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Uncertain Entropy as a Risk Measure in Multi-Objective Portfolio Optimization

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ABSTRACT

As we are looking for knowledge of stock future returns in portfolio optimization, we are practically faced with two principal concepts: Uncertainty and Information about variables. This paper attempts to introduce a pragmatic bi-objective investment model based on uncertainty, instead of probability space and information theory, instead of variance and other moments as a risk measure for portfolio optimization. Not only is uncertainty space expected to be more in line with investment theory, but also, applying and learning this approach seems more straightforward and practical for novice investors. The proposed model simultaneously maximizes the uncertain mean of stock returns and minimizes uncertain entropy as a measure of portfolio risk. The uncertain zigzag distribution has been used for variables to avoid the complexity of fitting distributions for data. This uncertain mean-entropy portfolio optimization (UMEPO) has been solved by three meta-heuristic methods of multi-objective optimization: NSGA-II, MOPSO, and MOICA. Finally, it was observed that the optimal portfolio obtained from the proposed model has a higher return and a lower entropy as a risk measure compared to the same model in the probability space.

1 Introduction

All investors are interested in knowing how to manage their portfolios to maximize profits while maintaining the value of their assets and reducing risk. The reviewed and presented models are generally in the space of probability and require access and analysis of historical data. However, access and accuracy of the information, the validity of models based on historical data for predicting future stock behaviour, the complexity of calculations, and the hardship of extracting correct data distribution are among the problems of the proposed models that make these researches difficult for use public investors. The value of any asset is a function of its "expected return" and "risk", therefore the models that researchers used for estimate, are a function of these two elements. The first person to take this issue seriously was Markowitz [41][41] . His modern theory deals with two concepts of "expected return" and "risk" and used them to select the optimal portfolio. In the practical application of this theory, investors encountered problems and conducted research to address each of these problems. The following are some of the problems that make Markowitz's model difficult to use:

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(1) In this model, it is assumed that stock returns have a normal distribution, but it is usually observed that this assumption does not apply and in this case, using variance as a risk measure does not have the necessary performance;
(2) The complexity of calculating variance as a risk measure when the number of stocks is many;
(3) In many cases, we are dealing with stocks that are offered for the first time and we have no historical data about it, access to historical data is difficult or the data provided is not correct. The high sensitivity of the model result in the accuracy of the input data can lead to deviation in the results;
(4) It isn't easy to get the distribution of stocks and even if we estimate the distribution of data, there is a possibility that it does not correspond to the actual distribution of data.
The need to use a solution that can solve all or most of the problems at the same time seems necessary. Using the entropy instead of variance as a measure of risk, was founded as an appropriate solution for the existing problems. Firstly, McGill in [42][42] proposed it, and then it was developed by other researchers. Recently, in [44] introduced a multi-objective model that minimized "Shannon Entropy" and used the experimental probability generating function called return-entropy portfolio optimization (REPO) then showed that using entropy as a risk criterion could solve the existing problems in the Markowitz model. In the model presented in this research, the following achievements and innovations are considered goals:

- Using entropy instead of variance as a measure of risk,
- Minimizing risk when maximizing the assets return at the same time, as one of the main goals of the equation and not a constraint of the equation,
- Solving the problem in the uncertain environment instead of the probability environment because it is closer to the performance of the stock market,
- Considering asset return as an uncertain variable and not a random variable due to its greater correspondence with the behavior of this variable,
- The use of the uncertain zigzag distribution, because it is not only more consistent with the real behavior of asset returns, but its use is applicable to all investors who do not have specialized knowledge of mathematics and statistics and especially to novice investors,
- Solving this model as a multi-objective equation instead of simplifying and turning it into a single-objective equation,
- Solving the multi-objective equation through 3 meta-heuristic methods that have a high ability to solve complex multi-objective equations and optimal efficiency then comparison the results of its.

2 Practical Problems with prior research

This probability generating function, which is presented in [44] to find the probability of a portfolio with a continuous distribution, divides the data range into several intervals by inferring from the experimental probability distribution and calculates the frequency of data that is placed in each of these intervals. In other words, this method converts continuous data distribution into discrete distribution and uses the entropy discrete distribution formula, which itself can be one of the disadvantages of using this method. Although the expected value-entropy optimization model, removes the problems in the Markowitz model, solving this model in the probability space, especially with empirical probability generating functions, can have the following problems:
(1) The portfolio distribution of n assets is not necessarily equal to the sum of the distribution of each of them, and therefore we are faced with the difficulties of finding a portfolio distribution when there are many assets, especially when they are not independent and identically distributed. In some cases, it is even impossible to achieve portfolio distribution;

(2) Data segmentation is difficult, especially in cases where the amount of data is abundant. For example, if we have 10 stocks and we have collected their historical data for the last three years, it is difficult and longsome to divide this data for each stock;
(3) The quality of the data distribution depends on the correct data segmentation. Therefore, finding the appropriate number of intervals and the proper limits of each interval is very important and difficult too. Also, the interval selection determines the severity of the entropy bias.
(4) It is possible that even after finding the appropriate number of intervals, the frequency of data placed in each partition is the same or does not differ significantly. Therefore, in this case, the estimated density function has not the required quality and cannot well determine the data distribution;
(5) As mentioned above, if the obtained probability distributions of the data for different stocks be the same, the entropy of each of them will be the same and the possibility of using entropy as a risk measure will be taken away from us. Because entropy only works with probabilities and does not use the values of variables;
(6) The entropy of the sum of several variables is not equal to the sum of their entropy, and due to the complexity of calculating the entropy with the joint distribution of assets, the distribution of the sum of assets has been suggested in [44].
Hence finding a solution that can solve all or most of the problems that occurred when using entropy as a risk measure, at the same time, seems necessary. On the other hand, in portfolio optimization models, we seek to estimate the future behaviour of stocks to select the best portfolio based on it. Therefore, in most cases, we do not know the distribution of stock returns, even in cases where based on historical data we are finding the data distribution. We are not sure if the stocks will follow this distribution in the future or not. Especially in volatile stocks and markets such as crypto currencies and due to the numerous shocks to the economy such as the corona epidemic. In this case, the use of probability space no longer has the required efficiency and the uncertainty space can be the way forward. Therefore, a precise prediction of the probability of occurrences of each return does not work for us, nor can it be completely valid. We do not need to know the exact point-to-point distribution of stocks to predict future stock behaviour. It is enough to know the range of fluctuation for each stock. For example, knowing how much the minimum possible return, the maximum return, or 50% of the future return is less or more than a certain value, can be enough to make a decision. In order to deal with an indeterminate quantity (such as stock price), Uncertainty theory proposed a distribution function that determines the degree to which the quantity falls into the left side of the current point x. when we think it is entirely impossible that the stock returns fall into the left side of the current point, the distribution function takes a value of 0, when we are 20% sure that the stock returns fall into the right side and 80% sure that the stock returns fall into the left side, the distribution function takes a value of 0.8.
Our knowledge of cumulative distribution can be a good and sufficient guide for stock selection, and on the other hand, it is easy to understand for any ordinary person who does not know academic knowledge about technical stock analysis. Thus we don't need to know statistics. Therefore, in the space of uncertainty, using the concept of degree of belief, uncertainty distribution, and uncertain entropy, the problems that existed in the space of probability are removed. We no longer need to obtain the probability distribution of the data and access the historical data and analyze them, segmentation of the data. Also, if the probability density function of the data is the same, the uncertain entropy will not be the same, and the entropy of the sum of the two variables in the uncertainty space is equal to the sum of their entropy. This article focuses on the topic of "Risk Assessment" and uses the concepts of the "Uncertainty Theory" and "Information Theory" to reduce the problems that exist in the practical application of "Markowitz Theory". The combination and application of "Uncertainty Theory" and "Information

Theory" and the proximity of their concepts to the fact of investment, can have practical and effective dimensions in improving the process of analysis and optimal portfolio management. Therefore, the need to conduct research and provide a model that does not have the stated deficiencies seems to be necessary for the following reasons:

- Fixing the stated problems in the Markowitz model,
- Fixing the problems stated in the use of entropy as a measure of risk in previous models,
- The hardness of access to historical data and their analysis,
- Incompatibility of the behaviour of variables in the stock market with the space of probability and its coordination with the space of uncertainty,
- Problems of considering asset returns as random variables when in fact they are uncertain variables,
- Mathematical and statistical complexities in finding the appropriate distribution of variables and the uncertainty of the distribution fitted to the data,
- Non-practically and inapplicability of the previous models for the community of real investors especially for new investors due to their complexities,
- The low accuracy of the single-objective model in solving the stock portfolio optimization model.

It also uses uncertain multi-objective programming to solve its multi-objective equation. Article 3, (See [28][28]),investigates the effect of liquidity and the diversity and multiplicity of stocks in obtaining the optimal portfolio. It uses the entropy weight of each stock for the diversity of stocks in the portfolio and the turnover rate for liquidity. It considers the normal uncertain distribution for stock returns and the experimental uncertain distribution as the turnover rate distribution. Finally, it shows that the two conditions of liquidity and diversification affect the selection of the optimal portfolio. In article 4, (See [29]),like the Markowitz model, the variance is used as a measure of risk. In his equation, the entropy of stock weights used as a diversification index and used the value at risk (VaR) measure. Also, this article has considered a situation where we have a combination of random and uncertain variables. Article 5, (See [58]),introduces and defines Tsallis entropy in uncertainty space and puts this entropy as a condition of his single objective optimization model. In other words, instead of using Shannon entropy (logarithmic), it uses Tsallis entropy as the constraint of its model. It is necessary to explain that in this article, asset returns are considered as an uncertain set and not an uncertain variable, and in its numerical example, it uses the "uncertain triangular set" for the set of asset returns. Article 6 is done in the space of random variables and not in the environment of uncertainty (See [4][4]). It has used stable distributions as the margin of portfolio returns and different specifications and calibrations of parametric copula functions to investigate the dependence structure between assets. In the presented model, the main goal is to minimize the risk criterion, which uses the value at risk (VaR) measure and uses the maximization of the expected value of return on assets as a condition of the equation. The focus of this article is on presenting two multi-objective models based on the MOPSOs algorithm to solve the portfolio optimization model and the superiority of these presented models over the NSGAII and SPEA2 algorithms. Therefore, our article, due to the combination of " Uncertainty Theory" and "Information Theory", uses entropy as a measure of risk and as the main goal of the equation, solving the equation in the space of uncertainty, which in addition to the simplicity of its application for all investors, It has more similar to the real behavior of the portfolio and also, the solution of the model presented as a multi-objective equation using three methods of the best available methods in solving multi-objective equations is superior to the mentioned articles. 4 presented the concept of uncertainty theory, uncertainty distribution, uncertain expected value, and uncertain entropy.

Title	1-Mean-risk model for uncertain portfolio selection with background risk[62][62]	2- Cross-entropy Based Multi-objective Uncertain Portfolio Selection Problem[25][25]	3-A new mean-variance-entropy model for uncertain portfolio optimization with liquidity and diversification [28]	4-A new uncertain random portfolio optimization model for complex sys-	5-Uncertain Portfolio Optimization based on Tsallis Entropy of Uncertain Sets[58][58]	6-Multi-objective portfolio optimization considering the dependence structure of as-	Our Article :Uncertain Entropy as a Risk Measure in Multi-Objective Portfo-
Return (maximum)	Expected value of Portfolio with background risk E [ξ1 x 1 + ξ2 x 2 + ξ3 x 3 + . . . + ξn x n + rb]	Expected value of Portfolio	Expected value of Portfolio	Expected value of Portfolio	Expected value of Portfolio	-	Expected value of Portfolio
Risk (minimum)	-	Variance of Portfolio	Variance of Portfolio	Variance of Portfolio	-	VaR of Portfolio	Entropy of Portfolio
Constraints	R (x1, x2, . . . , xn, rb ; r) ≤ α (r)	Cross Entropy	Entropy of Proportion as Diversification and Turnove Rates as Liquidity	VaR as Downside Risks and Entropy Proportion as Diversification	Tsallis Entropy	Expected Value of Portfolio	-
Model	Linear Programming Problem	Uncertain Multi-Objective Programming (UMOP)	Single Objective Optimization Model	-	Single Objective Optimization Model	Multi-Objective Programming (UMO)	Multi-Objective Programming (UMO)
Space/variable	Uncertain	Uncertain	Uncertain	Uncertain and Random	Uncertain Set	Random	Uncertain
Distribution	Uncertain Zigzag& Uncertain Normal	Uncertain Zigzag	Uncertain Normal for asset returns Empirical Uncertainty Distribut on for Turnover Rates	-	Triangular Uncertain Set	Stable Distribution	Uncertain Zigzag
Method of Solutions	Simplex Method	NSGA-II and AbYSS	fmincon Algorithm	NSGA-II algorithm	Not Mentioned	MOPSO, NSGA-II, SPEA2	MOICA, MOPSO, NSGA-II

Section 5 introduces the bi-objective optimization model that in it use entropy as a risk measure and expected value as a return measure and then details the featured method of this paper: uncertain mean-entropy portfolio optimization (UMEPO). A numerical example using UMEPO is demonstrated in Section 6 and finally, in Section 7 conclusions are discussed.

3 Literature Review

Various authors worked on the Modern Portfolio Theory that was introduced by Markowitz in [41][41]. The following is a brief overview of previous studies in this domain. Researchers have proposed various solutions to the problems of the mean-variance model, the most well-known of which is the Black-Litterman model. (eeS[7][7], [8][8]). In these efforts, the theory of postmodern portfolio (PMPT) was formed that uses more statistical concepts such as higher moments, skewness, and kurtosis, in the literature. (See [51][51]). The Information Theory (eeS[54][54], [55][55]) was first proposed by Shannon, an American mathematician working in the field of electronic engineering. The first application of entropy as the main feature of information theory instead of variance goes back to the study of McGill (See [42][42]) and Garner (See [14]). Its expansion into the literature on stock optimization was articulated by Philippatus (See [48][48]), then research on this subject was promoted.
Rompolis in [52][52] reviewed the actual future risk density of stocks and other assets based on the maximum entropy rule. Lassance and Vrins in [26][26] showed that Rényi Entropy is a powerful alternative to the risk measure and can pave new paths in portfolio selection theory using higher moments. In [9] developed the Risk Parity model, a novel risk diversification approach to portfolio selection. In [1], Amini presented a model under non-normality assumption that used a higher moment and the entropy for diversification. Tahmasebi in [56] proposed a risk measurement under minimal entropy martingale measure. Jourshari in [49]introduces an optimal Robust mean-cvar model. In [44], Mercurio et al. by introducing the return-entropy portfolio optimization (REPO) method, using the mean-entropy function instead of the mean-variance function in the Markowitz optimization model (MVPO) and shows that in most cases it is better than the mean-variance model. In [4] Babaei et al. provided the application of stable distributions in the space of random variables for multi-objective portfolio optimization. There is a lot of research on portfolio optimization in fuzzy, randomness and fuzziness, and stochastic-fuzzy methodology for considering the fact that the financial market is an uncertain space (See [27][27],[63][62], [13], [50], [6], [19][19], [21][20], [17][17],[31][31], [57]). But, an actual situation, there are many cases where for describing uncertainty neither randomness nor fuzziness cannot be responsive. In 2007 "Uncertainty Theory" was founded by Liu in [33][33] that it rationally deals with personal belief degrees and was subsequently studied by him (See [35], [36][36]) and many researchers in finance and economics (See [16], [60][60], [61]).
Yan in [59][59], considered stock returns as uncertain variables in his mean-variance portfolio selection models. A portfolio selection problem under Liu's uncertainty theory framework; was first addressed by Huang in [23][23]. Subsequently, a mean-risk model for uncertain portfolio selection was introduced by Huang (See [20]). In [29][29] and [38][38], the value-at-risk criterion is applied in the uncertain portfolio. In [39] and [62][62], background risks are considered in addition to the risk measure, in the portfolio model
A new framework of the mean-entropy-skewness portfolio selection problem has been introduced by Bhattacharyya et al. in [5][5] that has used the transaction cost under constraints on the maximum and minimum allowable capital invested in stocks, short and long-term returns, the number of assets in the portfolio, and dividends. Mehralizade et al. in [43][43] investigated portfolio selection in the uncertainty environment based on the risk curve. Sajedi et al. in [53][53] analyzed the impact of Order υ Entropy and Cross Entropy in the portfolio optimization model. Ning et al. in [46][46] have considered a mean-variance portfolio selection problem with triangular entropy as a constraint under uncertainty theory.

Abtahi et al. in [3] used asymmetric entropy while introducing the Skew-Normal Uncertainty Distribution in the Portfolio optimization model.

Huang and Di in [24][24] developed an uncertain portfolio selection problem by considering the return and risk associated with background assets. Kar et al. in [25][25] proposed a mean-variance-cross entropy uncertain portfolio selection problem. Gao et al. in [15] presented the application of the partial similarity measure of uncertain random variables in the portfolio optimization model. Majumder et al. in [40] have presented a bi-objective mean-entropy portfolio selection problem under uncertainty space that used triangular entropy as a risk measure. Elliptic entropy, Partial Exponential Entropy, and Tsallis Entropy have been used in [10], [37][37], and [58] respectively instead of Shannon entropy. In [30][30] introduced an uncertain portfolio selection problem, the value–variance–entropy model where optimistic value, variance and entropy are used for measuring investment return, risk, and diversification, respectively. Li and Zhang in [28][28] presented the mean-variance-entropy model for an uncertain portfolio optimization problem have presented by taking into account four criteria viz., risk, return, liquidity, and diversification degree of a portfolio. For further review, a table of comparison of some previous research with our presented model is given.

Article 1, (See [62][62]),it examines the effect of background and risk history on stock portfolio optimization. It solves the presented model in two cases, considering the background and without considering the risk background. For a numerical example, he considers two symmetric normal distributions and asymmetric zigzag distribution and solves his equation in the form of linear programming using a simple method and shows in two cases that considering the risk background is effective in choosing the optimal portfolio. In Article 2, (See [25][25]),like the Markowitz model, variance is used as a measure of risk, the disadvantages of which are explained in the text of the article, while in our article, the existing problems are solved by using entropy as a measure of risk. In Article 2, the criterion of "mutual entropy" is used to examine the convergence between assets. It also uses uncertain multi-objective programming to solve its multi-objective equation. Article 3, (See [28][28]),investigates the effect of liquidity and the diversity and multiplicity of stocks in obtaining the optimal portfolio. It uses the entropy weight of each stock for the diversity of stocks in the portfolio and the turnover rate for liquidity. It considers the normal uncertain distribution for stock returns and the experimental uncertain distribution as the turnover rate distribution. Finally, it shows that the two conditions of liquidity and diversification affect the selection of the optimal portfolio. In article 4, (See [29]),like the Markowitz model, the variance is used as a measure of risk. In his equation, the entropy of stock weights used as a diversification index and used the value at risk (VaR) measure. Also, this article has considered a situation where we have a combination of random and uncertain variables.

Article 5, (See [58]),introduces and defines Tsallis entropy in uncertainty space and puts this entropy as a condition of his single objective optimization model. In other words, instead of using Shannon entropy (logarithmic), it uses Tsallis entropy as the constraint of its model. It is necessary to explain that in this article, asset returns are considered as an uncertain set and not an uncertain variable, and in its numerical example, it uses the "uncertain triangular set" for the set of asset returns. Article 6 is done in the space of random variables and not in the environment of uncertainty (See [4][4]). It has used stable distributions as the margin of portfolio returns and different specifications and calibrations of parametric copula functions to investigate the dependence structure between assets. In the presented model, the main goal is to minimize the risk criterion, which uses the value at risk (VaR) measure and uses the maximization of the expected value of return on assets as a condition of the equation. The focus of this article is on presenting two multi-objective models based on the MOPSOs algorithm to solve the portfolio optimization model and the superiority of these presented models over the NSGAII and

Method of Solutions	Distribution	Space/variable	Model	Constraints	Risk (minimum)	Return (maximum)	Title
Simplex Method	Uncertain Zig-zag& Uncertain Normal	Uncertain	Linear Programming Problem	**R** $(x_1, x_2, \ldots, x_n, r_b; r) \leq \alpha(r)$	-	Expected value of Portfolio with background risk **E** [$\xi_1 x_1 + \xi_2 x_2 +$	1-Mean-risk model for uncertain portfolio selection with background risk[62][62]
NSGA-II and AbYSS	Uncertain Zigzag	Uncertain	Uncertain Multi-Objective Programming (UMOP)	Cross Entropy	Variance of Portfolio	Expected value of Portfolio	2- Cross-entropy Based Multi-objective Uncertain Portfolio Selection Problem[25][25]
fmincon Algorithm	Uncertain Normal for asset returns Empirical Uncertainty Distribut	Uncertain	Single Objective Optimization Model	Entropy of Proportion as Diversification and Turnove Rates as Liquidity	Variance of Portfolio	Expected value of Portfolio	3-A new mean-variance-entropy model for uncertain portfolio optimization with liquidity and diversification [28[
NSGA-II algorithm	-	Uncertain and Random	-	VaR as Downside Risks and Entropy Proportion as Diversification	Variance of Portfolio	Expected value of Portfolio	4-A new uncertain random portfolio optimization model for complex systems with downside risks and diversification[29][29]
Not Mentioned	Triangular Uncertain Set	Uncertain Set	Single Objective Optimization Model	Tsallis Entropy	-	Expected value of Portfolio	5-Uncertain Portfolio Optimization based on Tsallis Entropy of Uncertain Sets[58][58]
MOPSO, NSGA-II, SPEA2	Stable Distribution	Random	Multi-Objective Programming (UMO)	Expected Value of Portfolio	VaR of Portfolio	-	6-Multi-objective portfolio optimization considering the dependence structure of asset [4][4]
MOICA, MOPSO, NSGA-II	**Uncertain Zigzag**	**Uncertain**	**Multi-Objective Programming (UMO)**	**-**	**Entropy of Portfolio**	**Expected value of Portfolio**	**Our Article :Uncertain Entropy as a Risk Measure in Multi-Objective Portfolio Optimization**

SPEA2 algorithms. It also uses uncertain multi-objective programming to solve its multi-objective equation. Article 3, (See [28][28]),investigates the effect of liquidity and the diversity and multiplicity of stocks in obtaining the optimal portfolio. It uses the entropy weight of each stock for the diversity of stocks in the portfolio and the turnover rate for liquidity. It considers the normal uncertain distribution for stock returns and the experimental uncertain distribution as the turnover rate distribution. Finally, it shows that the two conditions of liquidity and diversification affect the selection of the optimal portfolio. In article 4, (See [29]),like the Markowitz model, the variance is used as a measure of risk. In his equation, the entropy of stock weights used as a diversification index and used the value at risk (VaR) measure. Also, this article has considered a situation where we have a combination of random and uncertain variables.

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4 Uncertain Theory

In [33][33], Liu introduced the uncertainty theory. Let that the triplet (Γ, $\mathcal{L}$, $\mathcal{M}$) is uncertainty space, that in this space, Γ be a nonempty set, $\mathcal{L}$ be an σ-algebra over Γ and $\mathcal{M}$ be an uncertain measure. $\mathcal{M}$ will be assigned each event Λ in $\mathcal{L}$ to the belief degree (not frequency) of an uncertain event that may happen. Liu, in [33], satisfies the following four axioms In order to define $\mathcal{M}\{\bullet\}$.

Axiom 1. (Normality) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2. (Self -Duality) For any event Λ, $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$.

Axiom 3. (Subadditivity) for every countable sequence of Λ_1, Λ_2, …, we have

$$\mathcal{M}\{\cup_{i=1}^{\infty} \Lambda_i\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}. \tag{1}$$

Axiom 4. (Product Axiom) Let (Γ_k , $\mathcal{L}_k$, $\mathcal{M}_k$) be uncertainty spaces for k =1, 2, … The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} \le \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}, \tag{2}$$

Where Λ_k are arbitrarily chosen events from$\mathcal{L}_k$ for k = 1, 2, …, respectively.

4.1 Uncertainty Distribution

Definition 1(See[33][33]): The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \le x\}, \tag{3}$$

for any real number x.

Theorem 1(Peng-Iwamura Theorem (See[47][47]): A function Φ(x): $\mathcal{R}\to$ (0, 1) is an uncertainty distribution if and only if it is a monotone increasing function except Φ(x) ≡0 and Φ(x) ≡1.

Definition 2 (See[34][34]): An uncertain variable ξ is called empirical if it has an empirical uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & if\ x \le x_1 \\ \alpha_i + \dfrac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i}, & if\ x_i \le x \le x_{i+1}, 1 \le i \le n \\ 1, & if\ x \ge x_n \end{cases} \tag{4}$$

where $x_1 < x_2 < \cdots < x_n$ *and* $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_n \le 1$.

Definition 3 (See[34][34]): An uncertain variable ξ is called zigzag if it has a zigzag uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & if\ x \le a \\ \frac{x-a}{2(b-a)}, & if\ a \le x \le b \\ \frac{x+c-2b}{2(c-b)}, & if\ b \le x \le c \\ 1, & if\ x \ge c \end{cases}, \tag{5}$$

denoted by Ƶ(a; b; c) where a, b, c are real numbers with a < b < c.(Fig. 1). Should be noted that "zigzag uncertainty distribution "is a special case of empirical uncertainty distribution where $x_1 = a, x_2 = b, x_3 = c, n = 3, \alpha_1 = 0, \alpha_2 = \frac{1}{2}, \alpha_3 = 1$.

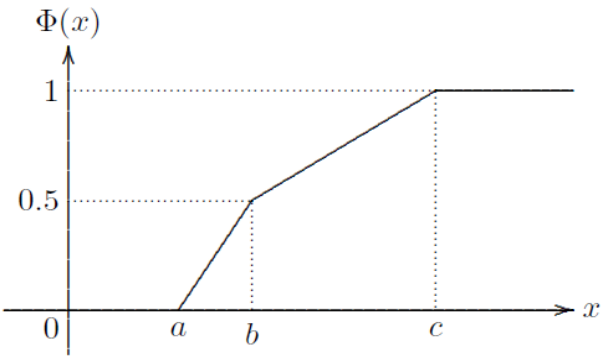


Fig. 1: Zigzag Uncertainty Distribution

Theorem 2 (See[35][35]): Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function f $(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$ then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable with inverse uncertainty distribution

$$\psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right). \quad (6)$$

***Theorem 3 (See [34][34])*:** Assume that ξ_1 and ξ_2 are independent zigzag uncertain variables Ƶ (a_1, b_1, c_1) and Ƶ (a_2; b_2; c_2), respectively. Then the sum Ƶ$_1$+ Ƶ$_2$ is also a zigzag uncertain variable Ƶ ($a_1 + a_2$, $b_1 + b_2$, $c_1 + c_2$), i.e.,

$$Ƶ(a_1, b_1, c_1) + Ƶ(a_2; b_2; c_2) = Ƶ(a_1 + a_2, b_1 + b_2, c_1 + c_2), \quad (7)$$

the multiplication of a zigzag uncertain variable Ƶ (a, b, c) and a scalar number $k > 0$ is also a zigzag uncertain variable Ƶ (ka, kb, kc), i.e.,

$$k \cdot Ƶ(a, b, c) = Ƶ(ka, kb, kc). \quad (8)$$

4.2 Uncertain Expected Value

Theorem 4 (See [33]: Let ξ be an uncertain variable with uncertainty distribution Φ then

$$E[\xi] = \int_0^{+\infty}\left(1 - \Phi(x)\right)dx - \int_{-\infty}^{0} \Phi(x)dx. \quad (9)$$

Theorem 5 (See[35]): Let ξ and η be independent uncertain variables with finite expected values. Then for any real numbers a and b, we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \quad (10)$$

4.3 Uncertain Entropy

Definition 4 (See[32][32]): Suppose that ξ is an uncertain variable with uncertainty distribution Φ. Then its entropy is defined by

$$H[\xi] = \int_{-\infty}^{+\infty} S\left(\Phi(x)\right)dx, \quad (11)$$

where $S(t) = -t\ln t - (1-t)\ln(1-t)$.

Theorem 6 (See[11][11]): Let ξ and η be independent uncertain variables. Then for any real numbers a and b, we have

$$H[a\xi + b\eta] = |a|H[\xi] + |b|H[\eta]. \quad (12)$$

5 Bi-Objective Optimization Model

Suppose that, there are n possible assets to select in a portfolio. Define variable r_i , i=1,2,..,n as return of asset i. Obviously, we do not know the amount of assets return, and as explained, r_i is an uncertain variable that has an uncertain distribution Φ_i. Therefore R= (r_1, r_2, …, r_n) is a vector of return of assets that can select in a portfolio with an uncertain distribution Φ_p =(Φ_1, Φ_2,…, Φ_n).

Let W= (w_1, w_2, …, w_n) a vector of the weight of assets. These weights can determine which possible assets include in the final portfolio and what is the portion of each of them and $w_i \geq 0, i = 1,2, \ldots, n, \sum_{i=1}^{n} w_i = 1$.

$$\text{Return of portfolio} = R_p = W^t R = \sum_{i=1}^{n} w_i \cdot r_i. \quad (13)$$

Now we proposed a bi-objective model for optimizing the portfolio. Our aim is that our selected portfolio has a maximum return and also has a minimum risk. Thus we use the expected value and the entropy respectively as a measure of return and risk of a portfolio.

$$\begin{cases} Min\ H(R_P) \\ Max\ E(R_P) \\ s.t \\ \sum_{i=1}^{n} w_i = 1 \\ w_i \geq 0, i = 1,2,\dots,n \end{cases} \tag{14}$$

if let $R_p = \sum_{i=1}^{n} w_i \,.\, r_i$ as shown in equation (13) and applying it in des equations (10) and (12), since our coefficients $w_i \geq 0, i = 1,2,\dots,n$, we have

$$H(R_P) = \sum_{i=1}^{n} w_i\, H(r_i), \tag{15}$$

$$E(R_P) = \sum_{i=1}^{n} w_i\, E(r_i). \tag{16}$$

and also by applying equation (11) in equation (15), our bi-objective model (Equation (14)) can be shown as

$$\begin{cases} Min\ H(R_P) = \sum_{i=1}^{n} w_i\, H(r_i) = \sum_{i=1}^{n} w_i \int_{-\infty}^{+\infty} S(\Phi_i)dr \\ Max\ E(R_P) = \sum_{i=1}^{n} w_i\, E(r_i) \\ s.t \\ \sum_{i=1}^{n} w_i = 1 \\ w_i \geq 0, i = 1,2,\dots,n \end{cases} \tag{17}$$

as the same way, if we use definition ***4*** in $H(r_i)$ and Theorem ***4*** in $E(R_P)$, our bi-objective model (Equation (14)) can be rewritten as follows:

$$\begin{cases} Min\quad H(R_P) = \sum_{i=1}^{n} w_i \int_{-\infty}^{+\infty} -\Phi_i \ln(\Phi_i) - (1-\Phi_i)\ln(1-\Phi_i)\, dr \\ Max\quad E(R_P) = \sum_{i=1}^{n} w_i \int_{0}^{+\infty} \big(1-\Phi_i(r)\big)dr - \int_{-\infty}^{0} \Phi_i(r)dr \\ s.t \\ \sum_{i=1}^{n} w_i = 1 \\ w_i \geq 0, i = 1,2,\dots,n \end{cases} \tag{18}$$

finally, by replacing distributions of each assets as Φ_i, we can solve this bi-objective model, simply.

6 A Bi-Objective Optimization Model Example

6.1 Numerical Example in Uncertainty Space

In this section, we try our proposed portfolio selection model, presented in Equation (14), with a numerical example. As we explained before, in uncertain theory we don't need historical data and its analysis and just enough to use the belief of experts and determine our distribution of data. In uncertainty space, we use the belief degree evaluated by experts to determine the distribution of data. For this purpose, we selected the securities of 5 companies and demand the experts explain their beliefs about distributions of these 5 selected stocks after March 2021.

Many distributions are defined in the uncertainty space, and each of them has its own characteristics. The simplest distribution that can be applied to stock data is the empirical uncertainty distribution. Due to the proximity of this distribution to the probability generating function presented in Mercurio's research (See [44][44]), this distribution has been used. On the other hand, because we are interested to avoid the complexities of probability distribution and easy access to the distribution of data for all people, especially novice investors, the proposed distribution function is an "Uncertain Zigzag Distribution" function. The "Zigzag Uncertainty Distribution" is a special case of the empirical uncertainty distribution. To find this distribution function, it is sufficient to express the minimum and maximum returns that we believe a stock can take. Then we determine the measure that experts believe 50% of the stock returns will be above this amount. People may not know how data is distributed, but it is possible for each person to specify these three numbers (belief degree).

The distribution of the stocks based on the opinions of experts is given in Table **1**. In other words, when we let GOOG ~Ƶ (-0.056, 0.003, 0.087), it means that according to experts' belief, it is completely impossible that the quantity of GOOG returns be less than -0.056, they are 50% sure that the quantity of GOOG returns be less than 0.003 and 50% sure that the quantity of GOOG returns to be greater than it, then it is completely impossible that the quantity of GOOG returns to be greater than 0.087.

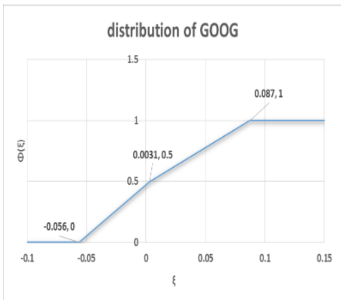
Table 1: Distribution of Securities

Securities name	Uncertain return rate ξ_i
GOOG	$Ƶ\ (-0.056, 0.003, 0.087)$
NFLX	$Ƶ\ (-0.086, 0.001, 0.169)$
VIAC	$Ƶ\ (-0.273, 0.009, 0.149)$
CALX	$Ƶ\ (-0.117, 0.002, 0.286)$
TSLA	$Ƶ\ (-0.211, 0.005, 0.196)$

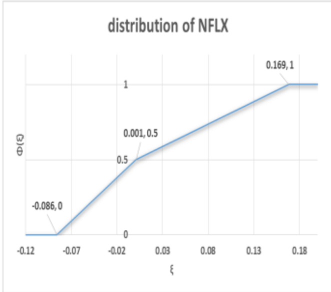
Table 2: The five randomly selected securities and their uncertain expected value and uncertain entropy.

Securities name	Uncertain Expected value	Uncertain Entropy
GOOG	0.0093	0.072
NFLX	0.0211	0.127
VIAC	-0.0268	0.211
CALX	0.0435	0.202
TSLA	-0.0008	0.203

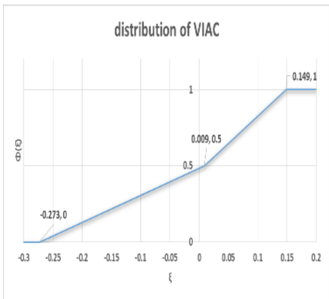
Scanning of the distribution function of stock shows that Google and Netflix have a larger minimum value than the others. On the other hand, Caltex and Tesla have a larger maximum value. The graph of the distribution function of these stocks is shown below (Figure **2**). The performance of the stocks seems close and makes the decision, difficult. Which of these stocks can make a portfolio with the highest returns and the lowest risk? This is the question that our model wants to address.



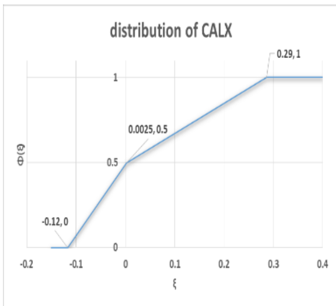
(**a**)



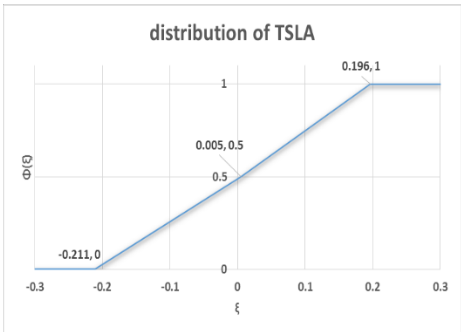
(**b**)



(c)



(d)



(**e**)

Fig. 2: Zigzag Uncertainty Distribution of (a) GOOG (b) NLFX (c) VIAC (d) CALX (e) TSLA base on belief degree of experts.is a figure.

When the conditions and constraints of the real world are considered, the optimal problem of creating a portfolio is not easily solved through mathematical methods. For this reason, the use of innovative methods has been one of the most important topics discussed in recent times. The researches indicate that these three strong meta-heuristic methods, Non-Dominated Sorting Genetic Algorithm-II (NSGA-II) (See [12][12]), Multi-Objective Particle Swarm Optimization (MOPSO) (See [45][45]) and Multi-Objective Imperialist Competitive Algorithm (MOICA) (See [2][3]) show significantly higher efficiency than other methods. All of these methods are reliable and popular for model optimization and they use the iteration strategy of the solution to obtain the optimal solution. NSGA-II is one of the multi-objective genetic algorithms that is well known for fast sorting and elitism. This technique can simultaneously optimizes each objective without being dominated by any other solution. When faced with complex optimization problems, a multi-objective particle swarm (MOPSO) helps us access premature convergence. Cause of the popularity of this method is its simplicity, low computation cost, and high efficiency in complex. MOICA is well-known for its great performance in computational time and for maintaining a diverse population of solutions. The proposed model in Equations (14) is implemented with these three methods. The following results are programmed in Matlab R2015b.The results are given in Table **3**.

Table 3: Results of solving uncertain mean-entropy portfolio optimization, the optimal weight of stocks, uncertain expected value, and uncertain entropy by three meta-heuristic methods.

Method	Optimal weight of stocks	Optimal Uncertain Expected value	Optimal Uncertain Entropy
MOICA	(0.36 , 0.25 , 0.10, 0.079 , 0.20)	0.00918	0.137
MOPSO	(0.39, 0.34, 0 , 0.073, 0.198)	0.0137	0.126
NSGA-II	(1, 0 , 0, 0 ,0)	0.0093	0.072

Examination of the weights obtained from each method shows that the lowest entropy levels belong to the NSGA-II, MOPSO, and MOICA respectively. However, the NSGA-II method puts only one stock in its portfolio, which according to the opinions of experts in this field, selecting only one stock in the portfolio increase concentration risk, and the diversity of the portfolio has always been desirable for investors. Also, the highest expected value of portfolio returns belongs to the MOPSO method, NSGA-II, and MOICA, respectively. As mentioned, portfolio diversity is one of the criteria for evaluating the quality of a portfolio. Of course, this does not necessarily mean that all stocks should be included in the portfolio, but the goal is to have a reasonable diversity that makes it easier to achieve our goals. From this point of view, the MOICA method, MOPSO, and NSGA-II have the highest number of shares, respectively.

Regardless of the mathematical methods and models that are offered academically for stock portfolio optimization, choosing of stock portfolio primarily depends on the level of investor risk appetite. On the other hand, it is the amount of investment risk appetite that determines which of the following methods and combinations of stocks can be the choice of each person. As you know the fact of investment is associated with risk. Risk is an integral part of investment and cannot be avoided and only, can be managed. It is clear that the relationship between return and risk, two concepts considered by investors, is direct and as the rate of return increases, the amount of risk also increases. So you cannot achieve a high return until you accept the high risk. But the fundamental question is how much risk can be accepted to achieve the return?

Some prioritize choosing the highest return portfolio, some the lowest risk portfolio, others a diverse portfolio of all available stocks, and some combinations of these. Therefore, the choice between these three proposed portfolios in Table **3**, can be different according to the risk appetite of individuals. But it seems that the portfolio presented by the NSGA-II, despite having the lowest entropy, is not suitable due to having a single stock. Between the two others presented portfolios, because the MOPSO has less entropy and a higher expected value than the MOICA, it seems that it can be a more suitable portfolio. In fact, since portfolio diversity should be commensurate with lower risk and higher returns, and the MOPSO has only one stock less than the MOICA, the results of the MOPSO method can be offered as the proposed model. As mentioned earlier, this selection may not be the same for all people with different risk appetites. The available options provided by the MOPSO model are shown in Figure 2. Each person can choose a combination of weights based on their risk appetite from the numbers shown in Figure **3**.

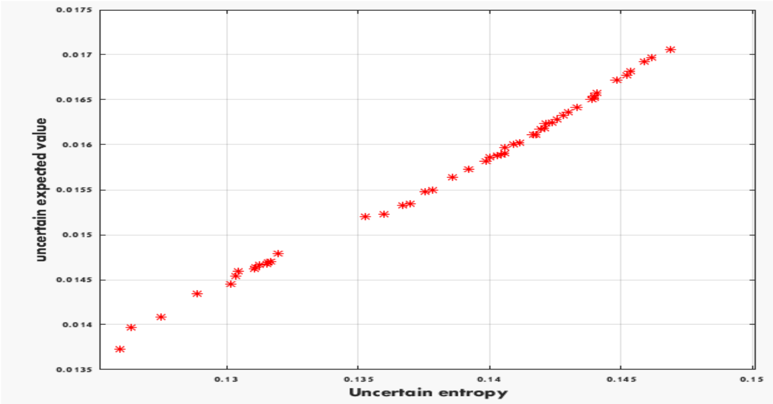


Fig. 3: Solutions of the uncertain mean-entropy portfolio optimization for the MOPSO method

6.2 Numerical Example in Probability Space

Now, to examine the efficiency of the uncertainty space compared to the probability space and also to show the performance of the space of uncertainty, we will solve the proposed model in the probability space once again. For this purpose, instead of considering stock return as an uncertain variable and obtaining the uncertain distribution function and their uncertain expected value and uncertain entropy, we gathered daily stock price information for one year from March 30, 2020, to March 29, 2021, for our 5 selected securities. Then, to solve the problem in the probability space, we need to know the probability distribution of the portfolio return to calculate the entropy. For this purpose, we use the empirical probability generating function provided by Mercurio et al. (See [44]). Then run the three multi-objective optimization methods, NSGA-II, MOPSO, and MOICA on it. The results of this study are given in Table **4**.

Table 4: Results of solving return-entropy portfolio optimization, the optimal weight of stocks, expected value, and entropy by three meta-heuristic methods.

Method	Optimal weight of stocks	Optimal Expected value	Optimal Entropy
MOICA	(0.34, 0.021, 0.029, 0.20 , 0.40)	0.0059	1.6094
MOPSO	(0.048, 0, 0.32 , 0.38, 0.24)	0.0068	1.6094
NSGA-II	(0.22, 0.082 , 0.12, 0.28 , 0.29)	0.0059	1.6094

It is necessary to explain that due to the dependence of the empirical distribution on the segmentation of the data, we divided the range of data into 5 parts in all stocks, and because there is the same frequency in these 5 intervals, the entropy of these stocks is equal to each other. As mentioned before, this is one of the problems in the probability space. As can be seen, the optimal portfolio presented in each of these 3 methods in probability space has a lower return and higher entropy compared to the same method in the uncertainty space. In other words, it can be concluded that the assumption of the uncertain distribution of stock returns instead of obtaining an experimental data distribution improves model performance. Therefore, it can be deduced that as the uncertainty theory has been said, when the estimated probability distribution was not close enough to the cumulative frequency, the use of uncertainty space performs better.

7 Conclusions

In this article, we present a model for portfolio optimization in uncertainty space. The goals of this bi-objective model are to maximize the uncertain expected value of return and minimize uncertain entropy

as the measure of return and risk, respectively. Since in many cases there is no access to historical data, hardness to obtain an accurate distribution of stock returns, fitted distribution cannot well describe data, the future behavior of stocks is uncertain and not determined by prior behavior, complexity of calculation joint distribution and entropy of portfolio with many stocks, we proposed one model base on Uncertainty Theory that solve these problems. Five securities are supposed for the proposed portfolio selection model. Returns of these securities are considered as uncertain variables with Zigzag Uncertainty Distribution. The proposed bi-objective equation is solved by three strong multi-objective meta-heuristic techniques: NSGA-II, MOPSO, and MOICA. The numerical example illustrated that a portfolio constructed based on the MOPSO method has lower entropy and higher returns. This model is also solved in the probability space, which is observed that solving the proposed model in the uncertainty space has less entropy and higher returns than the model solved in the probability space.
Consequently, the results of this research show that solving the stock portfolio optimization model in the uncertainty environment and considering the stock return as an uncertain variable is more consistent with its actual behavior than solving this model in the probability space and considering the asset return as a random variable. Also, replacing the entropy of asset return as a risk measure solves many problems in the models that use others risk measures such as variance. On the other hand, maximizing the return of the portfolio and minimizing the entropy of the portfolio at the same time and solving the model as a multi-objective model helps us in achieving the optimal portfolio. The results of the numerical example showed that among the 3 meta-heuristic methods presented, the MOPSO method provides a diverse portfolio with higher efficiency and less risk than other methods. The uncertain mean-entropy portfolio optimization model (UMEPO) can be improved with constraints that depend on the risk appetite of investors. Constraints such as the minimum acceptable risk, the minimum expected return, etc. This model can also be used in a situation where there are random variables and uncertain variable at the same time.

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