.5418	10.9546	11.1449	11.1888	11.2378	11.3287	11.4248	11.4939	11.5158
	Loy et al. [20]	11.542	10.955	11.145	11.189	11.238	11.329	11.425	11.494	11.516
6	FSDT	16.8965	16.0369	16.3168	16.3808	16.4529	16.5869	16.7269	16.8267	16.8588
	Loy et al. [20]	16.897	16.037	16.317	16.381	16.453	16.587	16.727	16.827	16.859
7	FSDT	23.2437	22.0608	22.4468	22.5349	22.6327	22.4537	23.0108	23.1469	23.1918
	Loy et al. [20]	23.244	22.061	22.447	22.535	22.633	22.454	23.011	23.147	23.192
8	FSDT	30.5728	29.0169	29.5238	29.6408	29.7700	30.0139	30.2667	30.4459	30.5049
	Loy et al. [20]	30.573	29.017	29.524	29.641	29.770	30.014	30.267	30.446	30.505
9	FSDT	38.879	36.9018	37.5477	37.6956	37.8608	38.1708	38.4918	38.7200	38.7948
	Loy et al. [20]	38.881	36.902	37.548	37.696	37.861	38.171	38.492	38.720	38.795
10	FSDT	48.1675	45.7158	46.5169	46.7000	46.9037	47.2878	47.6858	47.9679	48.0608
	Loy et al. [20]	48.168	45.716	46.517	46.700	46.904	47.288	47.686	47.968	48.061

Table 4. Natural frequencies for Type I FGCS versus the length-to radius ratio $(h/R = 0.002)$.

$\frac{L}{R}$	Theory	Stainless steel	Nickel	N					
				0.5	0.7	1	2	5	15
0.2	FSDT	439.35(20)	417.53(20)	432.11(20)	430.44(20)	428.61(20)	425.15(20)	421.59(20)	419.16(20)
	Loy et al. [20]	439.36(20)	417.54(20)	432.12(20)	430.46(20)	428.62(20)	425.16(20)	421.60(20)	419.17(20)
0.5	FSDT	175.47(15)	166.74(15)	172.56(15)	171.92(15)	171.17(15)	169.80(15)	168.36(15)	167.40(15)
	Loy et al. [20]	175.49(15)	166.76(15)	172.59(15)	171.93(15)	171.19(15)	169.81(15)	168.38(15)	167.41(15)
1	FSDT	87.330(11)	82.992(11)	85.88(11)	85.560(11)	85.193(11)	84.504(11)	83.796(11)	83.315(11)
	Loy et al. [20]	87.331(11)	82.993(11)	85.890(11)	85.561(11)	85.195(11)	84.506(11)	83.798(11)	83.316(11)
2	FSDT	43.372(8)	41.216(8)	42.655(8)	42.492(8)	42.310(8)	41.968(8)	41.616(8)	41.377(8)
	Loy et al. [20]	43.373(8)	41.217(8)	42.656(8)	42.493(8)	42.311(8)	41.969(8)	41.618(8)	41.378(8)
5	FSDT	16.916(5)	16.077(5)	16.637(5)	16.574(5)	16.503(5)	16.370(5)	16.233(5)	16.140(5)
	Loy et al. [20]	16.917(5)	16.079(5)	16.639(5)	16.576(5)	16.505(5)	16.371(5)	16.234(5)	16.141(5)
10	FSDT	8.6033(4)	8.1721(4)	8.4590(4)	8.4263(4)	8.3903(4)	8.3227(4)	8.2532(4)	8.2051(4)
	Loy et al. [20]	8.6035(4)	8.1723(4)	8.4591(4)	8.4265(4)	8.3904(4)	8.3228(4)	8.2533(4)	8.2052(4)
20	FSDT	4.2632(3)	4.0487(3)	4.1910(3)	4.1746(3)	4.1567(3)	4.1233(3)	4.0891(3)	4.0652(3)
	Loy et al. [20]	4.2633(3)	4.0489(3)	4.1911(3)	4.1749(3)	4.1569(3)	4.1235(3)	4.0892(3)	4.0653(3)
50	FSDT	1.4917(2)	1.4166(2)	1.4664(2)	1.4606(2)	1.4543(2)	1.4426(2)	1.4307(2)	1.4223(2)
	Loy et al. [20]	1.4918(2)	1.4167(2)	1.4665(2)	1.4608(2)	1.4545(2)	1.4428(2)	1.4308(2)	1.4225(2)
100	FSDT	0.5593(1)	0.5323(1)	0.5501(1)	0.5479(1)	0.54560(1)	0.54113(1)	0.5368(1)	0.5340(1)
	Loy et al. [20]	0.5595(1)	0.5325(1)	0.5502(1)	0.5480(1)	0.54561(1)	0.54115(1)	0.5368(1)	0.5341(1)

Table 5. Natural frequencies for Type II FGCS versus the length-to radius ratio $(h/R = 0.002)$.

$\frac{L}{R}$	Theory	Stainless steel	Nickel	N					
				0.5	0.7	1	2	5	15
0.2	FSDT	439.35(20)	417.52(20)	424.19(20)	425.77(20)	427.61(20)	431.13(20)	434.91(20)	437.53(20)
	Loy et al. [20]	439.36(20)	417.54(20)	424.20(20)	425.80(20)	427.62(20)	431.15(20)	434.93(20)	437.57(20)
0.5	FSDT	175.47(15)	166.74(15)	169.42(15)	170.05(15)	170.77(15)	172.18(15)	173.70(15)	174.73(15)
	Loy et al. [20]	175.49(15)	166.76(15)	169.43(15)	170.06(15)	170.79(15)	172.20(15)	173.71(15)	174.76(15)
1	FSDT	87.330(11)	82.992(11)	84.315(11)	84.633(11)	84.993(11)	85.696(11)	86.446(11)	86.972(11)
	Loy et al. [20]	87.331(11)	82.993(11)	84.316(11)	84.634(11)	84.995(11)	85.697(11)	86.448(11)	86.974(11)
2	FSDT	43.372(8)	41.216(8)	41.873(8)	42.033(8)	42.211(8)	42.560(8)	42.932(8)	43.193(8)
	Loy et al. [20]	43.373(8)	41.217(8)	41.875(8)	42.033(8)	42.212(8)	42.561(8)	42.934(8)	43.195(8)
5	FSDT	16.916(5)	16.077(5)	16.333(5)	16.394(5)	16.464(5)	16.601(5)	16.746(5)	16.847(5)
	Loy et al. [20]	16.917(5)	16.079(5)	16.335(5)	16.396(5)	16.466(5)	16.602(5)	16.748(5)	16.849(5)
10	FSDT	8.6033(4)	8.1721(4)	8.3048(4)	8.3363(4)	8.3720(4)	8.4410(4)	8.5147(4)	8.5671(4)
	Loy et al. [20]	8.6035(4)	8.1723(4)	8.3050(4)	8.3365(4)	8.3722(4)	8.4411(4)	8.5148(4)	8.5672(4)
20	FSDT	4.2631(3)	4.0487(3)	4.1151(3)	4.1307(3)	4.1484(3)	4.1826(3)	4.2190(3)	4.2450(3)
	Loy et al. [20]	4.2633(3)	4.0489(3)	4.1152(3)	4.1309(3)	4.1486(3)	4.1827(3)	4.2191(3)	4.2451(3)
50	FSDT	1.4917(2)	1.4166(2)	1.4399(2)	1.4453(2)	1.4515(2)	1.4634(2)	1.4762(2)	1.4853(2)
	Loy et al. [20]	1.4918(2)	1.4167(2)	1.4400(2)	1.4455(2)	1.4517(2)	1.4636(2)	1.4763(2)	1.4854(2)
100	FSDT	0.5593(1)	0.5323(1)	0.5411(1)	0.54322(1)	0.5454(1)	0.5501(1)	0.5547(1)	0.5576(1)
	Loy et al. [20]	0.5595(1)	0.5325(1)	0.5412(1)	0.54324(1)	0.5456(1)	0.5502(1)	0.5548(1)	0.5578(1)

Table 6. Natural frequencies for Type I FGCS versus the thickness ratio $(L/R=20)$.

$\frac{h}{R}$	Theory	Stainless steel	Nickel	N					
				0.5	0.7	1	2	5	15
0.001	FSDT	2.7917(3)	2.6535(3)	2.7460(3)	2.7354(3)	2.7237(3)	2.7016(3)	2.6791(3)	2.6637(3)
	Loy et al. [20]	2.7919(3)	2.6537(3)	2.7461(3)	2.7356(3)	2.7239(3)	2.7018(3)	2.6792(3)	2.6639(3)
0.005	FSDT	5.4991(2)	5.2281(2)	5.4092(2)	5.3886(2)	5.3654(2)	5.3220(2)	5.2774(2)	5.2477(2)
	Loy et al. [20]	5.4992(2)	5.2283(2)	5.4094(2)	5.3887(2)	5.3656(2)	5.3221(2)	5.2776(2)	5.2478(2)
0.007	FSDT	6.379(2)	6.0630(2)	6.2743(2)	6.2505(2)	6.2237(2)	6.1734(2)	6.1217(2)	6.0865(2)
	Loy et al. [20]	6.380(2)	6.0631(2)	6.2746(2)	6.2506(2)	6.2239(2)	6.1736(2)	6.1219(2)	6.0867(2)
0.01	FSDT	7.9331(2)	7.5356(2)	7.8000(2)	7.7685(2)	7.7365(2)	7.6742(2)	7.6102(2)	7.5660(2)
	Loy et al. [20]	7.9333(2)	7.5358(2)	7.8001(2)	7.7700(2)	7.7367(2)	7.6744(2)	7.6104(2)	7.5661(2)
0.02	FSDT	13.550(1)	12.896(1)	13.323(1)	13.271(1)	13.213(1)	13.105(1)	13.000(1)	12.934(1)
	Loy et al. [20]	13.552(1)	12.898(1)	13.325(1)	13.273(1)	13.215(2)	13.107(2)	13.001(2)	12.936(2)
0.03	FSDT	13.555(1)	12.901(1)	13.328(1)	13.276(1)	13.218(1)	13.111(1)	13.004(1)	12.940(1)
	Loy et al. [20]	13.557(1)	12.902(1)	13.330(1)	13.278(1)	13.220(1)	13.112(1)	13.006(1)	12.941(1)
0.04	FSDT	13.561(1)	12.907(1)	13.334(1)	13.282(1)	13.225(1)	13.118(1)	13.012(1)	12.946(1)
	Loy et al. [20]	13.563(1)	12.909(1)	13.336(1)	13.284(1)	13.226(1)	13.119(1)	13.013(1)	12.948(1)
0.05	FSDT	13.570(1)	12.916(1)	13.343(1)	13.291(1)	13.232(1)	13.125(1)	13.020(1)	12.953(1)
	Loy et al. [20]	13.572(1)	12.917(1)	13.345(1)	13.293(1)	13.235(1)	13.127(1)	13.021(1)	12.956(1)

Table 7. Natural frequencies for Type II FGCS versus the thickness ratio $(L/R=20)$.

$\frac{h}{R}$	Theory	Stainless steel	Nickel	N					
				0.5	0.7	1	2	5	15
0.001	FSDT	2.7917(3)	2.6535(3)	2.6956(3)	2.7059(3)	2.7173(3)	2.738(3)	2.7638(3)	2.7806(3)
	Loy et al. [20]	2.7919(3)	2.6537(3)	2.6958(3)	2.7060(3)	2.7175(3)	2.740(3)	2.7640(3)	2.7807(3)
0.005	FSDT	5.4991(2)	5.2281(2)	5.3107(2)	5.3306(2)	5.3534(2)	5.3977(2)	5.4451(2)	5.4775(2)
	Loy et al. [20]	5.4992(2)	5.2283(2)	5.3109(2)	5.3308(2)	5.3536(2)	5.3979(2)	5.4452(2)	5.4777(2)
0.007	FSDT	6.378(2)	6.0630(2)	6.1596(2)	6.1828(2)	6.2092(2)	6.2604(2)	6.3153(2)	6.3537(2)
	Loy et al. [20]	6.380(2)	6.0631(2)	6.1598(2)	6.1830(2)	6.2094(2)	6.2606(2)	6.3155(2)	6.3539(2)
0.01	FSDT	7.9331(2)	7.5356(2)	7.6581(2)	7.6871(2)	7.7201(2)	7.7835(2)	7.8514(2)	7.8998(2)
	Loy et al. [20]	7.9333(2)	7.5358(2)	7.6583(2)	7.6873(2)	7.7202(2)	7.7837(2)	7.8516(2)	7.8999(2)
0.02	FSDT	13.550(1)	12.896(1)	13.106(1)	13.155(1)	13.213(1)	13.323(1)	13.436(1)	13.506(1)
	Loy et al. [20]	13.552(1)	12.898(1)	13.107(1)	13.157(1)	13.215(1)	13.325(1)	13.437(1)	13.508(1)
0.03	FSDT	13.554(1)	12.901(1)	13.111(1)	13.161(1)	13.218(1)	13.327(1)	13.441(1)	13.512(1)
	Loy et al. [20]	13.557(1)	12.902(1)	13.112(1)	13.162(1)	13.219(1)	13.329(1)	13.442(1)	13.513(1)
0.04	FSDT	13.561(1)	12.907(1)	13.116(1)	13.167(1)	13.225(1)	13.334(1)	13.446(1)	13.518(1)
	Loy et al. [20]	13.563(1)	12.909(1)	13.118(1)	13.169(1)	13.226(1)	13.336(1)	13.448(1)	13.520(1)
0.05	FSDT	13.570(1)	12.916(1)	13.125(1)	13.175(1)	13.232(1)	13.342(1)	13.456(1)	13.527(1)
	Loy et al. [20]	13.572(1)	12.917(1)	13.126(1)	13.177(1)	13.234(1)	13.344(1)	13.457(1)	13.528(1)

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Appendix A

The strains and curvatures in cylindrical coordinate can be expressed as

$$\in_{11}^{0}=\frac{\partial u_1}{\partial x}$$
$$\in_{22}^{0}=\frac{1}{a}\frac{\partial u_2}{\partial \theta}+\frac{u_3}{a}$$
$$\in_{12}^{0}=\frac{\partial u_2}{\partial x}+\frac{1}{a}\frac{\partial u_1}{\partial \theta}$$
$$k_{11}=\frac{\partial \phi_1}{\partial x}$$
$$k_{22}=\frac{1}{a}\frac{\partial \phi_2}{\partial \theta}$$
$$k_{12}=\frac{\partial \phi_2}{\partial x}+\frac{1}{a}\frac{\partial \phi_1}{\partial \theta}$$
$$\gamma_{13}^{0}=\phi_1+\frac{\partial u_3}{\partial x}$$
$$\gamma_{23}^{0}=\phi_2+\frac{1}{a}\frac{\partial u_3}{\partial \theta}$$

The stress resultants are expressed as

$$N_{11}=A_{11}\frac{\partial u_1}{\partial x}+B_{11}\frac{\partial \phi_1}{\partial x}+A_{12}\frac{1}{a}\frac{\partial u_2}{\partial \theta}+A_{12}\frac{1}{a}u_3+B_{12}\frac{1}{a}\frac{\partial \phi_2}{\partial \theta}$$
$$N_{22}=A_{12}\frac{\partial u_1}{\partial x}+B_{12}\frac{\partial \phi_1}{\partial x}+B_{22}\frac{1}{a}\frac{\partial \phi_2}{\partial \theta}++A_{22}\frac{1}{a}\frac{\partial u_2}{\partial \theta}+A_{22}\frac{1}{a}u_3$$
$$N_{12}=A_{66}\frac{\partial u_2}{\partial x}+A_{66}\frac{1}{a}\frac{\partial u_1}{\partial \theta}+B_{66}\frac{\partial \phi_2}{\partial x}+B_{66}\frac{1}{a}\frac{\partial \phi_1}{\partial \theta}$$
$$M_{11}=B_{11}\frac{\partial u_1}{\partial x}+B_{12}\frac{1}{a}\frac{\partial u_2}{\partial \theta}+B_{12}\frac{1}{a}u_3$$
$$M_{22}=B_{12}\frac{\partial u_1}{\partial x}+B_{22}\frac{1}{a}\frac{\partial u_2}{\partial \theta}+B_{22}\frac{1}{a}u_3$$
$$M_{12}=B_{66}\frac{\partial u_2}{\partial x}+B_{66}\frac{1}{a}\frac{\partial u_1}{\partial \theta}$$
$$Q_{13}=A_{55}\phi_1+A_{55}\frac{\partial u_3}{\partial x}$$
$$Q_{23}=A_{44}\phi_2+A_{44}\frac{1}{a}\frac{\partial u_3}{\partial \theta}$$

Appendix B

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} \end{bmatrix}$$

The coefficients of C_{ij}

$$C_{11}=-\frac{m^2\pi^2 a}{l^2}A_{11}-\frac{n^2}{a}A_{66}$$

$$C_{12}=\frac{mn\pi}{l}A_{12}+\frac{mn\pi}{l}A_{66}$$

$$C_{13}=\frac{m\pi}{l}A_{12}$$

$$C_{14}=-\frac{m^2\pi^2 a}{l^2}B_{11}-\frac{n^2}{a}B_{66}$$

$$C_{15}=\frac{mn\pi}{l}B_{12}+\frac{mn\pi}{l}B_{66}$$

$$C_{21}=\frac{mn\pi}{l}A_{12}$$

$$C_{22}=-\frac{n^2}{a}A_{22}-\frac{1}{a}A_{44}$$

$$C_{23}=-\frac{n}{a}A_{22}-\frac{n}{a}A_{44}$$

$$C_{24}=\frac{mn\pi}{l}B_{12}$$

$$C_{25}=-\frac{n^2}{a}B_{22}+A_{44}$$

$$C_{31}=-\frac{m\pi}{l}A_{22}$$

$$C_{33}=\frac{1}{a}A_{22}+\frac{m^2\pi^2 a}{l^2}A_{55}+\frac{n^2}{a}A_{44}$$

$$C_{34}=-\frac{m\pi}{l}B_{22}+\frac{m\pi a}{l}A_{55}$$

$$C_{35}=\frac{n}{a}B_{22}-nA_{44}$$

$$C_{41}=\frac{m^2\pi^2 a}{l^2}B_{11}+\frac{n^2}{a}B_{66}$$

$$C_{42}=-\frac{mn\pi}{l}B_{12}-\frac{mn\pi}{l}B_{66}$$

$$C_{43}=-\frac{m\pi}{l}B_{12}+\frac{m\pi a}{l}A_{55}$$

$$C_{44} = A_{55} a$$
$$C_{45} = 0$$
$$C_{51} = -\frac{mn\pi}{l} B_{12} - \frac{mn\pi}{l} B_{66}$$
$$C_{52} = \frac{n^2}{a} B_{22} + \frac{m^2 \pi^2 a}{l^2} B_{66}$$
$$C_{53} = -nA_{44}$$
$$C_{54} = 0$$
$$C_{55} = 0$$

The coefficients of M_{ij}

$$[M] = \begin{bmatrix} -I_0 & 0 & 0 & I_1 & 0 \\ 0 & -I_0 & 0 & 0 & -I_1 \\ 0 & 0 & -I_0 & 0 & 0 \\ 0 & 0 & -I_1 & -I_2 & 0 \\ 0 & -I_1 & 0 & 0 & -I_2 \end{bmatrix}$$