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Vibration of functionally graded cylindrical shells based on higher order shear deformation theory

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Abstract

In this research, free vibration analysis of cylindrical shells compose of functionally graded material (FGM) is considered. The governing equations of a FG cylindrical shells are derived based on the higher order shear deformation theory. Assuming that the material properties vary as a power form of the thickness coordinate variable z and using the Hamilton's principle, the system of fundamental partial differential equations of motion is established. The objective is to study the natural frequencies, the influence of constituent volume fractions and the effects of configurations of the constituent materials on the frequencies. The results show that the frequencies are affected by the constituent volume fractions and the configurations of the constituent materials. The present analysis is validated by comparing results with those available in the literature.

Keywords: Vibration; Functionally graded material; cylindrical shells; Higher order

1. Introduction

In recent years functionally graded materials (FGMs) have gained considerable importance as materials to be used in extremely high temperature environments such as nuclear reactors and high-speed spacecraft industries (Yamanouchi et al., 1990). FGMs were first introduced by a group of scientists in Sendai Japan in 1984 (Koizumi, 1997). FGMs are new inhomogeneous materials, in which the mechanical properties vary smoothly and continuously from one surface to the other. This is achieved by gradually varying the volume fraction of the constituent materials. This continuous change in composition results in the graded properties of FGMs (Reddy and Cheng, 2001). This gradation in properties of the material reduces thermal stresses, residual stresses and stress concentration factors (Reddy et al., 1999). Typically these materials are made from a mixture of ceramic and metal or from a combination of different materials. The ceramic constituent of the material provides the high-temperature resistance due to its low thermal conductivity. The ductile metal constituent on the other hand, prevents fracture caused by stresses due to the high temperature gradient in a very short period of time. Furthermore a mixture of ceramic and metal with a continuously varying volume fraction can be easily manufactured (Fukui, 1991). This eliminates interface problems of composite materials, and thus, the stress distributions are smooth. Studies on FGMs have been extensive but are largely confined to analysis of thermal stresses and deformations. Tanigawa et al. (1997) derived a one dimensional temperature solution for a nonhomogeneous plate in the transient state and optimized the material compositions by introducing a laminated composite model. The optimal composition profile problems of the FGM to decrease the thermal stresses and thermal stress intensity factor were discussed by Noda (Noda and Fuchiyama, 1995; Noda, 1999). He concluded that when the continuously changing composition

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between ceramics and metals can be selected pertinently, thermal stresses in the FGM are drastically decreased. Javaheri and Eslami (2002) presented the thermal buckling of rectangular FGM plate based on the high order plate theories. The buckling analysis of circular FGM plates is given by Najafizadeh and Eslami (2002a, 2002b).

Studies on vibration of cylindrical shells are extensive. Many of these studies are for isotropic and composite shells. Among those who have carried out studies on the vibration of cylindrical shells include (Arnold and Warburton ,1949), (Ludwig and Krieg, 1981), (Chung, 1981), (Soedel, 1980), (Forsberg, 1964), (Bhimaraddi, 1984), (Soldatos and Hajigeoriou, 1990), and (Loy and Lam, 1997).

Studies on vibration of cylindrical shells made of FGMs are limited. Loy et al. (1999) presented Rayleigh-Ritz solutions for free vibration of simply supported FGM cylindrical shells. Pradhan et al. (2000) discussed the effects of boundary Conditions and volume fractions on the natural frequencies of the FGM cylindrical shells.

In the present work, vibration of functionally graded cylindrical shells based on the third order shear deformation theory are studied. The objective is to study the frequency characteristics, the influence of the constituent volume fractions, and the affects of the configurations of the constituent materials on the natural frequencies. To validate the analysis, results for simply supported cylindrical shells are compared with Loy et al. The comparisons show that the present results agreed well with those in the literature.

2. Functionally gradient materials

Functionally gradient materials (FGMs) are obtained by combining two or more materials. Most of the functionally gradient materials are employed in high-temperature environments and many of the constituent materials may possess temperature-dependent properties. The material properties P can be expressed as a function of temperature as (Touloukian, 1967)

$$P = P_0(P_{-1}T^{-1} + 1 + P_1T + P_2T^2 + P_3T^3) \tag{1}$$

where P_0, P_{-1}, P_1, P_2 and P_3 are the coefficients of temperature T(K) expressed in Kelvin and are unique to the constituent materials. The material properties P of FGMs are function of the material properties and volume fraction of the constituent materials and are expressed as

$$P = \sum_{j=1}^{K} P_j V_{f_j} \tag{2}$$

where P_j and V_{f_j} are, respectively, the material property and volume fraction of the constituent material j. The volume fractions of all the constituent materials should add up to one.

$$\sum_{j=1}^{K} V_{f_j} = 1 \tag{3}$$

For a cylindrical shell with a uniform thickness h and a reference surface at its middle surface, the volume fraction can be written as

$$V_f = (\frac{z+\frac{h}{2}}{h})^N, \qquad N \geq 0, N = \inf inity \tag{4}$$

where N is the power-law index that takes value greater than or equal to zero. For a functionally gradient material with two constituent materials, the young's modulus E, Poisson ratio υ and the mass density ρ can be expressed as

$$E = (E_1 - E_2)(\frac{2z+h}{2h})^N + E_2 \tag{5}$$

$$\upsilon = (\upsilon_1 - \upsilon_2)(\frac{2z+h}{2h})^N + \upsilon_2 \tag{6}$$

$$\rho = (\rho_1 - \rho_2)(\frac{2z+h}{2h})^N + \rho_2 \tag{7}$$

From these equations, it is interesting to note that at $z=\frac{h}{2}$, FGM material properties are same as those of material 2. While at $z=\frac{h}{2}$, FGM material properties are the same as those of material 1. Thus, the material properties vary continuously from material 2 at the inner surface of the cylindrical shell to material 1 at the outer surface of the cylindrical shell. Fig. 1 shows a geometric definition of the shell.

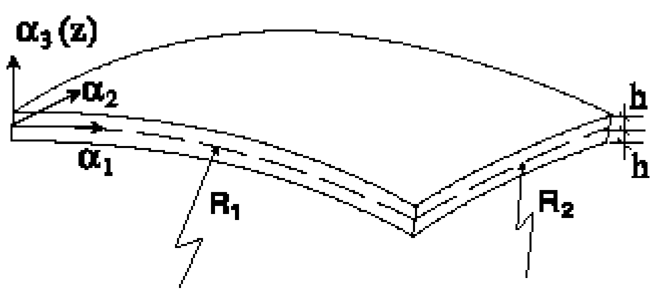

Fig. 1. Configuration of a generic shell.

3. Strains - displacement relationships

The strain-displacement relationships for a thin shell are (Soedel, 1981)

$$\in_{11}=\frac{1}{A_1(1+\frac{\alpha_3}{R_1})}\left[\frac{\partial U_1}{\partial\alpha_1}+\frac{U_2}{A_2}\frac{\partial A_1}{\partial\alpha_2}+U_3\frac{A_1}{R_1}\right] \tag{8}$$

$$\in_{22}\ \frac{1}{A_2(1+\frac{\alpha_3}{R_2})}\left[\frac{\partial U_2}{\partial\alpha_2}+\frac{U_1}{A_1}\frac{\partial A_2}{\partial\alpha_1}+U_3\frac{A_2}{R_2}\right] \tag{9}$$

$$\in_{33}=\frac{\partial U_3}{\partial\alpha_3} \tag{10}$$

$$\in_{12}=\frac{A_1(1+\frac{\alpha_3}{R_1})}{A_2(1+\frac{\alpha_3}{R_2})}\frac{\partial}{\partial\alpha_2}\left(\frac{U_1}{A_1(1+\frac{\alpha_3}{R_1})}\right)+\frac{A_2(1+\frac{\alpha_3}{R_2})}{A_1(1+\frac{\alpha_3}{R_1})}\frac{\partial}{\partial\alpha_1}\left(\frac{U_2}{A_2(1+\frac{\alpha_3}{R_2})}\right) \tag{11}$$

$$\in_{13}=A_1(1+\frac{\alpha_3}{R_1})\frac{\partial}{\partial\alpha_3}\left(\frac{U_1}{A_1(1+\frac{\alpha_3}{R_1})}\right)+\frac{1}{A_1(1+\frac{\alpha_3}{R_1})}\frac{\partial U_3}{\partial\alpha_1}$$

(12)

$$\in_{23}=A_2(1+\frac{\alpha_3}{R_2})\frac{\partial}{\partial\alpha_3}\left(\frac{U_2}{A_2(1+\frac{\alpha_3}{R_2})}\right)+\frac{1}{A_2(1+\frac{\alpha_3}{R_2})}\frac{\partial U_3}{\partial\alpha_2} \tag{13}$$

$$A_1=\left|\frac{\partial\bar{r}}{\partial\alpha_1}\right| \quad (14) \qquad , \qquad A_2=\left|\frac{\partial\bar{r}}{\partial\alpha_2}\right| \quad (15)$$

where A_1 and A_2 are the fundamental form parameters or Lame parameters, U_1, U_2 and U_3 are the displacement at any point ($\alpha_1,\alpha_2,\alpha_3$), R_1 and R_2 are the radii of curvature related to α_1 and α_2, respectively.

The third-order theory of Reddy (Reddy et al. 1999) used in the present study is based on the following displacement field

$$\begin{cases}U_1=u_1(\alpha_1,\alpha_2)+\alpha_3.\phi_1(\alpha_1,\alpha_2)+\alpha_3^2.\psi_1(\alpha_1,\alpha_2)+\alpha_3^3.\beta_1(\alpha_1,\alpha_2)\\U_2=u_2(\alpha_1,\alpha_2)+\alpha_3.\phi_1(\alpha_1,\alpha_2)+\alpha_3^2.\psi_2(\alpha_1,\alpha_2)+\alpha_3^3.\beta_2(\alpha_1,\alpha_2)\\U_3=u_3(\alpha_1,\alpha_2)\end{cases} \tag{16}$$

These equations can be reduced by satisfying the stress-free conditions on the top and bottom faces of the laminates, which are equivalent to $\in_{13}=\in_{23}=0$ at $z=\pm\frac{h}{2}$. Thus,

$$\begin{cases}U_1=u_1(\alpha_1,\alpha_2)+\alpha_3.\phi_1(\alpha_1,\alpha_2)-C_1.\alpha_3^3(-\frac{u_1}{R_1}+\phi_1+\frac{\partial u_3}{A_1\partial\alpha_1})\\U_2=u_2(\alpha_1,\alpha_2)+\alpha_3.\phi_2(\alpha_1,\alpha_2)-C_1.\alpha_3^3(-\frac{u_2}{R_2}+\phi_2+\frac{\partial u_3}{A_2\partial\alpha_2})\\U_3=u_3(\alpha_1,\alpha_2)\end{cases} \tag{17}$$

where $C_1 = \dfrac{4}{3h^2}$. Substituting Eq. (17) into nonlinear strain-displacement relation (8) to (13) gives

$$\begin{Bmatrix} \in_{11} \\ \in_{22} \\ \in_{12} \end{Bmatrix} = \begin{Bmatrix} \in_{11}^0 \\ \in_{22}^0 \\ \in_{12}^0 \end{Bmatrix} + \alpha_3 \begin{Bmatrix} k_{11} \\ k_{22} \\ k_{12} \end{Bmatrix} + \alpha_3^3 \begin{Bmatrix} k'_{11} \\ k'_{22} \\ k'_{12} \end{Bmatrix} \tag{18}$$

$$\begin{Bmatrix} \in_{13} \\ \in_{23} \end{Bmatrix} = \begin{Bmatrix} \gamma_{13}^0 \\ \gamma_{23}^0 \end{Bmatrix} + \alpha_3^2 \begin{Bmatrix} \gamma_{13}^2 \\ \gamma_{23}^2 \end{Bmatrix} + \alpha_3^2 \begin{Bmatrix} \gamma_{13}^3 \\ \gamma_{23}^3 \end{Bmatrix} \tag{19}$$

where

$$\begin{Bmatrix} \in_{11}^0 \\ \in_{22}^0 \\ \in_{12}^0 \end{Bmatrix} = \begin{Bmatrix} (\frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{u_2}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{u_3}{R_1}) \\ (\frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2} + \frac{u_1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{u_3}{R_2}) \\ \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}(\frac{u_2}{A_2}) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}(\frac{u_1}{A_1}) \end{Bmatrix}, \begin{Bmatrix} k'_{11} \\ k'_{22} \\ k'_{12} \end{Bmatrix} = \begin{Bmatrix} (\frac{1}{A_1}\frac{\partial \phi_1}{\partial \alpha_1} + \frac{\phi_2}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}) \\ (\frac{1}{A_2}\frac{\partial \phi_2}{\partial \alpha_2} + \frac{\phi_1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}) \\ (\frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}(\frac{\phi_2}{A_2}) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}(\frac{\phi_1}{A_1})) \end{Bmatrix} \tag{20}$$

$$\begin{Bmatrix} k'_{11} \\ k'_{22} \\ k'_{12} \end{Bmatrix} = -C_1 \begin{Bmatrix} (\frac{1}{A_1}(-\frac{\partial u_1}{R_1\partial \alpha_1} + \frac{\partial \phi_1}{\partial \alpha_1} + \frac{\partial^2 u_3}{A_1\partial \alpha_1^2} - \frac{\partial A_1}{A_1^2\partial \alpha_1}\frac{\partial u_3}{\partial \alpha_1}) + \frac{\partial A_1}{\partial \alpha_2}\frac{1}{A_1A_2}(-\frac{u_2}{R_2} + \phi_2 + \frac{\partial u_3}{A_2\partial \alpha_2})) \\ (\frac{1}{A_2}(-\frac{\partial u_2}{R_2\partial \alpha_2} + \frac{\partial \phi_2}{\partial \alpha_2} + \frac{\partial^2 u_3}{A_2\partial \alpha_2^2} - \frac{\partial A_2}{A_2^2\partial \alpha_2}\frac{\partial u_3}{\partial \alpha_2}) + \frac{\partial A_2}{\partial \alpha_1}\frac{1}{A_1A_2}(-\frac{u_1}{R_1} + \phi_1 + \frac{\partial u_3}{A_1\partial \alpha_1})) \\ (\frac{A_2}{A_1}(-\frac{\partial}{R_2\partial \alpha_1}(\frac{u_2}{A_2}) + \frac{\partial}{\partial \alpha_1}(\frac{\phi_2}{A_2}) + \frac{1}{A_2^2}\frac{\partial^2 u_3}{\partial \alpha_1\partial \alpha_2} - \frac{1}{A_2^4}\frac{\partial A_2^2}{\partial \alpha_1}\frac{\partial u_3}{\partial \alpha_2}) + \\ + \frac{A_1}{A_2}(-\frac{\partial}{R_1\partial \alpha_2}(\frac{u_1}{A_1}) + \frac{\partial}{\partial \alpha_2}(\frac{\phi_1}{A_1}) + \frac{1}{A_1^2}\frac{\partial^2 u_3}{\partial \alpha_1\partial \alpha_2} - \frac{1}{A_1^4}\frac{\partial A_1^2}{\partial \alpha_2}\frac{\partial u_3}{\partial \alpha_1})) \end{Bmatrix} \tag{21}$$

$$\begin{Bmatrix} \gamma_{13}^0 \\ \gamma_{23}^0 \end{Bmatrix} = \begin{Bmatrix} (\phi_1 - \frac{u_1}{R_1} + \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha_1}) \\ (\phi_2 - \frac{u_2}{R_2} + \frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2}) \end{Bmatrix}, \tag{22}$$

$$\begin{Bmatrix} \gamma_{13}^2 \\ \gamma_{23}^2 \end{Bmatrix} = 3C_1 \begin{Bmatrix} (-\frac{u_1}{R_1} + \phi_1 + \frac{\partial u_3}{A_1\partial \alpha_1}) \\ (-\frac{u_2}{R_2} + \phi_2 + \frac{\partial u_3}{A_2\partial \alpha_2} \end{Bmatrix} \tag{23}$$

$$\begin{Bmatrix} \gamma_{13}^3 \\ \\ \gamma_{23}^3 \end{Bmatrix} = C_1 \begin{Bmatrix} \dfrac{(-\dfrac{u_1}{R_1} + \phi_1 + \dfrac{\partial u_3}{A_1 \partial \alpha_1})}{R_1} \\ \dfrac{(-\dfrac{u_2}{R_2} + \phi_2 + \dfrac{\partial u_3}{A_2 \partial \alpha_2})}{R_2} \end{Bmatrix} \tag{24}$$

where $(\varepsilon^0, \gamma^0)$ are the membrane strains , and $(k, k', \gamma^2, \gamma^3)$ are the bending strains, known as the curvatures .

4. Stress- strain relationships

For a thin shell, the stress -strain relationship are defined as (Soedel, 1981)

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \in_{11} \\ \in_{22} \\ \in_{23} \\ \in_{13} \\ \in_{12} \end{Bmatrix} - \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \alpha.\Delta \mathrm{T} \right\} \tag{25}$$

where

$$Q_{11} = Q_{22} = \frac{E}{1-\upsilon^2}, \quad Q_{12} = \upsilon Q_{11}, \quad Q_{44} = Q_{55} = Q_{66} = \frac{E}{2(1+\upsilon)} \tag{26}$$

and $\Delta \mathrm{T}$ is the temperature change from a stress-free state which the present study equal to zero. Defining:

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, G_{ij}, H_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij} (1, \alpha_3, \alpha_3^2, \alpha_3^3, \alpha_3^4, \alpha_3^5, \alpha_3^6) d\alpha_3 \tag{27}$$

where Q_{ij} are a function of z for functionally gradient materials. Here A_{ij} denote the extensional stiffnesses, D_{ij} the bending stiffnesses, B_{ij} the bending-extensional coupling stiffnesses and $E_{ij}, F_{ij}, G_{ij}, H_{ij}$ are the extensional, bending, coupling, and higher-order stiffnesses .

5. The stress resultants

The force and moment resultants are expressed as

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} d\alpha_3 \quad (28), \qquad \begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \alpha_3 . d\alpha_3 \quad (29)$$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix} d\alpha_3 \qquad (30), \qquad \begin{Bmatrix} P_{11} \\ P_{22} \\ P_{12} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \alpha_3^3 . d\alpha_3 \qquad (31)$$

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix} \alpha_3^2 . d\alpha_3 \qquad \begin{Bmatrix} P_{13} \\ P_{23} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix} \alpha_3^3 . d\alpha_3 \quad (32), \qquad (33)$$

Thus

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11}(\epsilon_{11}^0 + \alpha_3 k_{11} + \alpha_3^3 k'_{11}) + Q_{12}(\epsilon_{22}^0 + \alpha k_{22} + \alpha_3^3 k'_{22}) \\ Q_{12}(\epsilon_{11}^0 + \alpha_3 k_{11} + \alpha_3^3 k'_{11}) + Q_{22}(\epsilon_{22}^0 + \alpha_3 k_{22} + \alpha_3^3 k'_{22}) \\ Q_{66}(\epsilon_{12}^0 + \alpha_3 k_{12} + \alpha_3^3 k'_{12}) \end{bmatrix} \qquad (34)$$

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \begin{bmatrix} A_{11}\epsilon_{11}^0 + B_{11}k_{11} + E_{11}k'_{11} + A_{12}\epsilon_{22}^0 + B_{12}k_{22} + E_{12}k'_{22} \\ A_{12}\epsilon_{11}^0 + B_{12}k_{11} + E_{12}k'_{11} + A_{12}\epsilon_{22}^0 + B_{12}k_{22} + E_{12}k'_{22} \\ A_{66}\epsilon_{12}^0 + B_{66}k_{12} + E_{66}k'_{12} \end{bmatrix} \qquad (35)$$

$$\begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \begin{bmatrix} B_{11}\epsilon_{11}^0 + D_{11}k_{11} + F_{11}k'_{11} + B_{12}\epsilon_{22}^0 + D_{12}k_{22} + F_{12}k'_{22} \\ B_{12}\epsilon_{11}^0 + D_{12}k_{11} + F_{12}k'_{11} + B_{12}\epsilon_{22}^0 + D_{12}k_{22} + F_{12}k'_{22} \\ B_{66}\epsilon_{12}^0 + D_{66}k_{12} + F_{66}k'_{12} \end{bmatrix} \qquad (36)$$

$$\begin{Bmatrix} P_{11} \\ P_{22} \\ P_{12} \end{Bmatrix} = \begin{bmatrix} (E_{11}\epsilon_{11}^0 + F_{11}k_{11} + H_{11}k'_{11}) + (E_{12}\epsilon_{22}^0 + F_{12}k_{22} + H_{12}k'_{22}) \\ (E_{12}\epsilon_{11}^0 + F_{12}k_{11} + H_{12}k'_{11}) + (E_{12}\epsilon_{22}^0 + F_{12}k_{22} + H_{12}k'_{22}) \\ E_{66}\epsilon_{12}^0 + F_{66}k_{12} + H_{66}k'_{12} \end{bmatrix} \qquad (37)$$

$$\begin{Bmatrix} P_{13} \\ P_{23} \end{Bmatrix} = \begin{Bmatrix} E_{55}\gamma_{13}^0 + G_{55}\gamma_{13}^2 + H_{55}\gamma_{13}^3 \\ E_{44}\gamma_{23}^0 + G_{44}\gamma_{23}^2 + H_{44}\gamma_{23}^3 \end{Bmatrix} \qquad (38)$$

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} D_{55}\gamma_{13}^0 + F_{55}\gamma_{13}^2 + G_{55}\gamma_{13}^3 \\ D_{44}\gamma_{23}^0 + F_{44}\gamma_{23}^2 + G_{44}\gamma_{23}^3 \end{Bmatrix} \qquad (39)$$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} A_{55}\gamma_{13}^0 + D_{55}\gamma_{13}^2 + E_{55}\gamma_{13}^3 \\ A_{44}\gamma_{23}^0 + D_{44}\gamma_{23}^2 + E_{44}\gamma_{23}^3 \end{Bmatrix} \qquad (40)$$

6. The equations of motion

The equations of motion for vibration of shell can be derived by using Hamilton's principle which is described by

$$\delta\int_{t_1}^{t_2}(\Pi - K)dt = 0 \quad , \qquad \Pi = U - V \qquad (41)$$

where K , Π , U and V are the total kinetic, potential, strain and loading energies; t_0 and t_1 are arbitrary time. The kinetic, strain and loading energies of a cylindrical shell can be written as

$$K=\frac{1}{2}\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}\rho(\dot U_1^2+\dot U_2^2+\dot U_3^2)d\forall \tag{42}$$

$$U=\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}(\sigma_{11}\in_{11}+\sigma_{22}\in_{22}+\sigma_{12}\in_{12}+\sigma_{13}\in_{13}+\sigma_{23}\in_{23})d\forall \tag{43}$$

$$V=\int_{\alpha_1}\int_{\alpha_2}(q_1\delta U_1+q_2\delta U_2+q_3\delta U_3)A_1A_2d\alpha_1d\alpha_2 \tag{44}$$

The infinitesimal volume is given by

$$d\forall=A_1A_2d\alpha_1d\alpha_2d\alpha_3 \tag{45}$$

Substituting Eqs. (44), (45) and (46) into Eq. (43) we get the equations of motions

$$-\frac{\partial(N_{11}A_2)}{\partial\alpha_1}+N_{22}\frac{\partial A_2}{\partial\alpha_1}-\frac{\partial(N_{12}A_1^2)}{A_1\partial\alpha_2}-\frac{Q_{13}}{R_1}A_1A_2-\frac{\partial}{\partial\alpha_1}(\frac{P_{11}C_1A_2}{R_1})+\frac{P_{22}C_1}{R_1}\frac{\partial A_2}{\partial\alpha_1}-\frac{\partial}{\partial\alpha_2}(\frac{P_{12}C_1A_1^2}{R_1})\frac{1}{A_1}$$
$$+\frac{3C_1R_{13}}{R_1}A_1A_2-\frac{C_1P_{13}A_1A_2}{R_1^2}=-(\ddot u_1I_\circ+\ddot\phi_1I_1+[-C_1(-\frac{\ddot u_1}{R_1}+\ddot\phi_1+\frac{\partial\ddot u_3}{A_1\partial\alpha_1})+\frac{C_1\ddot u_1}{R_1}\ ]I_3+\frac{C_1\ddot\phi_1}{R_1}I_4$$
$$-\frac{C_1^2}{R_1}(-\frac{\ddot u_1}{R_1}+\ddot\phi_1+\frac{\partial\ddot u_3}{A_1\partial\alpha_1})I_6-q_1) \tag{46}$$

$$\frac{\partial(N_{22}A_1)}{\partial\alpha_2}-N_{11}\frac{\partial A_1}{\partial\alpha_2}+\frac{\partial(N_{12}A_2^2)}{A_2\partial\alpha_1}+\frac{Q_{23}}{R_2}A_1A_2+\frac{\partial}{\partial\alpha_2}(\frac{P_{22}C_1A_1}{R_2})-\frac{P_{11}C_1}{R_2}\frac{\partial A_1}{\partial\alpha_2}+\frac{\partial}{\partial\alpha_1}(\frac{P_{12}C_1A_2^2}{R_2})\frac{1}{A_2}$$
$$-\frac{3C_1R_{23}}{R_2}A_1A_2+\frac{C_1P_{23}A_1A_2}{R_2^2}=(\ddot u_2I_\circ+\ddot\phi_2I_1-\Bigg[-c_1(\ -\frac{\ddot u_2}{R_2}+\ddot\phi_2+\frac{\partial\ddot u_3}{A_2\partial\alpha_2})+\frac{C_1\ddot u_2}{R_2}\ \Bigg]I_3+\frac{C_1\ddot\phi_2}{R_2}I_{42}+$$
$$-\frac{C_1^2}{R_2}(-\frac{\ddot u_2}{R_2}+\ddot\phi_2+\frac{\partial u_3}{A_2\partial\alpha_2})I_6-q_2)\ ] \tag{47}$$

$$(-\frac{\partial^2(P_{11}C_1A_2/A_1)}{\partial\alpha_1^2}+N_{11}\frac{A_1A_2}{R_1}+\frac{\partial}{\partial\alpha_2}(\frac{C_1P_{11}}{A_2}\frac{\partial A_1}{\partial\alpha_2})+N_{22}\frac{A_1A_2}{R_2}-\frac{\partial^2(P_{22}A_1C_1/A_2)}{\partial\alpha_2^2}-\frac{\partial(Q_{13}A_2)}{\partial\alpha_1}+\frac{\partial(3C_1R_{13}A_2)}{\partial\alpha_1}$$
$$-\frac{\partial}{\partial\alpha_1}(\frac{P_{13}C_1A_2}{R_1})-\frac{\partial(Q_{23}A_1)}{\partial\alpha_2}+\frac{\partial(3C_1R_{23}A_1)}{\partial\alpha_2}-\frac{\partial}{\partial\alpha_2}(\frac{C_1P_{23}A_1}{R_2})-\frac{\partial}{\partial\alpha_1}(\frac{P_{11}C_1A_2}{A_1^2}\frac{\partial A_1}{\partial\alpha_1})-\frac{\partial}{\partial\alpha_2}(P_{22}C_1\frac{A_1}{A_2^2}\frac{\partial A_2}{\partial\alpha_2}))=$$
$$-\{\ddot u_3I_\circ++C_1\left[\frac{\partial}{\partial\alpha_1}(\frac{u_1}{A_1})+\frac{\partial}{\partial\alpha_2}(\frac{u_2}{A_2})\right]I_3+C_1\left[\frac{\partial}{\partial\alpha_1}(\frac{\ddot\phi_1}{A_1})+\frac{\partial}{\partial\alpha_2}(\frac{\ddot\phi_2}{A_2})\right]I_4-C_1^2I_6\Bigg[(-\frac{\partial}{R_2\partial\alpha_2}(\frac{\ddot u_2}{A_2})$$
$$(48)+\frac{\partial}{\partial\alpha_2}(\frac{\ddot\phi_2}{A_2})+\frac{1}{A_2}\frac{\partial^2\ddot u_3}{\partial\alpha_2^2}-\frac{\partial A_2}{A_1^2\partial\alpha_2}\frac{\partial\ddot u_3}{\partial\alpha_2})+(-\frac{\partial}{R_1\partial\alpha_1}(\frac{\ddot u_1}{A_1})+\frac{\partial}{\partial\alpha_1}(\frac{\ddot\phi_1}{A_1})+\frac{1}{A_1}\frac{\partial^2\ddot u_3}{\partial\alpha_1^2}-\frac{\partial A_1}{A_1^2\partial\alpha_1}\frac{\partial\ddot u_3}{\partial\alpha_1})\Bigg]-q_3\}$$

$$-\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+\frac{\partial(C_1P_{11}A_2)}{\partial\alpha_1}+M_{22}\frac{\partial A_2}{\partial\alpha_1}-C_1P_{22}\frac{\partial A_2}{\partial\alpha_1}-\frac{\partial(M_{12}A_1^2)}{A_1\partial\alpha_2}+\frac{\partial(P_{12}C_1A_1^2)}{A_1\partial\alpha_2}-3C_1R_{13}A_1A_2+A_1A_2Q_{13}$$

$$(49)+\frac{C_1P_{13}}{R_1}A_1A_2=-\Big[\ddot{u}_1I_1+\ddot{\phi}_1I_2-C_1\ddot{u}_1I_3+(-2C_1\ddot{\phi}_1+C_1\frac{\ddot{u}_1}{R_1}--\frac{C_1}{A_1}\frac{\partial\ddot{u}_3}{\partial\alpha_1})I_4+C_1^2(-\frac{\ddot{u}_1}{R_1}+\ddot{\phi}_1+\frac{\partial\ddot{u}_3}{A_1\partial\alpha_1})I_6\Big]$$

$$-\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+\frac{\partial(C_1A_1P_{22})}{\partial\alpha_2}+M_{11}\frac{\partial A_1}{\partial\alpha_2}-C_1P_{11}\frac{\partial A_1}{\partial\alpha_2}-\frac{\partial(M_{12}A_2^2)}{A_2\partial\alpha_1}+\frac{\partial(P_{12}C_1A_2^2)}{A_2\partial\alpha_1}-3C_1R_{23}A_1A_2+A_1A_2Q_{23}$$

$$+C_1\frac{\ddot{u}_2}{R_2}-\frac{C_1}{A_2}\frac{\partial\ddot{u}_3}{\partial\alpha_2})I_4+C_1^2(\frac{\ddot{u}_2}{R_2}+\ddot{\phi}_2+\frac{\partial\ddot{u}_3}{A_2\partial\alpha_2})I_6\Big]+\frac{C_1P_{23}}{R_2}A_1A_2=-\Big[\ddot{u}_2I_1+\ddot{\phi}_2I_2-C_1\ddot{u}_2I_3+(-2C_1\ddot{\phi}_2+$$

$$+C_1\frac{\ddot{u}_2}{R_2}-\frac{C_1}{A_2}\frac{\partial\ddot{u}_3}{\partial\alpha_2})I_4+C_1^2(\frac{\ddot{u}_2}{R_2}+\ddot{\phi}_2+\frac{\partial\ddot{u}_3}{A_2\partial\alpha_2})I_6\Big] \qquad (50)$$

7. The equations of motion for vibration of functionally graded cylindrical shells

The curvilinear coordinates and fundamental form parameters for the cylindrical shell are

$$,\alpha_1=x \quad \alpha_2=\theta \;,\; \alpha_3=\alpha_3 \;,\;, A_1=1 \;,\; A_2=a \;,\; \frac{1}{R_1}=0 \qquad R_2=a \qquad (51)$$

Thus, the equations of motions for the cylindrical shell are converted to

$$(52)\; a\frac{\partial N_{11}}{\partial x}+\frac{\partial N_{12}}{\partial\theta}=-q_1+I_0\ddot{u}_1+(I_1-C_1I_3)\ddot{\phi}_1-C_1I_3\frac{\partial\ddot{u}_3}{\partial x}$$

$$\frac{\partial N_{22}}{\partial\theta}+C_1\frac{\partial P_{12}}{\partial x}+Q_{23}-3C_1R_{23}+C_1P_{23}=(I_0+2\frac{C_1}{a}I_3+\frac{C_1^2}{a^2}I_6)\ddot{u}_2+$$

$$+(I_1-C_1I_3+\frac{C_1}{a}I_4-\frac{C_1^2}{a}I_6)\ddot{\phi}_2-(\frac{C_1}{a}I_3-\frac{C_1^2}{a^2}I_6)\frac{\partial\ddot{u}_3}{\partial\theta}-q_2 \qquad (53)$$

$$-C_1a\frac{\partial^2P_{11}}{\partial x^2}+N_{22}-\frac{C_1}{a}\frac{\partial^2P_{22}}{\partial\theta^2}-2C_1\frac{\partial^2P_{12}}{\partial x\partial\theta}-a\frac{\partial Q_{13}}{\partial x}+3C_1a\frac{\partial R_{13}}{\partial x}-\frac{\partial Q_{23}}{\partial\theta}+3C_1\frac{\partial R_{23}}{\partial\theta}-\frac{C_1}{a}\frac{\partial P_{23}}{\partial\theta}=$$

$$+(-\frac{C_1}{a}I_4+\frac{C_1^2}{a}I_6)\frac{\partial\ddot{\phi}_2}{\partial\theta}-\frac{C_1^2}{a^2}I_6\frac{\partial\ddot{u}_2}{\partial\theta}-C_1I_3\frac{\partial u_1}{\partial x}-\frac{C_1}{a}I_3\frac{\partial u_2}{\partial\theta}+(-C_1I_4+C_1^2I_6)\frac{\partial\ddot{\phi}_1}{\partial x}+$$

$$+C_1^2I_6\frac{\partial^2\ddot{u}_3}{\partial x^2}+\frac{C_1^2}{a}I_6\frac{\partial^2\ddot{u}_3}{\partial\theta^2}-\ddot{u}_3I_0-q_3 \qquad (54)$$

$$-a\frac{\partial M_{11}}{\partial x}+C_1a\frac{\partial P_{11}}{\partial x}-\frac{\partial M_{12}}{\partial\theta}+C_1\frac{\partial P_{12}}{\partial\theta}-3C_1R_{13}a+aQ_{13}=-I_1\ddot{u}_3+C_1I_3\ddot{u}_1+$$

$$+(-I_2+2C_1I_4-C_1^2I_6)\ddot{\phi}_1+(C_1I_4-C_1^2I_6)\frac{\partial\ddot{u}_3}{\partial x} \qquad (55)$$

$$-\frac{\partial M_{22}}{\partial\theta}-C_1\frac{\partial P_{22}}{\partial\theta}-a\frac{\partial M_{12}}{\partial x}+C_1a\frac{\partial P_{12}}{\partial x}-3C_1R_{23}a+aQ_{23}+C_1R_{23}=$$

$$(56)$$

$$(-I_1+C_1I_3-\frac{C_1}{a}I_4)\ddot{u}_2+(-I_2+2C_1I_4)\ddot{\phi}_2-\frac{C_1}{a}I_4\frac{\partial\ddot{u}_3}{\partial\theta}$$

(see Appendix 1 , 2)

8. Analysis

It is assumed that the boundary conditions for circular cylindrical shell with simply supported are such that

$$
\begin{aligned}
&u_3(0,\theta,t)=0, && u_3(l,\theta,t)=0 \\
&u_\theta(0,\theta,t)=0, && u_\theta(l,\theta,t)=0 \\
&M_{xx}(0,\theta,t)=0, && M_{xx}(l,\theta,t)=0 \\
&N_{xx}(0,\theta,t)=0, && N_{xx}(l,\theta,t)=0
\end{aligned}
\tag{57}
$$

The displacement fields which satisfy these boundary conditions can be written as

$$
\left.
\begin{aligned}
u_1 &= A\cos\frac{m\pi x}{l}\cos n(\theta-\phi)e^{j\omega t} \\
u_2 &= B\sin\frac{m\pi x}{l}\sin n(\theta-\phi)e^{j\omega t} \\
u_3 &= C\sin\frac{m\pi x}{l}\cos n(\theta-\phi)e^{j\omega t} \\
\phi_1 &= D\cos\frac{m\pi x}{l}\cos n(\theta-\phi)e^{j\omega t} \\
\phi_2 &= E\sin\frac{m\pi x}{l}\sin n(\theta-\phi)e^{j\omega t}
\end{aligned}
\right\}
\tag{58}
$$

where A, B, C, D and E are the constants denoting the amplitudes of vibration, n and m are the axial and circumferential wave numbers, and $\omega\,(\mathrm{rad}\mathbf{s}^{-1})$ is the natural angular frequency of the vibration . For free vibration we have

$$
q_1 = q_2 = q_3 = 0 \tag{59}
$$

Substituting Eqs. (60) into Eqs. (54) through (58) we get

$$
[C]\begin{Bmatrix} A \\ B \\ C \\ D \\ E \end{Bmatrix} - \omega^2 [M]\begin{Bmatrix} A \\ B \\ C \\ D \\ E \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{60}
$$

where

$$
[C]=\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55}
\end{bmatrix} \tag{61}
$$

$$[M]=\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} \end{bmatrix} \quad (62)$$

The coefficients c_{ij} and m_{ij} are listed in the appendix 3 and 4. Eq. (62) is solved by imposing nontrivial solutions and equating the characteristic determinant $\left(C_{ij} - M_{ij}\omega^2\right) = 0$. Expanding the characteristic determinant the following polynomial is obtained.

$$\det\left(C_{ij} - M_{ij}\omega^2\right) = 0 \quad (63)$$

9. Results and discussion

To validate the analysis, results for simply supported cylindrical shells are compared with Loy (1999), see Tables 2 to 9. The comparisons show that the present results agreed well with those in the literature.

In this paper studies are presented on the vibration of a simply supported functionally graded (FG) cylindrical shells. The functionally gradient material (FGM) considered is composed of stainless steel and nickel and its properties are graded in thickness direction according to a volume fraction power-law distribution. Results for the frequency characteristics, the influence of the constituent volume fractions V_f and the effects of the FGM configuration are presented. The influence of constituent volume fractions is studied by varying the value of the power-law exponent N. The effects of the FGM configuration are studied by studying the frequencies of two FG cylindrical shells:

Type I: FG cylindrical shell has nickel on its inner surface and stainless steel on its outer surface

Type II: FG cylindrical shell has stainless steel on its inner surface and nickel on its outer surface.

The material properties for stainless steel and nickel, calculated at T=300K, are presented in Table 1. The natural frequencies first decreased and than increased with circumferential wave numbers n. The frequencies for higher axial modes m are higher than those for lower axial modes. Thus, the fundamental frequencies occur at m=1. It is interesting to note that the value of the power-law exponent N did not affect the value of the circumferential wave number at which the fundamental natural frequency might occur. For example, in Table 2, for $\frac{L}{R}$ = 20, the fundamental frequencies for N = 0.5, 0.7, 1, 2, 5 and 15 all occur at circumferential wave number n=3. The natural frequencies of a short cylindrical shell. small $\frac{L}{R}$ ratios, are higher than those of a long shell, and the natural frequencies of a thick cylindrical shell, big h/R ratios, are higher than those for a thin shell**.**

Tables 2 and 4 show the variations of the natural frequencies (Hz) with the circumferential wave numbers n for a Type I FG cylindrical shell. The columns N^{ss} and N^{N} show the natural frequencies for a stainless steel cylindrical shell and a nickel cylindrical shell, respectively. As N increased**,** the natural frequencies decreased. The decreased in the natural frequencies from $N=1$ to $N=15$ is about 2.3**%** at n=1 and about 2.4**%** at n=10. When N is small, the natural frequencies approached those of N^{ss} and when N is large they approached those of N^{N}. Hence, the natural frequencies for $N>0$ fell between those of N^{ss} and N^{N} for a given circumferential wave number n.

Tables 3 and 5 show the variations of the natural frequencies (Hz) with the circumferential wave numbers n for a Type II FG cylindrical shell. The influence of N or constituent volume fraction on the natural frequencies in the opposite of a Type I FG cylindrical shell. Unlike a Type I FG cylindrical shell where the natural frequencies decreased with N**,** the natural frequencies for a Type II FG cylindrical shell increased with N. The increase in natural frequencies from $N=1$ to $N=15$ in about 2.3**%** at n=1 and about 2.4**%** at n=10. Thus the influence of the constituent volume fractions for a Type II FG cylindrical shell is different from a Type I FG cylindrical shell.

Comparing the frequencies in tables 2 and 4 with those in Tables 3 and 5, it can be seen that for $N>1$, the natural frequencies for a Type II FG cylindrical shell are higher than a Type I FG cylindrical shell. On the other hand, for $N<1$, the frequencies for a Type I FG cylindrical shell are higher than a Type II FG cylindrical shell. For example, for $N=15$ at n=10 and $\frac{h}{R}$=0.002, the natural frequency for a Type II FG cylindrical shell is about 4.67**%** higher than a Type I FG cylindrical shell. For $N=0.5$ at n=10 and $\frac{h}{R}$=0.002, the natural frequency for a Type I FG cylindrical shell is 1.66**%** higher than a Type II FG cylindrical shell. Thus the natural frequencies are affected by the configuration of the constituent materials in the functionally graded cylindrical shells.

Tabels 6 and 7 show the variations of the fundamental natural frequencies (Hz) with $\frac{L}{R}$ ratios for Type I and Type II FG cylindrical shells. The numbers in the brackets indicate the circumferential wave numbers at which the fundamental frequencies occur. For Type I FG cylindrical shell, the fundamental frequencies decreased with N, and for Type II FG cylindrical shell, the fundamental frequencies increased with N. The difference in the fundamental frequencies between $N=1$ and $N=15$ about 2.2**%** for Type I and Type II FG cylindrical shells. The fundamental natural frequencies for Type I and Type II FG cylindrical shells occur at the same circumferential wave numbers. For all values of N, the fundamental natural frequencies fall between those for N^{ss} and N^{N}. Tables 8 and 9 show the variations in the fundamental natural frequencies (Hz) with $\frac{h}{R}$ ratios for Type I and Type II FG cylindrical shells. For Type I FG cylindrical shell the fundamental frequencies decrease with N, and for the Type II FG cylindrical shell, the fundamental frequencies increased with N. for all N the fundamental natural frequencies fall between those for N^{ss} and $N^{N.}$**.**

Thus the constituent volume fractions and the configurations of the constituent materials affect the natural frequencies.

Table 1
Properties of materials

Coefficents	Stainless steel			Nickel		
	$E(Nm^{-2})$	υ	$\rho(Kgm^{-3})$	$E(Nm^{-2})$	υ	$\rho(Kgm^{-3})$
P0	201.04×10^{9}	0.3262	8166	223.95×10^{9}	0.3100	8900
P-1	0	0	0	0	0	0
P1	3.079×10^{-4}	-2.002×10^{-4}	0	-2.794×10^{-4}	0	0
P2	-6.534×10^{-7}	3.797×10^{-7}	0	-3.998×10^{9}	0	0
P3	0	0	0	0	0	0
	2.07788×10^{11}	0.317756	8166	2.05098×10^{11}	0.3100	8900

Table 2

Vibration of natural frequencies (Hz) against circumferential wave number

n $\left(\frac{L}{R}=20, \frac{h}{R}=0.002, m=1\right)$

Tape I FG Cylindrical Shell										
n	Method	$N^{ss}=0$	$N^{N}=0$	N=0.5	$N=0.7$	$N=1$	$N=2$	$N=5$	$N=15$	$N=30$
1	3 order	13.5476	12.8935	13.3208	13.268	13.2107	13.1028	12.9978	12.9326	12.9135
	Loy	13.548	12.894	13.321	13.269	13.211	13.103	12.998	12.933	12.914
2	3 order	4.5918	4.3686	4.5165	4.4992	4.4800	4.4432	4.4065	4.3832	4.3763
	Loy	4.5920	4.3690	4.5168	4.4994	4.480	4.4435	4.4068	4.3834	4.3765
3	3 order	4.2630	4.0484	4.1909	4.1746	4.1565	4.1233	4.0890	4.0650	4.0573
	Loy	4.2633	4.0489	4.1911	4.1749	4.1569	4.1235	4.0891	4.0653	4.0576
4	3 order	7.2248	6.8573	7.0970	7.0687	7.0382	6.9817	6.9248	6.8851	6.8722
	Loy	7.2250	6.8577	7.0972	7.0691	7.0384	6.9820	6.9251	6.8856	6.8726
5	3 order	11.5417	10.9544	11.3352	11.2900	11.2407	11.1509	11.0607	10.9986	10.9778
	Loy	11.542	10.955	11.336	11.290	11.241	11.151	11.061	10.999	10.978
6	3 order	16.8962	16.0369	16.5937	16.5268	16.4547	16.3225	16.1916	16.1009	16.0708
	Loy	16.897	16.037	16.594	16.527	16.455	16.323	16.192	16.101	16.071
7	3 order	23.2435	22.0607	22.8258	22.7345	22.6345	22.4537	22.2726	22.1478	22.1078
	Loy	23.244	22.061	22.826	22.735	22.635	22.454	22.273	22.148	22.108
8	3 order	30.5727	29.0168	30.0227	29.9028	29.7706	29.5328	29.2958	29.1317	29.0776
	Loy	30.573	29.017	30.023	29.903	29.771	29.533	29.296	29.132	29.078
9	3 order	38.879	36.9017	38.1807	38.0277	37.8618	37.5586	37.2567	37.0475	36.9807
	Loy	38.881	36.902	38.181	38.028	37.862	37.559	37.257	37.048	36.981
10	3 order	48.1673	45.7157	47.3009	47.1109	46.9049	46.5287	46.1546	45.8967	45.8124
	Loy	48.168	45.716	47.301	47.111	46.905	46.529	46.155	45.897	45.813

Table 3

Vibration of natural frequencies (Hz) against circumferential wave number n $\left(\frac{L}{R}=20, \frac{h}{R}=0.002, m=1\right)$

Tape II FG Cylindrical Shell										
n	Method	$N^{ss}=0$	$N^{N}=0$	N=0.5	$N=0.7$	$N=1$	$N=2$	$N=5$	$N=15$	$N=30$
1	3 order	13.5476	12.8935	13.1027	13.1537	13.2107	13.1265	13.4327	13.5048	13.5224
	Loy	13.548	12.894	13.103	13.154	13.211	13.321	13.433	13.505	13.526
2	3 order	4.5918	4.3686	4.4380	4.4548	4.4740	4.5112	4.5502	4.5755	4.5832
	Loy	4.5920	4.3690	4.4382	4.4550	4.4742	4.5114	4.5504	4.5759	4.5836
3	3 order	4.2630	4.0484	4.1149	4.1307	4.1484	4.1824	4.2189	4.2447	4.2533
	Loy	4.2633	4.0489	4.1152	4.1309	4.1486	4.1827	4.2191	4.2451	4.2536
4	3 order	7.2248	6.8573	6.9752	7.0024	7.0327	7.0903	7.1508	7.1941	7.2082
	Loy	7.2250	6.8577	6.9754	7.0026	7.0330	7.0905	7.1510	7.1943	7.2085
5	3 order	11.5417	10.9544	11.1448	11.1885	11.2376	11.3285	11.4246	11.4938	11.5157
	Loy	11.542	10.955	11.145	11.189	11.238	11.329	11.425	11.494	11.516
6	3 order	16.8962	16.0369	16.3167	16.3807	16.4528	16.5868	16.7268	16.8265	16.8586
	Loy	16.897	16.037	16.317	16.381	16.453	16.587	16.727	16.827	16.859
7	3 order	23.2435	22.0607	22.4466	22.5348	22.6326	22.4535	23.0107	23.1467	23.1917
	Loy	23.244	22.061	22.447	22.535	22.633	22.454	23.011	23.147	23.192
8	3 order	30.5727	29.0168	29.5237	29.6407	29.7700	30.0138	30.2665	30.4458	30.5048
	Loy	30.573	29.017	29.524	29.641	29.770	30.014	30.267	30.446	30.505
9	3 order	38.879	36.9017	37.5475	37.6954	37.8607	38.1707	38.4916	38.7200	38.7946
	Loy	38.881	36.902	37.548	37.696	37.861	38.171	38.492	38.720	38.795
10	3 order	48.1673	45.7157	46.5168	46.7000	46.9035	47.2876	47.6857	47.9678	48.0607
	Loy	48.168	45.716	46.517	46.700	46.904	47.288	47.686	47.968	48.061

Table 4

Vibration of fundamental natural frequencies (Hz) against L/R ratios $\left(\frac{h}{R}=0.002, m=1\right)$

Tape I FG Cylindrical Shell									
L/R	Method	$N^{ss}=0$	$N^{N}=0$	N=0.5	$N=0.7$	$N=1$	$N=2$	$N=5$	$N=15$
0.2	3 order	439.34 (20)	417.51(20)	432.10(20)	430.43(20)	428.60(20)	425.14(20)	421.58(20)	419.14(20)
	Loy	439.36(20)	417.54(20)	432.12(20)	430.46(20)	428.62(20)	425.16(20)	421.60(20)	419.17(20)
0.5	3 order	175.46(15)	166.72(15)	172.54(15)	171.91(15)	171.15(15)	169.79(15)	168.35(15)	167.39(15)
	Loy	175.49(15)	166.76(15)	172.59(15)	171.93(15)	171.19(15)	169.81(15)	168.38(15)	167.41(15)
1	3 order	87.329(11)	82.991(11)	85.87(11)	85.559(11)	85.191(11)	84.502(11)	83.795(11)	83.314(11)
	Loy	87.331(11)	82.993(11)	85.890(11)	85.561(11)	85.195(11)	84.506(11)	83.798(11)	83.316(11)
2	3 order	43.371(8)	41.215(8)	42.654(8)	42.491(8)	42.309(8)	41.967(8)	41.615(8)	41.375(8)
	Loy	43.373(8)	41.217(8)	42.656(8)	42.493(8)	42.311(8)	41.969(8)	41.618(8)	41.378(8)
5	3 order	16.915(5)	16.076(5)	16.636(5)	16.573(5)	16.501(5)	16.369(5)	16.231(5)	16.139(5)
	Loy	16.917(5)	16.079(5)	16.639(5)	16.576(5)	16.505(5)	16.371(5)	16.234(5)	16.141(5)
10	3 order	8.6032(4)	8.1720(4)	8.4589(4)	8.4262(4)	8.3902(4)	8.3226(4)	8.2531(4)	8.2050(4)
	Loy	8.6035(4)	8.1723(4)	8.4591(4)	8.4265(4)	8.3904(4)	8.3228(4)	8.2533(4)	8.2052(4)
20	3 order	4.2630(3)	4.0485(3)	4.1909(3)	4.1745(3)	4.1565(3)	4.1232(3)	4.0889(3)	4.0651(3)
	Loy	4.2633(3)	4.0489(3)	4.1911(3)	4.1749(3)	4.1569(3)	4.1235(3)	4.0892(3)	4.0653(3)
50	3 order	1.4916(2)	1.4165(2)	1.4662(2)	1.4604(2)	1.4542(2)	1.4425(2)	1.4305(2)	1.4222(2)
	Loy	1.4918(2)	1.4167(2)	1.4665(2)	1.4608(2)	1.4545(2)	1.4428(2)	1.4308(2)	1.4225(2)
100	3 order	0.5592(1)	0.5322(1)	0.5500(1)	0.5478(1)	0.54559(1)	0.54112(1)	0.5368(1)	0.5339(1)
	Loy	0.5595(1)	0.5325(1)	0.5502(1)	0.5480(1)	0.54561(1)	0.54115(1)	0.5368(1)	0.5341(1)

Table 5
Vibration of fundamental natural frequencies (Hz) against L/R ratios

$$\left(\frac{h}{R}=0.002, m=1\right)$$

Tape II FG Cylindrical Shell									
L/R	Method	$N^{ss}=0$	$N^{N}=0$	N=0.5	$N=0.7$	$N=1$	$N=2$	$N=5$	$N=15$
0.2	3 order	439.34(20)	417.51(20)	424.18(20)	425.76(20)	427.60(20)	431.11(20)	434.90(20)	437.52(20)
	Loy	439.36(20)	417.54(20)	424.20(20)	425.80(20)	427.62(20)	431.15(20)	434.93(20)	437.57(20)
0.5	3 order	175.46(15)	166.72(15)	169.41(15)	170.04(15)	170.75(15)	172.17(15)	173.69(15)	174.72(15)
	Loy	175.49(15)	166.76(15)	169.43(15)	170.06(15)	170.79(15)	172.20(15)	173.71(15)	174.76(15)
1	3 order	87.329(11)	82.991(11)	84.314(11)	84.632(11)	84.991(11)	85.695(11)	86.444(11)	86.971(11)
	Loy	87.331(11)	82.993(11)	84.316(11)	84.634(11)	84.995(11)	85.697(11)	86.448(11)	86.974(11)
2	3 order	43.371(8)	41.215(8)	41.872(8)	42.033(8)	42.210(8)	42.559(8)	42.931(8)	43.192(8)
	Loy	43.373(8)	41.217(8)	41.875(8)	42.033(8)	42.212(8)	42.561(8)	42.934(8)	43.195(8)
5	3 order	16.915(5)	16.076(5)	16.331(5)	16.393(5)	16.463(5)	16.600(5)	16.745(5)	16.846(5)
	Loy	16.917(5)	16.079(5)	16.335(5)	16.396(5)	16.466(5)	16.602(5)	16.748(5)	16.849(5)
10	3 order	8.6032(4)	8.1720(4)	8.3047(4)	8.3362(4)	8.3719(4)	8.4409(4)	8.5146(4)	8.5670(4)
	Loy	8.6035(4)	8.1723(4)	8.3050(4)	8.3365(4)	8.3722(4)	8.4411(4)	8.5148(4)	8.5672(4)
20	3 order	4.2630(3)	4.0485(3)	4.1150(3)	4.1306(3)	4.1482(3)	4.1824(3)	4.2189(3)	4.2449(3)
	Loy	4.2633(3)	4.0489(3)	4.1152(3)	4.1309(3)	4.1486(3)	4.1827(3)	4.2191(3)	4.2451(3)
50	3 order	1.4916(2)	1.4165(2)	1.4399(2)	1.4451(2)	1.4514(2)	1.4633(2)	1.4761(2)	1.4852(2)
	Loy	1.4918(2)	1.4167(2)	1.4400(2)	1.4455(2)	1.4517(2)	1.4636(2)	1.4763(2)	1.4854(2)
100	3 order	0.5592(1)	0.5322(1)	0.5410(1)	0.54321(1)	0.5452(1)	0.5500(1)	0.5546(1)	0.5575(1)

	Loy	0.5595(1)	0.5325(1)	0.5412(1)	0.54324(1)	0.5456(1)	0.5502(1)	0.5548(1)	0.5578(1)

Table 6
Vibrational of fundamental natural frequencies (Hz) against h/R ratios

$$\left(\frac{L}{R}=20, m=1\right)$$

Tape I FG Cylindrical Shell									
h/R	Method	$N^{ss}=0$	$N^{N}=0$	N=0.5	$N=0.7$	$N=1$	$N=2$	$N=5$	$N=15$
0.001	3 order	2.7915(3)	2.6533(3)	2.7459(3)	2.7353(3)	2.7236(3)	2.7015(3)	2.6790(3)	2.6636(3)
	Loy	2.7919(3)	2.6537(3)	2.7461(3)	2.7356(3)	2.7239(3)	2.7018(3)	2.6792(3)	2.6639(3)
0.005	3 order	5.4990(2)	5.2280(2)	5.4091(2)	5.3885(2)	5.3653(2)	5.3219(2)	5.2773(2)	5.2476(2)
	Loy	5.4992(2)	5.2283(2)	5.4094(2)	5.3887(2)	5.3656(2)	5.3221(2)	5.2776(2)	5.2478(2)
0.007	3 order	6.378(2)	6.0629(2)	6.2742(2)	6.2504(2)	6.2236(2)	6.1733(2)	6.1216(2)	6.0864(2)
	Loy	6.380(2)	6.0631(2)	6.2746(2)	6.2506(2)	6.2239(2)	6.1736(2)	6.1219(2)	6.0867(2)
0.01	3 order	7.9330(2)	7.5355(2)	7.8000(2)	7.7680(2)	7.7364(2)	7.6741(2)	7.6101(2)	7.5659(2)
	Loy	7.9333(2)	7.5358(2)	7.8001(2)	7.7700(2)	7.7367(2)	7.6744(2)	7.6104(2)	7.5661(2)
0.02	3 order	13.549(1)	12.894(1)	13.321(1)	13.270(1)	13.212(1)	13.104(1)	13.000(1)	12.933(1)
	Loy	13.552(1)	12.898(1)	13.325(1)	13.273(1)	13.215(2)	13.107(2)	13.001(2)	12.936(2)
0.03	3 order	13.553(1)	12.900(1)	13.327(1)	13.275(1)	13.217(1)	13.110(1)	13.003(1)	12.939(1)
	Loy	13.557(1)	12.902(1)	13.330(1)	13.278(1)	13.220(1)	13.112(1)	13.006(1)	12.941(1)
0.04	3 order	13.560(1)	12.905(1)	13.333(1)	13.281(1)	13.224(1)	13.117(1)	13.011(1)	12.945(1)
	Loy	13.563(1)	121.909(1)	13.336(1)	13.284(1)	13.226(1)	13.119(1)	13.013(1)	12.948(1)
0.05	3 order	13.569(1)	12.915(1)	13.342(1)	13.290(1)	13.231(1)	13.124(1)	13.019(1)	12.954(1)
	Loy	13.572(1)	12.917(1)	13.345(1)	13.293(1)	13.235(1)	13.127(1)	13.021(1)	12.956(1)

Table 7
Vibrational of fundamental natural frequencies (Hz) against h/R ratios

$$\left(\frac{L}{R}=20, m=1\right)$$

Tape II FG Cylindrical Shell									
h/R	Method	$N^{ss}=0$	$N^{N}=0$	N=0.5	$N=0.7$	$N=1$	$N=2$	$N=5$	$N=15$
0.001	3 order	2.7915(3)	2.6533(3)	2.6955(3)	2.7058(3)	2.7172(3)	2.737(3)	2.7637(3)	2.7804(3)
	Loy	2.7919(3)	2.6537(3)	2.6958(3)	2.7060(3)	2.7175(3)	2.740(3)	2.7640(3)	2.7807(3)
0.005	3 order	5.4990(2)	5.2280(2)	5.3106(2)	5.3305(2)	5.3533(2)	5.3975(2)	5.4450(2)	5.4774(2)
	Loy	5.4992(2)	5.2283(2)	5.3109(2)	5.3308(2)	5.3536(2)	5.3979(2)	5.4452(2)	5.4777(2)
0.007	3 order	6.378(2)	6.0629(2)	6.1595(2)	6.1827(2)	6.2091(2)	6.2603(2)	6.3152(2)	6.3536(2)
	Loy	6.380(2)	6.0631(2)	6.1598(2)	6.1830(2)	6.2094(2)	6.2606(2)	6.3155(2)	6.3539(2)
0.01	3 order	7.9330(2)	7.5355(2)	7.6580(2)	7.6870(2)	7.7200(2)	7.7834(2)	7.8512(2)	7.8997(2)
	Loy	7.9333(2)	7.5358(2)	7.6583(2)	7.6873(2)	7.7202(2)	7.7837(2)	7.8516(2)	7.8999(2)
0.02	3 order	13.549(1)	12.894(1)	13.105(1)	13.154(1)	13.212(1)	13.322(1)	13.435(1)	13.504(1)
	Loy	13.552(1)	12.898(1)	13.107(1)	13.157(1)	13.215(1)	13.325(1)	13.437(1)	13.508(1)
0.03	3 order	13.553(1)	12.900(1)	13.110(1)	13.160(1)	13.217(1)	13.326(1)	13.440(1)	13.510(1)
	Loy	13.557(1)	12.902(1)	13.112(1)	13.162(1)	13.219(1)	13.329(1)	13.442(1)	13.513(1)
0.04	3 order	13.560(1)	12.905(1)	13.115(1)	13.165(1)	13.224(1)	13.333(1)	13.445(1)	13.517(1)
	Loy	13.563(1)	12.909(1)	13.118(1)	13.169(1)	13.226(1)	13.336(1)	13.448(1)	13.520(1)
0.05	3 order	13.569(1)	12.915(1)	13.124(1)	13.174(1)	13.231(1)	13.341(1)	13.455(1)	13.526(1)
	Loy	13.572(1)	12.917(1)	13.126(1)	13.177(1)	13.234(1)	13.344(1)	13.457(1)	13.528(1)

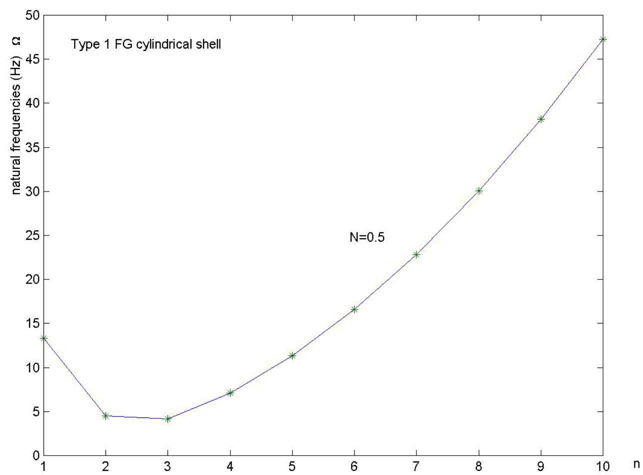

figure 6-3:variation of natural frequencies (Hz) against n (m=1,h/R=0.002,L/R=20)

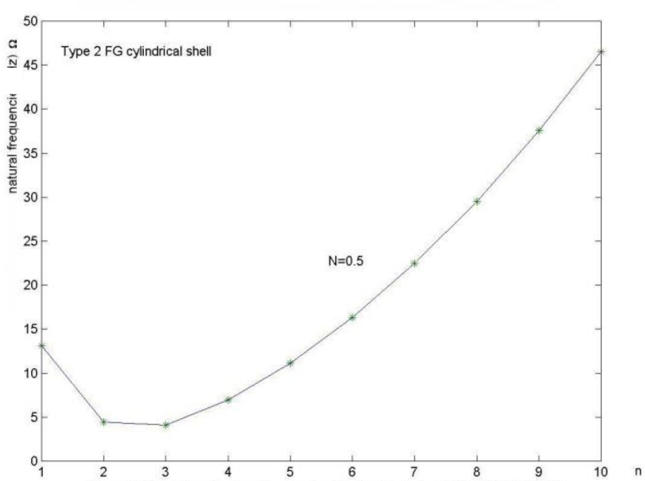

figure 6-12:variation of natural frequencies (Hz) against n (m=1,h/R=0.002,L/R=20)

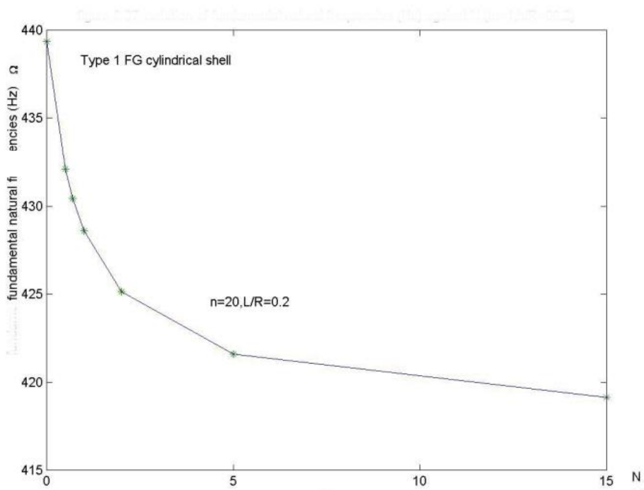

figure 6-37:variation of fundamental natural frequencies (Hz) against N (m=1,h/R=00.2)

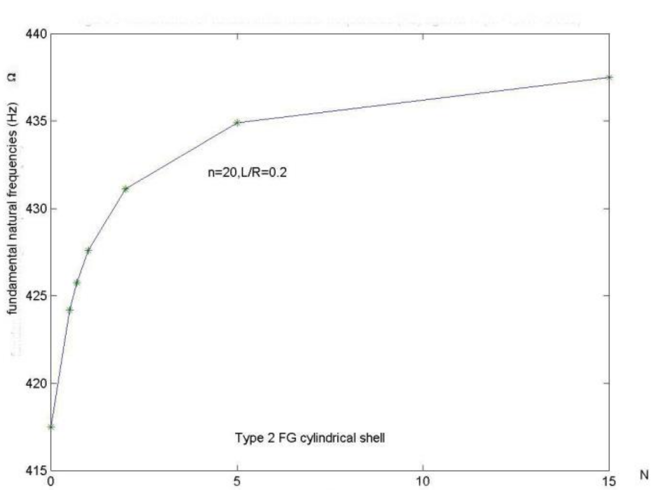

figure 6-46:variation of fundamental natural frequencies (Hz) against N (m=1,h/R=0.002)

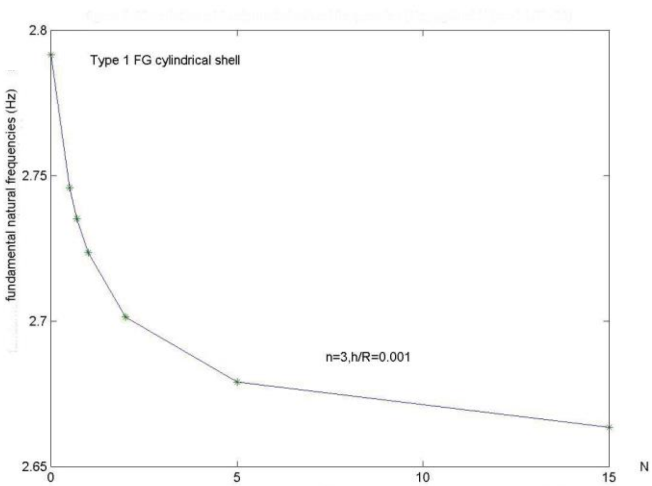

figure 6-55:variation of fundamental natural frequencies (Hz) against N (m=1,L/R=20)

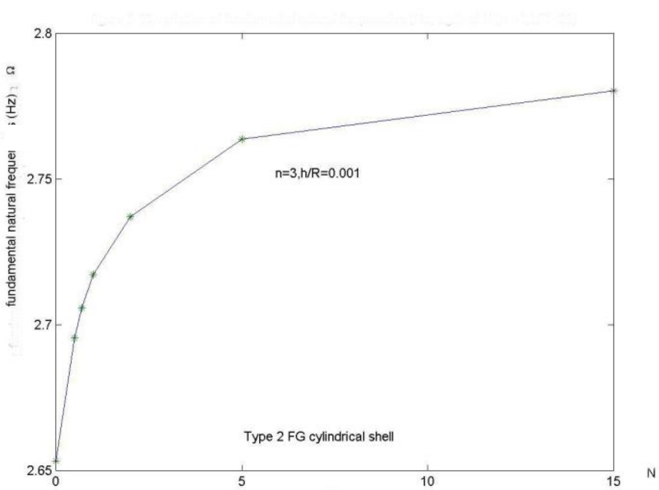

figure 6-63:variation of fundamental natural frequencies (Hz) against N (m=1,L/R=20)

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Appendix 1

Substituting Eq. (53) into Eqs. (21) to (25) gives

$$\epsilon_{11}^0 = \frac{\partial u_1}{\partial x}$$
$$\epsilon_{22}^0 = \frac{1}{a}\frac{\partial u_2}{\partial \theta} + \frac{u_3}{a}$$
$$\epsilon_{12}^0 = \frac{\partial u_2}{\partial x} + \frac{1}{a}\frac{\partial u_1}{\partial \theta}$$
$$k_{11} = \frac{\partial \phi_1}{\partial x}$$
$$k_{22} = \frac{1}{a}\frac{\partial \phi_2}{\partial \theta}$$
$$k_{12} = \frac{\partial \phi_2}{\partial x} + \frac{1}{a}\frac{\partial \phi_1}{\partial \theta}$$
$$k'_{11} = -C_1\frac{\partial \phi_1}{\partial x} - C_1\frac{\partial^2 u_3}{\partial x^2}$$
$$k'_{22} = \frac{C_1}{a^2}\frac{\partial u_2}{\partial \theta} - \frac{C_1}{a}\frac{\partial \phi_2}{\partial \theta} - \frac{C_1}{a}\frac{\partial^2 u_3}{\partial x^2}$$
$$k'_{12} = \frac{C_1}{a}\frac{\partial u_2}{\partial x} - C_1\frac{\partial \phi_2}{\partial x} - 2\frac{C_1}{a}\frac{\partial^2 u_3}{\partial x \partial \theta} - \frac{C_1}{a}\frac{\partial \phi}{\partial \theta}$$
$$\gamma_{13}^0 = \phi_1 + \frac{\partial u_3}{\partial x}$$
$$\gamma_{23}^0 = \phi_2 - \frac{u_2}{a} + \frac{1}{a}\frac{\partial u_3}{\partial \theta}$$
$$\gamma_{13}^2 = -3C_1\phi_1 - 3C_1\frac{\partial u_3}{\partial x}$$
$$\gamma_{23}^2 = 3\frac{C_1}{a}u_2 - 3C_1\phi_2 - 3\frac{C_1}{a}\frac{\partial u_3}{\partial \theta}$$
$$\gamma_{13}^3 = 0$$
$$\gamma_{23}^3 = -\frac{C_1}{a^2}u_2 + \frac{C_1}{a}\phi_2 + \frac{C_1}{a^2}\frac{\partial u_3}{\partial \theta}$$

Appendix 2

The stress resultants are expressed as

$$N_{11} = A_{11}\frac{\partial u_1}{\partial x} + B_{11}\frac{\partial \phi_1}{\partial x} - E_{11}C_1\frac{\partial \phi_1}{\partial x} - E_{11}C_1\frac{\partial^2 u_3}{\partial x^2} + A_{12}\frac{1}{a}\frac{\partial u_2}{\partial \theta} + A_{12}\frac{1}{a}u_3 + B_{12}\frac{1}{a}\frac{\partial \phi_2}{\partial \theta}$$

$$N_{22}=A_{12}\frac{\partial u_1}{\partial x}+B_{12}\frac{\partial \phi_1}{\partial x}+B_{22}\frac{1}{a}\frac{\partial \phi_2}{\partial \theta}+E_{12}\frac{C_1}{a^2}\frac{\partial u_2}{\partial \theta}-E_{12}\frac{C_1}{a}\frac{\partial \phi_2}{\partial \theta}-E_{12}\frac{C_1}{a}\frac{\partial^2 u_3}{\partial x^2}$$

$$-E_{22}\frac{C_1}{a}\frac{\partial \phi_2}{\partial \theta}+E_{22}\frac{C_1}{a^2}\frac{\partial u_2}{\partial \theta}-E_{12}C_1\frac{\partial \phi_1}{\partial x}-E_{12}C_1\frac{\partial^2 u_3}{\partial x^2}+A_{22}\frac{1}{a}\frac{\partial u_2}{\partial \theta}+A_{22}\frac{1}{a}u_3+$$

$$N_{12}=A_{66}\frac{\partial u_2}{\partial x}+A_{66}\frac{1}{a}\frac{\partial u_1}{\partial \theta}+B_{66}\frac{\partial \phi_2}{\partial x}+B_{66}\frac{1}{a}\frac{\partial \phi_1}{\partial \theta}-E_{22}\frac{C_1}{a}\frac{\partial^2 u_3}{\partial x^2}+E_{66}\frac{C_1}{a}\frac{\partial u_2}{\partial x}-E_{66}C_1\frac{\partial \phi_2}{\partial x}-$$

$$-2E_{66}\frac{C_1}{a}\frac{\partial^2 u_3}{\partial x\partial \theta}-E_{66}\frac{C_1}{a}\frac{\partial \phi_1}{\partial \theta}$$

$$M_{11}=B_{11}\frac{\partial u_1}{\partial x}+D_{11}\frac{\partial \phi_1}{\partial x}-F_{11}C_1\frac{\partial \phi_1}{\partial x}-F_{11}C_1\frac{\partial^2 u_3}{\partial x^2}+B_{12}\frac{1}{a}\frac{\partial u_2}{\partial \theta}+B_{12}\frac{1}{a}u_3+D_{12}\frac{1}{a}\frac{\partial \phi_2}{\partial \theta}$$

$$+F_{12}\frac{C_1}{a^2}\frac{\partial u_2}{\partial \theta}-F_{12}\frac{C_1}{a}\frac{\partial \phi_2}{\partial \theta}-F_{12}\frac{C_1}{a}\frac{\partial^2 u_3}{\partial x^2}$$

$$M_{22}=B_{12}\frac{\partial u_1}{\partial x}+D_{12}\frac{\partial \phi_1}{\partial x}-F_{12}C_1\frac{\partial \phi_1}{\partial x}-F_{12}C_1\frac{\partial^2 u_3}{\partial x^2}+B_{22}\frac{1}{a}\frac{\partial u_2}{\partial \theta}+B_{22}\frac{1}{a}u_3$$

$$+D_{22}\frac{1}{a}\frac{\partial \phi_2}{\partial \theta}+F_{22}\frac{C_1}{a^2}\frac{\partial u_2}{\partial \theta}-F_{22}\frac{C_1}{a}\frac{\partial \phi_2}{\partial \theta}-F_{22}\frac{C_1}{a}\frac{\partial^2 u_3}{\partial x^2}$$

$$M_{12}=B_{66}\frac{\partial u_2}{\partial x}+B_{66}\frac{1}{a}\frac{\partial u_1}{\partial \theta}+D_{66}\frac{\partial \phi_2}{\partial x}+D_{66}\frac{1}{a}\frac{\partial \phi_1}{\partial \theta}+F_{66}\frac{C_1}{a}\frac{\partial u_2}{\partial x}-$$

$$-F_{66}C_1\frac{\partial \phi_2}{\partial x}-2F_{66}\frac{C_1}{a}\frac{\partial^2 u_3}{\partial x\partial \theta}-F_{66}\frac{C_1}{a}\frac{\partial \phi_1}{\partial \theta}$$

$$P_{11}=E_{11}\frac{\partial u_1}{\partial x}+F_{11}\frac{\partial \phi_1}{\partial x}-H_{11}C_1\frac{\partial \phi_1}{\partial x}-H_{11}C_1\frac{\partial^2 u_3}{\partial x^2}+E_{12}\frac{1}{a}\frac{\partial u_2}{\partial \theta}+E_{12}\frac{u_3}{a}+$$

$$+F_{12}\frac{1}{a}\frac{\partial \phi_2}{\partial \theta}+H_{12}\frac{C_1}{a^2}\frac{\partial u_2}{\partial \theta}-H_{12}\frac{C_1}{a}\frac{\partial \phi_2}{\partial \theta}-H_{12}\frac{C_1}{a}\frac{\partial^2 u_3}{\partial x^2}$$

$$P_{22}=E_{12}\frac{\partial u_1}{\partial x}+F_{12}\frac{\partial \phi_1}{\partial x}-H_{12}C_1\frac{\partial \phi_1}{\partial x}-H_{12}C_1\frac{\partial^2 u_3}{\partial x^2}+E_{22}\frac{1}{\sigma}\frac{\partial u_2}{\partial \theta}+E_{22}\frac{u_3}{a}+$$

$$+F_{22}\frac{1}{a}\frac{\partial \phi_2}{\partial \theta}+H_{22}\frac{C_1}{a^2}\frac{\partial u_2}{\partial \theta}-H_{22}\frac{C_1}{a}\frac{\partial \phi_2}{\partial \theta}-H_{22}\frac{C_1}{a}\frac{\partial^2 u_3}{\partial x^2}$$

$$P_{12}=E_{66}\frac{\partial u_2}{\partial x}+E_{66}\frac{1}{a}\frac{\partial u_1}{\partial \theta}+F_{66}\frac{\partial \phi_2}{\partial x}+F_{66}\frac{1}{a}\frac{\partial \phi_1}{\partial \theta}+H_{66}\frac{C_1}{a}\frac{\partial u_2}{\partial x}-$$

$$-H_{66}C_1\frac{\partial \phi_2}{\partial x}-2H_{66}\frac{C_1}{a}\frac{\partial^2 u_3}{\partial x\partial \theta}-H_{66}\frac{C_1}{a}\frac{\partial \phi_1}{\partial \theta}$$

$$P_{13}=E_{55}\phi_1+E_{55}\frac{\partial u_3}{\partial x}-3G_{55}C_1\phi_1-3G_{55}C_1\frac{\partial u_3}{\partial x}$$

$$P_{23}=E_{44}\phi_2-E_{44}\frac{u_2}{a}+E_{44}\frac{1}{a}\frac{\partial u_3}{\partial \theta}+3G_{44}\frac{C_1}{a}u_2-3G_{44}C_1\phi_2-3G_{44}\frac{C_1}{a}\frac{\partial u_3}{\partial \theta}$$

$$-H_{44}\frac{C_1}{a^2}u_2+H_{44}\frac{C_1}{a}\phi_2+H_{44}\frac{C_1}{a^2}\frac{\partial u_3}{\partial \theta}$$

$$R_{13} = D_{55}\phi_1 + D_{55}\frac{\partial u_3}{\partial x} - 3F_{55}C_1\phi_1 - 3F_{55}C_1\frac{\partial u_3}{\partial x}$$

$$R_{23} = D_{44}\phi_2 - D_{44}\frac{u_2}{a} + D_{44}\frac{1}{a}\frac{\partial u_3}{\partial \theta} + 3F_{44}\frac{C_1}{a}u_2 - 3F_{44}C_1\phi_2 - 3F_{44}\frac{C_1}{a}\frac{\partial u_3}{\partial \theta}$$

$$-G_{44}\frac{C_1}{a^2}u_2 + G_{44}\frac{C_1}{a}\phi_2 + G_{44}\frac{C_1}{a^2}\frac{\partial u_3}{\partial \theta}$$

$$Q_{13} = A_{55}\phi_1 + A_{55}\frac{\partial u_3}{\partial x} - 3D_{55}C_1\phi_1 - 3D_{55}C_1\frac{\partial u_3}{\partial x}$$

$$Q_{23} = A_{44}\phi_2 - A_{44}\frac{u_2}{a} + A_{44}\frac{1}{a}\frac{\partial u_3}{\partial \theta} + 3D_{44}\frac{C_1}{a}u_2 - 3D_{44}C_1\phi_2 - 3D_{44}\frac{C_1}{a}\frac{\partial u_3}{\partial \theta}$$

$$-E_{44}\frac{C_1}{a^2}u_2 + E_{44}\frac{C_1}{a}\phi_2 + E_{44}\frac{C_1}{a^2}\frac{\partial u_3}{\partial \theta}$$

Appendix 3

In Eq. (63) are defined as follows: C_{ij} The coefficients

$$C_{11} = -\frac{m^2\pi^2 a}{l^2}A_{11} - \frac{n^2}{a}A_{66}$$

$$C_{12} = \frac{mn\pi}{l}A_{12} + \frac{mn\pi C_1}{al}E_{12} + \frac{mn\pi}{l}A_{66} + \frac{mn\pi C_1}{al}E_{66}$$

$$C_{13} = \frac{m^3\pi^3 aC_1}{l^3}E_{11} + \frac{m\pi}{l}A_{12} + \frac{m^3\pi^3 C_1}{l^3}E_{12} + 2\frac{mn^2\pi C_1}{al}E_{66}$$

$$C_{14} = -\frac{m^2\pi^2 a}{l^2}B_{11} + \frac{m^2\pi^2 aC_1}{l^2}E_{11} - \frac{n^2}{a}B_{66} + \frac{n^2 C_1}{a}E_{66}$$

$$C_{15} = \frac{mn\pi}{l}B_{12} - \frac{mn\pi C_1}{l}E_{12} + \frac{mn\pi}{l}B_{66} - \frac{mn\pi C_1}{l}E_{66}$$

$$C_{21} = \frac{mn\pi}{l}A_{12} + \frac{mn\pi C_1}{al}E_{66}$$

$$C_{22} = -\frac{n^2}{a}A_{22} - \frac{n^2 C_1}{a^2}E_{22} - \frac{m^2\pi^2 C_1}{l^2}E_{66} - \frac{m^2\pi^2 C_1^2}{al^2}H_{66} - \frac{1}{a}A_{44} + 6\frac{C_1}{a}D_{44} - \frac{C_1}{a^2}E_{44} -$$

$$-9\frac{C_1^2}{a}F_{44} + 3\frac{C_1^2}{a^2}G_{44} - \frac{C_1}{a}E_{44} + 3\frac{C_1^2}{a}G_{44} - \frac{C_1^2}{a^2}H_{44}$$

$$C_{23} = -\frac{m^2 n\pi^2 C_1}{l^2}E_{12} - \frac{n}{a}A_{22} - \frac{m^2 n\pi^2 C_1}{al^2}E_{22} - 2\frac{m^2 n\pi^2 C_1^2}{al^2}H_{66} - \frac{n}{a}A_{44} + 6\frac{nC_1}{a}D_{44}$$

$$-\frac{nC_1}{a^2}E_{44} - 9\frac{nC_1^2}{a}F_{44} + 3\frac{nC_1^2}{a^2}G_{44} - \frac{nC_1}{a}E_{44} + 3\frac{nC_1^2}{a}G_{44} - \frac{nC_1^2}{a^2}H_{44}$$

$$C_{24} = \frac{mn\pi}{l}B_{12} - \frac{mn\pi C_1}{l}E_{12} + \frac{mn\pi C_1}{al}F_{66} - \frac{mn\pi C_1^2}{al}H_{66}$$

$$C_{25} = -\frac{n^2}{a}B_{22} + \frac{n^2 C_1}{a}E_{22} - \frac{m^2\pi^2 C_1}{l^2}F_{66} + \frac{m^2\pi^2 C_1^2}{l^2}H_{66} + A_{44} - 6C_1 D_{44} + \frac{C_1}{a}E_{44} +$$

$$+9C_1^2F_{44}-3\frac{C_1^2}{a}G_{44}+C_1E_{44}-3C_1^2G_{44}+\frac{C_1^2}{a}H_{44}$$

$$C_{31}=-\frac{m^3\pi^3C_1a}{l^3}E_{11}-\frac{m\pi}{l}A_{22}-\frac{mn^2\pi C_1}{al}E_{12}-\frac{2mn^2\pi C_1}{al}E_{66}$$

$$C_{32}=\frac{m^2n\pi^2C_1}{l^2}E_{12}+\frac{m^2n\pi^2C_1^2}{al^2}H_{12}+\frac{n}{a}A_{22}+\frac{nC_1}{a^2}E_{22}+\frac{n^3C_1}{a^2}E_{22}+\frac{n^3C_1^2}{a^3}H_{22}+$$

$$+2\frac{m^2n\pi^2C_1}{l^2}E_{66}+2\frac{m^2n\pi^2C_1^2}{al^2}H_{66}+\frac{n}{a}A_{44}-6\frac{nC_1}{a}D_{44}+2\frac{nC_1}{a^2}E_{44}+$$

$$+9\frac{nC_1^2}{a}F_{44}-6\frac{nC_1^2}{a^2}G_{44}+\frac{nC_1^2}{a^3}H_{44}$$

$$C_{33}=\frac{m^4\pi^4C_1^2a}{l^4}H_{11}+\frac{m^2\pi^2C_1}{l^2}E_{12}+\frac{m^4\pi^4C_1^2}{l^4}H_{12}+\frac{m^2\pi^2C_1}{l^2}E_{22}+\frac{1}{a}A_{22}+$$

$$+\frac{m^2\pi^2C_1}{al^2}E_{22}+\frac{m^2n^2\pi^2C_1^2}{al^2}H_{12}+\frac{n^2C_1}{a^2}E_{22}+\frac{m^2n^2\pi^2C_1^2}{a^2l^2}H_{22}+4\frac{m^2\pi^2C_1^2}{al^2}H_{66}+$$

$$+\frac{m^2\pi^2a}{l^2}A_{55}-6\frac{m^2n^2C_1a}{l^2}D_{55}+9\frac{m^2\pi^2C_1^2a}{l^2}F_{55}+\frac{n^2}{a}A_{44}-6\frac{n^2C_1}{a}D_{44}+2\frac{n^2C_1}{a^2}E_{44}+$$

$$+9\frac{n^2c_1^2}{a}F_{44}-3\frac{n^2C_1^2}{a}G_{44}-3\frac{n^2C_1^2}{a^2}G_{44}+\frac{n^2C_1^2}{a^3}H_{44}$$

$$C_{34}=-\frac{m^3\pi^3C_1a}{l^3}F_{11}+\frac{m^3\pi^3C_1^2a}{l^3}H_{11}-\frac{m\pi}{l}B_{22}+\frac{m\pi C_1}{l}E_{22}-\frac{mn^2\pi C_1}{al}F_{12}+$$

$$+\frac{mn^2\pi C_1^2}{al}H_{12}-2\frac{mn^2\pi C_1}{al}F_{66}+2\frac{mn^2\pi C_1^2}{al}H_{66}+\frac{m\pi a}{l}A_{55}-6\frac{m\pi C_1a}{l}D_{55}+9\frac{m\pi C_1^2a}{l}F_{55}$$

$$C_{35}=\frac{m^2n\pi^2C_1}{l^2}F_{12}-\frac{m^2n\pi^2C_1^2}{l^2}H_{12}+\frac{n}{a}B_{22}-\frac{nC_1}{a}E_{22}+\frac{n^3C_1}{a^2}F_{22}-\frac{n^3C_1^2}{a}H_{22}$$

$$+2\frac{m^2n\pi^2C_1}{l^2}F_{66}-2\frac{m^2n\pi^2C_1^2}{l^2}H_{66}-nA_{44}+6nC_1D_{44}-\frac{nC_1}{a}E_{44}-9nC_1^2F_{44}$$

$$+6\frac{nC_1^2}{a}G_{44}-\frac{nC_1}{a}E_{44}-\frac{nC_1^2}{a^2}H_{44}$$

$$C_{41}=\frac{m^2\pi^2a}{l^2}B_{11}-\frac{m^2\pi^2C_1a}{l^2}E_{11}+\frac{n^2}{a}B_{66}+\frac{n^2}{a}D_{66}-\frac{n^2C_1}{a}E_{66}$$

$$C_{42}=-\frac{mn\pi}{l}B_{12}-\frac{mn\pi C_1}{al}F_{12}+\frac{mn\pi C_1}{l}E_{12}+\frac{mn\pi C_1^2}{al}H_{12}-\frac{mn\pi}{l}B_{66}-$$

$$-\frac{mn\pi C_1}{al}F_{66}+\frac{mn\pi C_1}{l}E_{66}+\frac{mn\pi C_1^2}{al}H_{66}$$

$$C_{43}=-\frac{m^3\pi^3C_1a}{l^3}F_{11}-\frac{m\pi}{l}B_{12}-\frac{m^3\pi^3C_1}{l^3}F_{12}+\frac{m^3\pi^3C_1^2a}{l^3}H_{11}+\frac{m\pi C_1}{l}E_{12}+$$

$$+\frac{m^3\pi^3C_1^2}{l^3}H_{12}-2\frac{mn^2\pi C_1}{al}F_{66}+2\frac{mn^2\pi C_1^2}{al}H_{66}-6\frac{m\pi C_1a}{l}D_{55}+9\frac{m\pi C_1^2}{l}F_{55}+\frac{m\pi a}{l}A_{55}$$

$$C_{44}=\frac{m^2\pi^2a}{l^2}D_{11}-2\frac{m^2\pi^2C_1a}{l^2}F_{11}+\frac{m^2\pi^2C_1^2a}{l^2}H_{11}-2\frac{n^2C_1}{a}F_{66}+\frac{n^2C_1^2}{a}H_{66}-$$
$$-6C_1aD_{55}+9C_1^2aF_{55}+A_{55}a$$
$$C_{45}=-\frac{mn\pi}{l}D_{12}+2\frac{mn\pi C_1}{l}F_{12}-\frac{mn\pi C_1^2}{l}H_{12}-2\frac{mn\pi C_1}{l}F_{66}-\frac{mn\pi C_1^2}{l}H_{66}$$
$$C_{51}=-\frac{mn\pi}{l}B_{12}-\frac{mn\pi C_1}{l}E_{12}-\frac{mn\pi}{l}B_{66}+\frac{mn\pi C_1}{l}E_{66}$$
$$C_{52}=\frac{n^2}{a}B_{22}+\frac{n^2C_1}{a^2}F_{22}+\frac{n^2C_1}{a}E_{22}+\frac{n^2C_1^2}{a^2}H_{22}+\frac{m^2\pi^2a}{l^2}B_{66}+\frac{m^2\pi^2C_1}{l^2}F_{66}-$$
$$-\frac{m^2n^2C_1a}{l^2}E_{66}-\frac{m^2\pi^2C_1^2}{l^2}H_{66}+6C_1D_{44}-9C_1^2F_{44}+3\frac{C_1^2}{a}G_{44}-A_{44}-\frac{C_1}{a}E_{44}-$$
$$-\frac{C_1}{a}D_{44}+3\frac{C_1^2}{a}F_{44}-\frac{C_1^2}{a^2}G_{44}$$
$$C_{53}=\frac{m^2n\pi^2C_1}{l^2}F_{12}+2\frac{m^2n\pi^2C_1}{l^2}F_{66}-2\frac{m^2n\pi^2C_1^2}{l^2}H_{66}+6nC_1D_{44}-9nC_1^2F_{44}$$
$$+3\frac{nC_1^2}{a}G_{44}-nA_{44}-\frac{nC_1}{a}E_{44}-\frac{nC_1}{a}D_{44}+3\frac{nC_1^2}{a}F_{44}-\frac{nC_1^2}{a^2}G_{44}$$
$$C_{54}=-\frac{mn\pi}{l}D_{12}+\frac{mn\pi C_1^2}{l}H_{12}-\frac{mn\pi}{l}D_{66}+2\frac{mn\pi C_1}{l}F_{66}-\frac{mn\pi C_1^2}{l}H_{66}$$
$$C_{55}=\frac{n^2}{a}D_{22}-\frac{n^2C_1^2}{a}H_{22}+\frac{m^2\pi^2a}{l^2}D_{66}-2\frac{m^2\pi^2C_1a}{l^2}F_{66}+\frac{m^2\pi^2C_1a}{l^2}H_{66}-$$
$$-6C_1aD_{44}+9C_1^2aF_{44}-3C_1^2G_{44}+aA_{44}+C_1E_{44}+C_1D_{44}-3C_1^2F_{44}+\frac{C_1^2}{a}G_{44}$$

Appendix 4

In Eq. (64) are defined as follows: M_{ij} The coefficients

$$m_{11}=-I_0$$
$$m_{12}=0$$
$$m_{13}=\frac{m\pi C_1}{l}I_3$$
$$m_{14}=I_1-C_1I_3$$
$$m_{15}=0$$
$$m_{21}=0$$
$$m_{22}=-I_0-2\frac{C_1}{a}I_1-\frac{C_1^2}{a^2}I_6$$
$$m_{23}=\frac{nC_1}{a}I_3-\frac{nC_1^2}{a^2}I_6$$
$$m_{24}=0$$

$$m_{25}=-I_1+C_1I_3-\frac{C_1}{a}I_4+\frac{C_1^2}{a}I_6$$

$$m_{31}=\frac{m\pi C_1}{l}I_3$$

$$m_{32}=-\frac{nC_1}{a}I_3$$

$$m_{33}=-I_0+(-\frac{n^2C_1^2}{a}-\frac{m^2\pi^2C_1^2}{l^2})I_6$$

$$m_{34}=\frac{m\pi C_1}{l}I_4-\frac{m\pi C_1}{l}I_6$$

$$m_{35}=\frac{nC_1}{a}I_4-\frac{nC_1^2}{a^2}I_6$$

$$m_{41}=C_1I_3$$

$$m_{42}=0$$

$$m_{43}=-I_1+\frac{m\pi C_1}{l}I_4-\frac{m\pi C_1^2}{l}I_6$$

$$m_{44}=-I_2+2C_1I_4-C_1^2I_6$$

$$m_{45}=0$$

$$m_{51}=0$$

$$m_{52}=-I_1+C_1I_3-\frac{C_1}{a}I_4-\frac{C_1^2}{a}I_6$$

$$m_{54}=0$$

$$m_{55}=-I_2+2C_1I_4-C_1^2I_6$$