Compact composition operators on real Banach spaces of complex-valued bounded Lipschitz functions

D. Alimohammadi, S. Sedgar

*Department of Mathematics, Faculty of Science, Arak University, Arak, 38156-8-8349, Iran.

Received 28 July 2014; Revised 13 September 2014; Accepted 29 September 2014.

Abstract. We characterize compact composition operators on real Banach spaces of complex-valued bounded Lipschitz functions on metric spaces, not necessarily compact, with Lipschitz involutions and determine their spectra.

© 2014 IAUCTB. All rights reserved.

Keywords: Compact operator, composition operator, Lipschitz function, Lipschitz involution, spectrum of an operator.

2010 AMS Subject Classification: Primary 46J10, 47B48; Secondary 46J15.

1. Introduction and Preliminaries

The symbol \( \mathbb{K} \) denotes a field that can be either \( \mathbb{R} \) or \( \mathbb{C} \). Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Banach spaces over \( \mathbb{K} \). We denote by \( BL_{\mathbb{K}}(\mathcal{X}, \mathcal{Y}) \) the Banach space of all bounded linear operators from \( \mathcal{X} \) into \( \mathcal{Y} \) over \( \mathbb{K} \) with the operator norm. Let us recall that \( T \in BL_{\mathbb{K}}(\mathcal{X}, \mathcal{Y}) \) is compact if the closure of \( T(E) \) is compact in \( \mathcal{Y} \) whenever \( E \) is a bounded set in \( \mathcal{X} \).

It is known that if \( \mathcal{X}, \mathcal{Y} \) and \( \mathcal{Z} \) are Banach spaces over \( \mathbb{K} \) and \( S \in BL_{\mathbb{K}}(\mathcal{X}, \mathcal{Y}) \) and \( T \in BL_{\mathbb{K}}(\mathcal{Y}, \mathcal{Z}) \), then \( T \circ S \) is compact if \( S \) or \( T \) is compact.

Let \( \mathcal{X} \) be a Banach space over \( \mathbb{K} \). Then \( BL_{\mathbb{K}}(\mathcal{X}, \mathcal{X}) \) is a unital Banach algebra over \( \mathbb{K} \) when \( ST = S \circ T \) for all \( S, T \in BL_{\mathbb{K}}(\mathcal{X}, \mathcal{X}) \). For \( T \in BL_{\mathbb{K}}(\mathcal{X}, \mathcal{X}) \), the spectrum of \( T \) is

*Corresponding author.
E-mail address: d-alimohammadi@araku.ac.ir (D. Alimohammadi).
denoted by $\sigma(T)$ and defined by
\[ \sigma(T) = \{ \lambda \in \mathbb{K} : \lambda I_X - T \text{ is not invertible in } BL_K(X, X) \}, \]
where $I_X : X \to X$ is the identity operator on $X$.

Let $X$ be a nonempty set, $V_K(X)$ be a vector space over $K$ of $K$-valued functions on $X$ and $\phi : X \to X$ be a map such that $f \circ \phi \in V_K(X)$ for all $f \in V_K(X)$. Then $C_{\phi, V_K}(X) : V_K(X) \to V_K(X)$ defined by $C_{\phi, V_K}(f) = f \circ \phi$ is a linear operator on $V_K(X)$ which is called the composition operator induced by $\phi$ on $V_K(X)$.

Let $X$ be a topological space. We denote by $C^b_K(X)$ the set of all $K$-valued bounded continuous functions on $X$. Then $C^b_K(X)$ is a unital commutative Banach algebra over $K$ under the pointwise operations and with the uniform norm
\[ \| f \|_\infty = \sup\{|f(x)| : x \in X\} \quad (f \in C^b_K(X)). \]

We denote by $C_K(X)$ the algebra of all $K$-valued continuous functions on $X$. Clearly, $C^b_K(X) = C_K(X)$ whenever $X$ is compact. We write $C^b(X)$ and $C(X)$ instead of $C^b_K(X)$ and $C_K(X)$, respectively.

Let $(X, d)$ and $(Y, \rho)$ be metric spaces. A map $\phi : X \to Y$ is called a Lipschitz mapping from $(X, d)$ into $(Y, \rho)$ if there exists a constant $M \geq 0$ such that $\rho(\phi(x), \phi(y)) \leq M d(x, y)$ for all $x, y \in X$. A map $\phi : X \to Y$ is called supercontractive from $(X, d)$ into $(Y, \rho)$ if
\[ \lim_{d(x, y) \to 0} \frac{\rho(\phi(x), \phi(y))}{d(x, y)} = 0, \]
that is, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\frac{\rho(\phi(x), \phi(y))}{d(x, y)} < \varepsilon$ whenever $x, y \in X$ and $0 < d(x, y) < \delta$.

Let $(X, d)$ be a metric space. A function $f : X \to K$ is called a $K$-valued Lipschitz function on $(X, d)$ if $f$ is a Lipschitz mapping from $(X, d)$ into the Euclidean metric space $K$. For a $K$-valued Lipschitz function $f$ on $(X, d)$, the Lipschitz number of $f$ on $(X, d)$ is denoted by $L_{(X, d)}(f)$ and defined by
\[ L_{(X, d)}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y \right\}. \]

We denote by $Lip_K(X, d)$ the set of all $K$-valued bounded Lipschitz functions on $(X, d)$. Clearly, $Lip_K(X, d)$ is a subalgebra of $C^b_K(X)$ and $1_X \in Lip_K(X, d)$, where $1_X$ is the constant function with value 1 on $X$. Moreover, $Lip_K(X, d)$ with the norm
\[ \| f \|_{X, L} = \max\{\| f \|_X, L_{(X, d)}(f)\} \]
is a Banach space and with the norm
\[ \| f \|_{Lip(X, d)} = \| f \|_X + L_{(X, d)}(f) \]
is a unital commutative Banach algebra over $K$. Since
\[ \| f \|_{X, L} \leq \| f \|_{Lip(X, d)} \leq 2 \| f \|_{X, L} \]
for all \( f \in \text{Lip}_K(X, d) \), we deduce that \( \| \cdot \|_{X, L} \) and \( \| \cdot \|_{\text{Lip}_K(X, d)} \) are equivalent norms on \( \text{Lip}_K(X, d) \). The set of all \( f \in \text{Lip}_K(X, d) \) for which \( f \) is supercontractive on \( (X, d) \), is denoted by \( \text{lip}_K(X, d) \). Clearly, \( \text{lip}_K(X, d) \) is a subalgebra of \( \text{Lip}_K(X, d) \) and \( 1_X \in \text{lip}_K(X, d) \). Moreover, \( \text{lip}_K(X, d) \) is a closed set in \( (\text{Lip}_K(X, d), \| \cdot \|_{X, L}) \) and \( (\text{Lip}_K(X, d), \| \cdot \|_{\text{Lip}_K(X, d)}) \). So \( (\text{lip}_K(X, d), \| \cdot \|_{X, L}) \) is a Banach space and \( (\text{lip}_K(X, d), \| \cdot \|_{\text{Lip}_K(X, d)}) \) is a unital commutative Banach algebra over \( \mathbb{K} \). We write \( \text{Lip}(X, d) \) and \( \text{lip}(X, d) \) instead of \( \text{Lip}_K(X, d) \) and \( \text{lip}_K(X, d) \), respectively. These algebras were first introduced by Sherbert in [8, 9]. Note that, if \( \phi : X \rightarrow X \) is a Lipschitz mapping then \( f \circ \phi \in \text{Lip}_K(X, d) \) (\( f \circ \phi \in \text{lip}_K(X, d) \), respectively) for all \( f \in \text{Lip}_K(X, d) \) (\( \text{lip}_K(X, d) \), respectively).

Let \( (X, d) \) be a pointed metric space with the base point \( e \in X \). We denote by \( \text{Lip}_0(X, d) \) the set of all \( \mathbb{K} \)-valued Lipschitz functions \( f \) on \( X \) such that \( f(e) = 0 \). Clearly, \( \text{Lip}_0(X, d) \) is a linear subspace of \( C(X) \). Moreover, \( \text{Lip}_0(X, d) \) with the norm \( L_{(X, d)}(\cdot) \) is a Banach space over \( \mathbb{K} \). Note that if \( \phi : X \rightarrow X \) is a base point preserving Lipschitz mapping, then \( f \circ \phi \in \text{Lip}_0(X, d) \) for all \( f \in \text{Lip}_0(X, d) \). We write \( \text{Lip}_0(X, d) \) instead of \( \text{Lip}_0(X, d) \). For further general facts about Lipschitz spaces \( \text{Lip}_K(X, d) \) and \( \text{lip}_K(X, d) \), we refer to [10].

Kamowitz and Scheinberg [5] characterized compact endomorphisms of complex Lipschitz algebras on compact metric spaces and determined their spectra.

Jiménez-Vargas and Villegas-Vallecillos [4] characterized compact composition operators on Banach spaces of Lipschitz functions \( \text{Lip}_K(X, d) \) with the norm \( \| \cdot \|_{X, L} \), \( \text{lip}_K(X, d) \) with the norm \( \| \cdot \|_{X, L} \), \( \| \cdot \|_{\text{Lip}_K(X, d)} \), \( \| \cdot \|_{\text{lip}_K(X, d)} \) and determined the spectrum of compact composition operators on \( \text{Lip}_K(X, d) \) and \( \text{lip}_K(X, d) \), where \( (X, d) \) is a metric space, not necessarily compact.

Let \( X \) be a topological space. A self-map \( \tau : X \rightarrow X \) is called a topological involution on \( X \) if \( \tau \) is continuous and \( \tau(\tau(x)) = x \) for all \( x \in X \).

Let \( X \) be a topological space and \( \tau \) be a topological involution on \( X \). The map \( \sigma : C^b(X) \rightarrow C^b(X) \) defined by \( \sigma(f) = \overline{f} \circ \tau \) is an algebra involution on the complex algebra \( C^b(X) \), which is called the algebra involution induced by \( \tau \) on \( C^b(X) \). Note that \( \| \sigma(f) \|_X = \| f \|_X \) for all \( f \in C^b(X) \).

We now define

\[ C^b(X, \tau) = \{ f \in C^b(X) : \sigma(f) = f \} \]

Then \( C^b(X, \tau) \) is a unital self-adjoint uniformly closed real subalgebra of \( C^b(X) \), \( i_X \notin C^b(X, \tau) \) where \( i_X \) is the constant function with value \( i \) on \( X \), \( C^b(X) = C^b(X, \tau) \oplus i C^b(X, \tau) \) and

\[ \max\{\| f \|_X, \| g \|_X \} \leq \| f + ig \|_X \leq 2 \max\{\| f \|_X, \| g \|_X \}, \]

for all \( f, g \in C^b(X, \tau) \). Moreover, \( C^b(X, \tau) = C^b_b(X) \) if \( \tau \) is the identity map on \( X \). Note that if \( X \) is compact, then \( C^b(X, \tau) = C(X, \tau) \), where \( C(X, \tau) = \{ f \in C(X) : \overline{f} \circ \tau = f \} \). Real Banach algebra \( C(X, \tau) \) was defined explicitly by Kulkarni and Limaye in [6]. For further general facts about \( C(X, \tau) \) and its real subalgebras, we refer to [7].

In this part we introduce real Lipschitz spaces \( \text{Lip}(X, d, \tau) \), \( \text{lip}(X, d, \tau) \) and \( \text{Lip}_0(X, d, \tau) \).

**Definition 1.1** Let \( (X, d) \) be a metric space. A self-map \( \tau : X \rightarrow X \) is called a Lipschitz involution on \( (X, d) \) if \( \tau(\tau(x)) = x \) and \( \tau \) is a Lipschitz mapping from \( (X, d) \) into \( (X, d) \).

Note that if \( \tau \) is a Lipschitz involution on \( (X, d) \), then \( \tau \) is a topological involution on
$(X,d)$ and $C \geq 1$ whenever $d(\tau(x),\tau(y)) \leq Cd(x,y)$ for all $x,y \in X$.

Let $(X,d)$ be a metric space, $\tau$ be a Lipschitz involution on $(X,d)$ and $\sigma$ be the algebra involution induced by $\tau$ on $C^b(X)$. We can easily show that $\sigma(Lip(X,d)) = Lip(X,d)$, $\sigma(\lip(X,d)) = lip(X,d)$, $L_{(X,d)}(\sigma(f)) \leq CL_{(X,d)}(f)$ for all $f \in Lip(X,d)$ and $\|\sigma(f)\|_{X,L} \leq C\|f\|_{X,L}$ for all $f \in Lip(X,d)$, where $C \geq 1$ and $d(\tau(x),\tau(y)) \leq Cd(x,y)$ for all $x,y \in X$. We now define

$$\text{Lip}(X,d,\tau) := \{ f \in Lip(X,d) : \sigma(f) = f \} \text{ and } \text{lip}(X,d,\tau) := \{ f \in lip(X,d) : \sigma(f) = f \}.$$ 

In fact, $\text{Lip}(X,d,\tau) = Lip(X,d) \cap C^b(X,\tau)$ and $\text{lip}(X,d,\tau) = lip(X,d) \cap C^b(X,\tau)$.

In the following result, we give some properties of $\text{Lip}(X,d,\tau)$ and $\text{lip}(X,d,\tau)$.

**Theorem 1.2** Let $(X,d)$ be a metric space and $\tau$ be a Lipschitz involution on $(X,d)$. Suppose that $A = Lip(X,d,\tau)$ and $B = Lip(X,d)$ ($A = lip(X,d,\tau)$ and $B = lip(X,d)$, respectively). Then:

(i) $A$ is a self-adjoint real subalgebra of $C^b(X,\tau)$ and $B$, $1_X \in A$ and $i_X \notin A$.

(ii) $B = A \oplus iA$.

(iii) For all $f,g \in A$ we have

$$\max\{\|f\|_{X,L},\|g\|_{X,L}\} \leq C\|f + ig\|_{X,L} \leq 2C\max\{\|f\|_{X,L},\|g\|_{X,L}\},$$

where $C \geq 1$ and $d(\tau(x),\tau(y)) \leq Cd(x,y)$ for all $x,y \in X$.

(iv) $A$ is closed in $(B, \| \cdot \|_{X,L})$ and so $(A, \| \cdot \|_{X,L})$ is a real Banach space.

(v) $f \circ \phi \in A$ for all $f \in A$ whenever $\phi : X \to X$ is a Lipschitz mapping from $(X,d)$ into $(X,d)$ with $\phi \circ \tau = \tau \circ \phi$.

(vi) $A = Lip_R(X,d)$ ($A = lip_R(X,d)$, respectively), if $\tau$ is the identity map on $X$.

Note that $\text{lip}(X,d,\tau)$ is a real subalgebra of $\text{Lip}(X,d,\tau)$ and a closed set in $(\text{Lip}(X,d,\tau), \| \cdot \|_{X,L})$.

Real Lipschitz algebras $\text{Lip}(X,d,\tau)$ and $\text{lip}(X,d,\tau)$ were first introduced in [1], whenever $(X,d)$ is a compact metric space. In this case, Ebadian and Ostadbashi [3] characterized compact endomorphisms of real Lipschitz algebras $\text{Lip}(X,d,\tau)$ with the norm $\| \cdot \|_{\text{Lip}(X,d)}$ and determined their spectra.

Let $(X,d)$ be a pointed metric space with a base point $e \in X$, $\tau$ be a base point-preserving Lipschitz involution on $(X,d)$ and $\sigma$ be the algebra involution induced by $\tau$ on $C^b(X)$. Then $L_{(X,d)}(\sigma(f)) \leq CL_{(X,d)}(f)$ for all $f \in Lip_0(X,d)$, where $C \geq 1$ and $d(\tau(x),\tau(y)) \leq Cd(x,y)$ for all $x,y \in X$. Therefore, $\sigma(Lip_0(X,d)) = Lip_0(X,d)$. We now define

$$\text{Lip}_0(X,d,\tau) = \{ f \in Lip_0(X,d) : \sigma(f) = f \}.$$ 

In fact, $\text{Lip}_0(X,d,\tau) = Lip_0(X,d) \cap C(X,\tau)$.

In the following result, we give some properties of $\text{Lip}_0(X,d,\tau)$.

**Theorem 1.3** Let $(X,d)$ be a pointed metric space and $\tau$ be a base point preserving Lipschitz involution on $(X,d)$. Then:
(i) $\text{Lip}_0(X, d, \tau)$ is a self-adjoint real subspace of $C^b(X, \tau)$ and $\text{Lip}_0(X, d)$, $1_X \notin \text{Lip}_0(X, d, \tau)$ and $i_1 X \notin \text{Lip}_0(X, d, \tau)$.

(ii) $\text{Lip}_0(X, d) = \text{Lip}_0(X, d, \tau) \oplus i \text{Lip}_0(X, d, \tau)$.

(iii) For all $f, g \in \text{Lip}_0(X, d, \tau)$ we have

$$\max\{L_{(X,d)}(f), L_{(X,d)}(g)\} \leq CL_{(X,d)}(f + ig) \leq 2C\max\{L_{(X,d)}(f), L_{(X,d)}(g)\},$$

where $C > 1$ and $d(\tau(x), \tau(y)) \leq Cd(x, y)$ for all $x, y \in X$.

(iv) $\text{Lip}_0(X, d, \tau)$ is closed in $(\text{Lip}_0(X, d), L_{(X,d)}(\cdot))$ and so $\text{Lip}_0(X, d, \tau)$ with the norm $L_{(X,d)}(\cdot)$ is a real Banach space.

(v) $f \circ \phi \in \text{Lip}_0(X, d, \tau)$ for all $f \in \text{Lip}_0(X, d, \tau)$, whenever $\phi : X \rightarrow X$ is a base point preserving Lipschitz mapping from $(X, d)$ into $(X, d)$ with $\phi \circ \tau = \tau \circ \phi$.

(vi) $\text{Lip}_0(X, d, \tau) = \text{Lip}_{0,R}(X, d)$, if $\tau$ is the identity map on $X$.

In Section 2, we characterize compact composition operators on real Lipschitz spaces $\text{Lip}(X, d, \tau), \| \cdot \|_{X,L}$, $(\text{lip}(X, d, \tau), \| \cdot \|_{X,L})$ and $(\text{Lip}_0(X, d, \tau), L_{(X,d)}(\cdot))$ and in Section 3 we determine the spectrum of compact composition operators on real Lipschitz spaces $\text{Lip}(X, d, \tau), \| \cdot \|_{X,L}$ and $(\text{lip}(X, d, \tau), \| \cdot \|_{X,L})$, whenever $(X, d)$ is a metric space, not necessarily compact and $\tau$ is a Lipschitz involution on $(X, d)$. In fact, we extend basic results of [3] and [4].

2. Compact composition operators

Let $X$ be a real linear space. The complexification of $X$ is the complex linear space $X_C := X \oplus iX$ with addition and scalar multiplication defined by

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \quad (x_1, y_1, x_2, y_2 \in X).$$

$$(\alpha + i\beta)(x + iy) = (\alpha x - \beta y) + i(\beta x + \alpha y) \quad (\alpha, \beta \in \mathbb{R}, x, y \in X).$$

Let $(X, \| \cdot \|)$ be a real Banach space. By a modification of [2, Proposition I.13.3], there exists a norm $\| \cdot \|$ on $X_C$ such that $\|x + i0\| = \|x\|$ for all $x \in X$, and

$$\max\{|x|, |y|\} \leq \|x + iy\| \leq 2\max\{|x|, |y|\},$$

for all $x, y \in X$, and so $(X_C, \| \cdot \|)$ is a complex Banach space.

**Theorem 2.1** Let $(X, \| \cdot \|)$ be a real Banach space, $X_C$ be the complexification of $X$ and $\| \cdot \|$ be a norm on $X_C$ with $\|f\| = \|f\|$ for all $f \in X$ and $C$ be a positive constant satisfying

$$\max\{|f|, |g|\} \leq C\|f + ig\| \leq 2C\max\{|f|, |g|\},$$

for all $f, g \in X$. Let $T \in BL_0(X, \mathfrak{F})$ and $T : X_C \rightarrow X_C$ be the mapping defined by $T(f + ig) = Tf + iTg \quad (f, g \in X)$. Then:
\( T' \in BL_C(\mathfrak{X}_C, \mathfrak{X}_C) \) and \( \|T'\| \leq 2C\|T\| \).

(ii) \( T' \) is compact if and only if \( T \) is compact.

(iii) \( T' \) is invertible in \( BL_C(\mathfrak{X}_C, \mathfrak{X}_C) \) if and only if \( T \) is invertible in \( BL_{\mathfrak{X}}(\mathfrak{X}, \mathfrak{X}) \).

(iv) \( T' = I_{\mathfrak{X}_C} \) if and only if \( T = I_{\mathfrak{X}} \).

(v) \( \sigma(T') \cap \mathbb{R} = \sigma(T) \).

**Proof.** Clearly \( T' \) is a complex linear map from \( \mathfrak{X}_C \) into \( \mathfrak{X}_C \). Since

\[
\|T'(f + ig)\| = \|Tf + iTg\| \leq \|Tf\| + \|Tg\|
\]

for all \( f, g \in \mathfrak{X} \), we deduce that \( T' \in BL_C(\mathfrak{X}_C, \mathfrak{X}_C) \) and \( \|T'\| \leq 2C\|T\| \). Hence, (i) holds.

To prove (ii), we first assume that \( T' \) is compact. Let \( \{f_n\}_{n=1}^{\infty} \) be a bounded sequence in \( (\mathfrak{X}, \|\cdot\|) \). Since \( \|f_n\| = \|f_n\| \) for all \( n \in \mathbb{N} \), we deduce that \( \{f_n\}_{n=1}^{\infty} \) is a bounded sequence in \( (\mathfrak{X}_C, \|\cdot\|) \). The compactness of \( T' \) implies that there exists a subsequence \( \{f_{n_k}\}_{k=1}^{\infty} \) of \( \{f_n\}_{n=1}^{\infty} \) such that \( \{T'f_{n_k}\}_{k=1}^{\infty} \) is a Cauchy sequence in \( (\mathfrak{X}_C, \|\cdot\|) \). Since

\[
\|Tf_{n_j} - Tf_{n_k}\| = \|T'f_{n_j} - T'f_{n_k}\|
\]

for all \( j, k \in \mathbb{N} \), we conclude that \( \{Tf_{n_k}\}_{k=1}^{\infty} \) is a Cauchy sequence in \( (\mathfrak{X}, \|\cdot\|) \). The completeness of \( (\mathfrak{X}, \|\cdot\|) \) implies that \( \{Tf_{n_k}\}_{k=1}^{\infty} \) is convergence in \( (\mathfrak{X}, \|\cdot\|) \). Therefore, \( T \) is compact.

We now assume that \( T \) is compact. Let \( \{h_n\}_{n=1}^{\infty} \) be a bounded sequence in \( (\mathfrak{X}_C, \|\cdot\|) \). Since \( \mathfrak{X}_C = \mathfrak{X} \oplus i\mathfrak{X} \), there exists unique elements \( f_n, g_n \in \mathfrak{X} \) such that \( h_n = f_n + ig_n \) for all \( n \in \mathbb{N} \). Since

\[
\max\{\|f_n\|, \|g_n\|\} \leq C\|f_n + ig_n\|
\]

for all \( n \in \mathbb{N} \), we deduce that \( \{f_n\}_{n=1}^{\infty} \) and \( \{g_n\}_{n=1}^{\infty} \) are bounded sequences in \( (\mathfrak{X}, \|\cdot\|) \). The compactness of \( T \) implies that there exist strictly increasing functions \( p : \mathbb{N} \rightarrow \mathbb{N} \) and \( q : \mathbb{N} \rightarrow \mathbb{N} \) and elements \( f \) and \( g \) in \( \mathfrak{X} \) such that

\[
\lim_{k \rightarrow \infty} \|f_{p(k)} - f\| = 0, \quad \lim_{k \rightarrow \infty} \|g_{q(k)} - g\| = 0.
\]

For each \( k \in \mathbb{N} \), set \( n_k = q(p(k)) \). Clearly, \( \{f_{n_k}\}_{k=1}^{\infty} \) is a subsequence \( \{f_n\}_{n=1}^{\infty} \),

\[
\lim_{k \rightarrow \infty} \|Tf_{n_k} - f\| = 0, \quad \{g_{n_k}\}_{k=1}^{\infty} \text{ is a subsequence } \{g_n\}_{n=1}^{\infty} \text{ and } \lim_{k \rightarrow \infty} \|Tg_{n_k} - g\| = 0.
\]

Clearly \( \{h_{n_k}\}_{k=1}^{\infty} \) is a subsequence of \( \{h_n\}_{n=1}^{\infty} \), \( f + ig \in \mathfrak{X}_C \) and

\[
\|T'h_{n_k} - (f + ig)\| \leq 2 \max\{\|Tf_{n_k} - f\|, \|Tg_{n_k} - g\|\}
\]

for all \( k \in \mathbb{N} \). Thus, \( \lim_{k \rightarrow \infty} \|T'h_{n_k} - (f + ig)\| = 0. \) Therefore, \( T' \) is compact. Hence (ii)
holds.

To prove (iii), we first assume that \( T' \) is invertible in \( BL_C(X_c, X_C) \). Then there exists \((T')^{-1} \in BL_C(X_c, X_C)\) such that \( T' \circ (T')^{-1} = (T')^{-1} \circ T' = I_{X_C} \). We now define the maps \( \Psi_1 : X \longrightarrow X_C \) and \( P_1 : X_C \longrightarrow X \) by

\[
\Psi_1(f) = f + i0 \quad (\forall f \in X) \quad \text{and} \quad P_1(f + ig) = f \quad (\forall f, g \in X).
\]

We can easily show that

\[
\Psi_1 \in BL_R(X, X_C), \quad \|\Psi_1\| \leq 2C, \quad P_1 \in BL_R(X_C, X) \quad \text{and} \quad \|P_1\| \leq C.
\]

Moreover, \( \Psi_1 \circ T = T' \circ \Psi_1 \) and \( T \circ P_1 = P_1 \circ T' \). Now, we have

\[
(P_1 \circ (T')^{-1} \circ \Psi_1) \circ T = I_X = T \circ (P_1 \circ (T')^{-1} \circ \Psi_1).
\]

Therefore, \( T \) is invertible in \( BL_R(X, X) \) and \( T^{-1} = P_1 \circ (T')^{-1} \circ \Psi_1 \).

We now assume that \( T \) is invertible in \( BL_R(X, X) \). Then there exists \( T^{-1} \in BL_R(X, X) \) such that \( T \circ T^{-1} = T^{-1} \circ T = I_X \). We now define the map \((T^{-1})' : X_C \longrightarrow X_C\) by

\[
(T^{-1})'(f + ig) = T^{-1}f + iT^{-1}g \quad (\forall f, g \in X).
\]

Then \((T^{-1})' \in BL_C(X_c, X_C)\) and \(\|(T^{-1})'\| \leq 2C\|T^{-1}\|\). Moreover,

\[
(T^{-1})' \circ T' = T' \circ (T^{-1})' = I_{X_C}.
\]

Therefore, \( T' \) is invertible in \( BL_C(X_c, X_C) \) and \((T')^{-1} = (T^{-1})'\). Hence, (iii) holds.

The proof of (iv) is obvious. From (iii) and (iv), we deduce that (v) holds.

Compact composition operators on Lipschitz spaces \((Lip_R(X, d), || \cdot ||_{X,L})\) characterized in [4] as the following.

**Theorem 2.2** (see [4, Theorem 1.1]). Let \((X, d)\) be a metric space and let \( \phi : X \longrightarrow X \) be a Lipschitz mapping from \((X, d)\) into \((X, d)\). Then the composition operator \( C_{\phi, Lip_R(X, d)} : Lip_R(X, d) \longrightarrow Lip_R(X, d) \) is compact if and only if \( \phi \) is supercontrative and \( \phi(X) \) is totally bounded in \((X, d)\).

In the following result, we characterize compact composition operators on real lipschitz spaces \((Lip(X, d, \tau), || \cdot ||_{X,L})\).

**Theorem 2.3** Let \((X, d)\) be a metric space, \( \tau \) be a Lipschitz involution on \((X, d)\) and \( \phi : X \longrightarrow X \) be a Lipschitz mapping from \((X, d)\) into \((X, d)\) such that \( \phi \circ \tau = \tau \circ \phi \). Then the composition operator \( C_{\phi, Lip(X, d, \tau)} : Lip(X, d, \tau) \longrightarrow Lip(X, d, \tau) \) is compact if and only if \( \phi \) is supercontractive and \( \phi(X) \) is totally bounded in \((X, d)\).

**Proof.** Since \( \tau \) is a Lipschitz involution on \((X, d)\), by Theorem 1.2, we deduce that
Let $\text{Lip}(X, d) = \text{Lip}(X, d, \tau) \oplus i \text{Lip}(X, d, \tau)$, there exists a constant $C \geq 1$ such that

$$\max\{\|f\|_{X,L}, \|g\|_{X,L}\} \leq C\|f + ig\|_{X,L} \leq 2C\max\{\|f\|_{X,L}, \|g\|_{X,L}\},$$

for all $f, g \in \text{Lip}(X, d, \tau)$ and $\text{Lip}(X, d, \tau)$ is a real Banach space. Hence, by Theorem 2.1, the compactness of $C_{\phi, \text{Lip}(X,d,\tau)} : \text{Lip}(X, d, \tau) \rightarrow \text{Lip}(X, d, \tau)$ is equivalent to the compactness of $(C_{\phi, \text{Lip}(X,d,\tau)})' : \text{Lip}(X, d) \rightarrow \text{Lip}(X, d)$ which is defined by

$$(C_{\phi, \text{Lip}(X,d,\tau)})(f + ig) = C_{\phi, \text{Lip}(X,d,\tau)}(f) + iC_{\phi, \text{Lip}(X,d,\tau)}(g)$$

for all $f, g \in \text{Lip}(X, d, \tau)$.

Since

$$(C_{\phi, \text{Lip}(X,d,\tau)})'(f + ig) = (f \circ \phi) + i(g \circ \phi)$$

$$= (f + ig) \circ \phi$$

$$= C_{\phi, \text{Lip}(X,d)}(f + ig)$$

for all $f, g \in \text{Lip}(X, d, \tau)$, we conclude that

$$(C_{\phi, \text{Lip}(X,d,\tau)})' = C_{\phi, \text{Lip}(X,d)}.$$ 

Thus, the compactness of $C_{\phi, \text{Lip}(X,d,\tau)} : \text{Lip}(X, d, \tau) \rightarrow \text{Lip}(X, d, \tau)$ is equivalent to the compactness of $C_{\phi, \text{Lip}(X,d)} : \text{Lip}(X, d) \rightarrow \text{Lip}(X, d)$, and this is equivalent to $\phi$ is supercontractive from $(X, d)$ into $(X, d)$ and $\phi(X)$ is totally bounded in $(X, d)$ by Theorem 2.2. Hence, the proof is complete. 

Note that Theorem 2.3 is a generalization of Theorem 2.2, whenever $\mathbb{K} = \mathbb{R}$.

We now show that the class of real Lipschitz spaces $(\text{Lip}(Y, \rho), \|\cdot\|_{Y,L})$ is larger than the class of complex Lipschitz spaces $(\text{Lip}(X, d), \|\cdot\|_{X,L})$ regarded as real Lipschitz spaces (Theorem 2.4, below), and the class of compact composition operators on real Lipschitz spaces $(\text{Lip}(Y, \rho, \tau), \|\cdot\|_{Y,L})$ is larger than the class of compact composition operators on complex Lipschitz spaces $(\text{Lip}(X, d), \|\cdot\|_{X,L})$ (Theorem 2.5, below).

**Theorem 2.4** Let $(X, d)$ be a metric space. Suppose that $Y = X \times \{0,1\}$ and $\rho$ is the metric on $Y$ defined by

$$\rho((x_1, j_1), (x_2, j_2)) = \max\{d(x_1, x_2), |j_1 - j_2|\}.$$ 

Let $\tau : Y \rightarrow Y$ be the self-map on $Y$ defined by

$$\tau(x, 0) = (x, 1) \quad (x \in X), \quad \tau(x, 1) = (x, 0) \quad (x \in X).$$

Then:

(i) $\tau$ is a Lipschitz involution on $(Y, \rho)$. 

(ii) The map $\Lambda : \text{Lip}(X,d) \to \text{Lip}(Y,\rho,\tau)$ defined by

$$(\Lambda f)(x,0) = f(x) \quad (f \in \text{Lip}(X,d), x \in X),$$

$$(\Lambda f)(x,1) = \overline{f(x)} \quad (f \in \text{Lip}(X,d), x \in X),$$

is an injective bounded real-linear operator from $(\text{Lip}(X,d), \| \cdot \|_{X,L})$ regarded as a real Banach space onto $(\text{Lip}(Y,\rho,\tau), \| \cdot \|_{Y,L})$, satisfying

$$\|f\|_{X,L} \leq \|\Lambda f\|_{Y,L} \leq 2\|f\|_{X,L}$$

for all $f \in \text{Lip}(X,d)$.

**Proof.** Clearly, $\tau(\tau(x,j)) = (x,j)$ for all $(x,j) \in Y$, and

$$\rho(\tau(x_1,j_1), \tau(x_2,j_2)) = \rho((x_1,j_1), (x_2,j_2))$$

for all $(x_1,j_1), (x_2,j_2) \in Y$. Hence, (i) holds.

It is easy to see that $\Lambda$ is well-defined and a real-linear operator from $\text{Lip}(X,d)$, regarded a real Banach space, into $\text{Lip}(Y,\rho,\tau)$. Let $g \in \text{Lip}(Y,\rho,\tau)$. We define the function $f : X \to \mathbb{C}$ by $f(x) = g(x,0)$. Then $f \in C^b(X)$, $\|f\|_X \leq \|g\|_Y$ and $L_{(X,d)}(f) \leq L_{(Y,\rho)}(g)$. Hence, $f \in \text{Lip}(X,d)$. Moreover,

$$(\Lambda f)(x,0) = f(x) = g(x,0),$$

$$(\Lambda f)(x,1) = \overline{f(x)} = \overline{g(x,0)} = (g \circ \tau)(x,0)$$

$$= g(\tau(x,0)) = g(x,1)$$

for all $x \in X$. Therefore, $\Lambda(f) = g$ and so $\Lambda$ is onto.

Let $f \in \text{Lip}(X,d)$. Clearly, $\|f\|_X = \|\Lambda f\|_Y$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Then

$$|f(x_1) - f(x_2)| = |(\Lambda f)(x_1,0) - (\Lambda f)(x_2,0)|$$

$$\leq L_{(Y,\rho)}(\Lambda f)\rho((x_1,0), (x_2,0))$$

$$= L_{(Y,\rho)}(\Lambda f)d(x_1,x_2).$$

Hence, $L_{(X,d)}(f) \leq L_{(Y,\rho)}(\Lambda f)$. Therefore,

$$\|f\|_{X,L} \leq \|\Lambda f\|_{Y,L}.$$
Now, let \((x_1, j_1), (x_2, j_2) \in Y\) with \((x_1, j_1) \neq (x_2, j_2)\). If \(j_1 = j_2\), then
\[
|\Lambda f(x_1, j_1) - (\Lambda f)(x_2, j_2)| = |f(x_1) - f(x_2)|
\leq L_{(X,d)}(f)d(x_1, x_2)
\leq 2\|f\|_{X,L}(d(x_1, j_1), (x_2, j_2)),
\]
and if \(j_1 \neq j_2\), then
\[
|\Lambda f(x_1, j_1) - (\Lambda f)(x_2, j_2)| = |f(x_1) - f(x_2)|
\leq 2\|f\|_X|j_1 - j_2|
\leq 2\|f\|_{X,L}(d(x_1, j_1), (x_2, j_2)).
\]
Thus,
\[
L_{(Y,\rho)}(\Lambda f) \leq 2\|f\|_{X,L}.
\]
On the other hand, we have
\[
\|\Lambda f\|_Y = \|f\|_X \leq 2\|f\|_{X,L}.
\]
Therefore,
\[
\|\Lambda f\|_{Y,L} \leq 2\|f\|_{X,L}.
\]
Hence, \((ii)\) holds.

\begin{theorem}
Let \((X, d)\) be a metric space, \(Y = X \times \{0, 1\}\), \(\rho\) be the metric on \(Y\) defined by \(\rho((x_1, j_1), (x_2, j_2)) = \max\{d(x_1, x_2), |j_1 - j_2|\}\) and \(\tau\) be the Lipschitz involution on \((Y, \rho)\) defined by
\[
\tau(x, 0) = (x, 1), \quad \tau(x, 1) = (x, 0), \quad (x \in X).
\]
Let \(\phi : X \longrightarrow X\) be a Lipschitz mapping from \((X, d)\) into \((X, d)\) and let \(\psi : Y \longrightarrow Y\) be the self-map on \(Y\) defined by
\[
\psi(x, 0) = (\phi(x), 0), \quad \psi(x, 1) = (\phi(x), 1) \quad (x \in X),
\]
Then:

\begin{enumerate}
\item \(\psi\) is a Lipschitz mapping from \((Y, \rho)\) into \((Y, \rho)\) such that \(\psi \circ \tau = \tau \circ \psi\).
\item The composition operator \(C_{\phi, Lip((X, d))} : Lip(X, d) \longrightarrow Lip(X, d)\) is compact if and only if the composition operator \(C_{\psi, Lip((Y, \rho, \tau))} : Lip(Y, \rho, \tau) \longrightarrow Lip(Y, \rho, \tau)\) is compact.
\end{enumerate}
\end{theorem}
Proof. Clearly, (i) holds. Let \( \Lambda : \text{Lip}(X, d) \to \text{Lip}(Y, \rho, \tau) \) defined by

\[
(\Lambda f)(x, 0) = f(x), \quad (\Lambda f)(x, 1) = \overline{f(x)} \quad (x \in X).
\]

By Theorem 2.4, \( \Lambda \) is an injective bounded real-linear operator from \( \text{Lip}(X, d) \) with the norm \( \| \cdot \|_{X, L} \) regarded as a real Banach space, onto the real Banach space \( \text{Lip}(Y, \rho, \tau) \) with the norm \( \| \cdot \|_{Y, L} \). We can easily show that

\[
\Lambda \circ C_{\phi, \text{Lip}(X, d)} = C_{\psi, \text{Lip}(Y, \rho, \tau)} \circ \Lambda. \quad (1)
\]

According to \( \Lambda \in B\text{L}^d(L\text{ip}(X, d), L\text{ip}(Y, \rho, \tau)) \) and (1), we deduce that the operator \( C_{\phi, \text{Lip}(X, d)} : L\text{ip}(X, d) \to L\text{ip}(X, d) \) is compact if and only if \( C_{\psi, \text{Lip}(Y, \rho, \tau)} : L\text{ip}(Y, \rho, \tau) \to L\text{ip}(Y, \rho, \tau) \) is compact. Hence, (ii) holds.

According to Theorems 2.4 and 2.5, it is clear that Theorem 2.3 is also a generalization of Theorem 2.2, whenever \( K = \mathbb{C} \).

In [4], Jiménez-Vargas and Villegas-Vallecillos obtained the analogous result for compact composition operators on little Lipschitz spaces \( (\text{lip}_K(X, d), \| \cdot \|_{X, L}) \) that satisfy a kind of uniform separation property.

Definition 2.6 (see [4, Definition 1.1]). Let \((X, d)\) be a metric space, not assumed to be compact. It is said that a linear subspace \( M \) of \( \text{lip}_K(X, d) \) separates the points uniformly on bounded subsets of \( X \) if for each bounded set \( K \subseteq X \), there exists a constant \( a \geq 1 \) (which may depend on \( K \)) such that for every \( x, y \in K \), some \( f \in M \) satisfies \( \| f \|_{X, L} \leq a \) and \( |f(x) - f(y)| = d(x, y) \).

Note that \( \text{lip}_K(X, d) \) satisfies aforementioned uniform separation property when \((X, d)\) is uniformly discrete (that is, \( \inf \{ d(x, y) : x \neq y \} > 0 \)), or when \((X, d)\) is a totally disconnected metric space [10, Example 3.1.6].

Theorem 2.7 (see [4, Theorem 1.3]). Let \((X, d)\) be a metric space and \( \phi : X \to X \) be a bounded Lipschitz mapping from \((X, d)\) into \((X, d)\). Assume that \( \text{lip}_K(X, d) \) separates points uniformly on bounded subsets of \( X \). Then the composition operator \( C_{\phi, \text{lip}_K(X, d)} : \text{lip}_K(X, d) \to \text{lip}_K(X, d) \) is compact if and only if \( \phi \) is supercontractive and \( \phi(X) \) is totally bounded in \((X, d)\).

In the following result, we characterize compact composition operators on real little Lipschitz spaces \( (\text{lip}(X, d, \tau), \| \cdot \|_{X, L}) \) when \( \text{lip}(X, d) \) satisfies aforementioned uniform separation property.

Theorem 2.8 Let \((X, d)\) be a metric space, \( \tau \) be a Lipschitz involution on \((X, d)\) and \( \phi : X \to X \) be a Lipschitz mapping from \((X, d)\) into \((X, d)\) with \( \phi \circ \tau = \tau \circ \phi \). Suppose that \( \text{lip}(X, d) \) separates points uniformly on bounded subsets of \( X \). Then the composition operator \( C_{\phi, \text{lip}(X, d, \tau)} : \text{lip}(X, d, \tau) \to \text{lip}(X, d, \tau) \) is compact if and only if \( \phi \) is supercontractive and \( \phi(X) \) is totally bounded in \((X, d)\).
Proof. Since $\tau$ is a Lipschitz involution on $(X, d)$, by Theorem 1.2, we deduce that $\text{lip}(X, d) = \text{lip}(X, d, \tau) \oplus i\text{lip}(X, d, \tau)$, there exists a constant $C \geq 1$ such that
\[
\max\{\|f\|_{X,L}, \|g\|_{X,L}\} \leq C\|f + ig\|_{X,L} \leq 2C \max\{\|f\|_{X,L}, \|g\|_{X,L}\}
\]
for all $f, g \in \text{lip}(X, d, \tau)$ and $\text{lip}(X, d, \tau)$ with the norm $\| \cdot \|_{X,L}$ is a real Banach space. Hence, by Theorem 2.1, the compactness of the operator $C_{\phi, \text{lip}(X,d,\tau)} : \text{lip}(X, d, \tau) \to \text{lip}(X, d)$ is equivalent to the compactness of the operator $(C_{\phi, \text{lip}(X,d,\tau)})' : \text{lip}(X, d) \to \text{lip}(X, d)$ which is defined by
\[
(C_{\phi, \text{lip}(X,d,\tau)})'(f + ig) = C_{\phi, \text{lip}(X,d,\tau)}(f) + iC_{\phi, \text{lip}(X,d,\tau)}(g)
\]
for all $f, g \in \text{lip}(X, d, \tau)$. It is easy to see that
\[
(C_{\phi, \text{lip}(X,d,\tau)})' = C_{\phi, \text{lip}(X,d)}.
\]
Since $\text{lip}(X, d)$ separates the points uniformly on bounded subsets of $X$ and $\phi$ is a bounded Lipschitz mapping from $(X, d)$ into $(X, d)$, by Theorem 2.7, the compactness of $C_{\phi, \text{lip}(X,d)}$ is equivalent to $\phi$ is supercontractive and $\phi(X)$ is totally bounded in $(X, d)$. Hence, the proof is complete. ■

Note that Theorem 2.8 is a generalization of [4, Theorem 1.3] whenever $\mathbb{K} = \mathbb{R}$.

We now show that the class of real little Lipschitz space $\text{lip}(Y, \rho, \tau)$ with the norm $\| \cdot \|_{Y,L}$ is larger than the class of complex little Lipschitz spaces $\text{lip}(X, d)$ with the norm $\| \cdot \|_{X,L}$ regarded as real Lipschitz spaces (Theorem 2.9, below) and the class of compact composition operators on $(\text{lip}(Y, \rho, \tau), \| \cdot \|_{Y,L})$ is larger than the class of compact composition operators on $(\text{lip}(X, d), \| \cdot \|_{X,L})$ (Theorem 2.10, below).

**Theorem 2.9** Let $(X, d)$ be a metric space, $Y = X \times \{0, 1\}$, $\rho$ be the metric on $Y$ defined by $\rho((x_1, j_1), (x_2, j_2)) = \max\{d(x_1, x_2), |j_1 - j_2|\}$ and $\tau$ be the Lipschitz involution on $(Y, \rho)$ defined by
\[
\tau(x, 0) = (x, 1), \quad \tau(x, 1) = (x, 0) \quad (x \in X).
\]
Then the map $\Gamma : \text{lip}(X, d) \to \text{lip}(Y, \rho, \tau)$ defined by
\[
(\Gamma f)(x, 0) = f(x), \quad (\Gamma f)(x, 1) = \overline{f(x)} \quad (f \in \text{lip}(X, d), x \in X),
\]
is an injective real-linear operator from $(\text{lip}(X, d), \| \cdot \|_{X,L})$ regarded as a real Banach space onto $(\text{lip}(Y, \rho, \tau), \| \cdot \|_{Y,L})$ satisfying
\[
\|f\|_{X,L} \leq \|\Gamma f\|_{Y,L} \leq 2\|f\|_{X,L},
\]
for all $f \in \text{lip}(X, d)$. 


Proof. Let $\Lambda : Lip(X, d) \rightarrow Lip(Y, \rho, \tau)$ defined by
\[
(\Lambda f)(x, 0) = f(x), \quad (\Lambda f)(x, 1) = \overline{f(x)} \quad (f \in Lip(X, d), x \in X).
\]

By Theorem 2.4, $\Lambda$ is an injective bounded real-linear operator from $Lip(X, d)$ with the norm $\| \cdot \|_{X, L}$ regarded as a real Banach space onto $Lip(Y, \rho, \tau)$ with the norm $\| \cdot \|_{Y, L}$ satisfying
\[
\|f\|_{X, L} \leq \|\Lambda f\|_{Y, L} \leq 2\|f\|_{X, L}
\]
for all $f, g \in Lip(X, d)$. We claim that
\[
\Lambda(lip(X, d)) = lip(Y, \rho, \tau).
\] (2)

Let $f \in lip(X, d)$. Then $f \in Lip(X, d)$ and so $\Lambda f \in Lip(Y, \rho, \tau)$. Let $\varepsilon > 0$ be given. There exists $\delta_0 > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$, whenever $x_1, x_2 \in X$ and $0 < d(x_1, x_2) < \delta_0$.

Set $\delta = \min\{\delta_0, 1/2\}$. If $(x_1, j_1), (x_2, j_2) \in Y$ with $0 < \rho((x_1, j_1), (x_2, j_2))$, then $0 < d(x_1, x_2) < \delta_0$ and $j_1 = j_2$, so we have
\[
\frac{|(\Lambda f)(x_1, j_1) - (\Lambda f)(x_2, j_2)|}{\rho((x_1, j_1), (x_2, j_2))} = \frac{|f(x_1) - f(x_2)|}{d(x_1, x_2)} < \varepsilon.
\]

Thus, $\Lambda f \in lip(Y, \rho, \tau)$.

Now, let $g \in lip(Y, \rho, \tau)$. Then $g \in Lip(Y, \rho, \tau)$ and so there exists $f \in Lip(X, d)$ such that $\Lambda f = g$. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $|g(x_1, j_1) - g(x_2, j_2)| < \varepsilon$, whenever $(x_1, j_1), (x_2, j_2) \in Y$ and $0 < \rho((x_1, j_1), (x_2, j_2)) < \delta$.

If $x_1, x_2 \in X$ with $0 < d(x_1, x_2) < \delta$, then $(x_1, 0), (x_2, 0) \in Y$ with $0 < \rho((x_1, 0), (x_2, 0)) < \delta$, and so
\[
\frac{|f(x_1) - f(x_2)|}{d(x_1, x_2)} = \frac{|(\Lambda f)(x_1, 0) - (\Lambda f)(x_2, 0)|}{\rho((x_1, 0), (x_2, 0))} < \varepsilon.
\]

Thus, $f \in lip(X, d)$ implies that $g \in \Lambda(lip(X, d))$. Hence, our claim is justified.

From (2) and definitions of $\Gamma$ and $\Lambda$, we conclude that $\Gamma$ is well-defined and $\Gamma = \Lambda|_{lip(X, d)}$. According to (2) and the above mentioned properties of $\Lambda$, we conclude that $\Gamma$ satisfies the required conditions.

Theorem 2.10 Let $(X, d)$ be a metric space, $Y = X \times \{0, 1\}$, $\rho$ be the metric on $Y$ defined by $\rho((x_1, j_1), (x_2, j_2)) = \max\{d(x_1, x_2), |j_1 - j_2|\}$ and $\tau$ be the Lipschitz involution on $(Y, \rho)$ defined by
\[
\tau(x, 0) = (x, 1), \quad \tau(x, 1) = (x, 0) \quad (x \in X).
\]
Let $\phi$ be a bounded Lipschitz mapping from $(X, d)$ into $(X, d)$ and the map $\psi : Y \rightarrow Y$ defined by

$$
\psi(x, 0) = (\phi(x), 0), \quad \psi(x, 1) = (\phi(x), 1) \quad (x \in X),
$$

Then:

(i) $\psi$ is a bounded Lipschitz mapping from $(Y, \rho)$ into $(Y, \rho)$ such that $\psi \circ \tau = \tau \circ \psi$.

(ii) The composition operator $C_{\psi, \text{lip}} : \text{lip}(X, d) \rightarrow \text{lip}(X, d)$ is compact if and only if the composition operator $C_{\psi, \text{lip}(Y, \rho, \tau)} : \text{lip}(Y, \rho, \tau) \rightarrow \text{lip}(Y, \rho, \tau)$ is compact.

**Proof.** By part (i) of Theorem 2.5, $\psi$ is a Lipschitz mapping from $(Y, \rho)$ onto $(Y, \rho)$ such that $\psi = \psi$. Since $\phi$ is bounded, there exists $x_1 \in X$ and $\delta_1 > 0$ such that

$$
\phi(X) \subseteq \{x \in X : d(x, x_1) < \delta_1\}.
$$

We assume that $y_1 = (x_1, 0)$ and $\gamma_1 = 1 + \delta_1$. It is easy to see that

$$
\psi(Y) \subseteq \{y \in Y : \rho(y, y_1) < \gamma_1\}.
$$

Therefore, $\psi$ is bounded. Hence, (i) holds.

Let $\Gamma : \text{lip}(X, d) \rightarrow \text{lip}(Y, \rho, \tau)$ defined by

$$
(\Gamma f)(x, 0) = f(x), \quad (\Gamma f)(x, 1) = f(x) \quad (f \in \text{lip}(X, d), x \in X).
$$

By Theorem 2.9, $\Gamma$ is an injective bounded real linear operator from $\text{lip}(X, d)$ with the norm $\| \cdot \|_{X,L}$ regarded as a real Banach space onto real Banach space $\text{lip}(Y, \rho, \tau)$ with the norm $\| \cdot \|_{Y,L}$. We can easily show that

$$
\Gamma \circ C_{\phi, \text{lip}(X, d)} = C_{\psi, \text{lip}(Y, \rho, \tau)} \circ \Gamma. \quad (3)
$$

According to $\Gamma \in \text{BL}_R(\text{lip}(X, d), \text{lip}(Y, \rho, \tau))$ and (2.3), we deduce that the operator $C_{\phi, \text{lip}(X, d)} : \text{lip}(X, d) \rightarrow \text{lip}(X, d)$ is compact if and only if $C_{\psi, \text{lip}(Y, \rho, \tau)} : \text{lip}(Y, \rho, \tau) \rightarrow \text{lip}(Y, \rho, \tau)$ is compact. Hence, (iii) holds.

According to Theorems 2.9 and 2.10, it is clear that Theorem 2.8 is also a generalization of [4, Theorem 1.3], whenever $\mathbb{K} = \mathbb{C}$.

The following result is concerning the compactness of composition operators on Lipschitz spaces $\text{Lip}_0,\mathbb{K}(X, d)$ obtained by Jiménez-Vargas and Víllegas-Vallecillos [4].

**Theorem 2.11** (see [4, Theorem 1.2]). Let $(X, d)$ be a base pointed metric space and $\phi : X \rightarrow X$ be a base point preserving Lipschitz mapping from $(X, d)$ into $(X, d)$. Then the composition operator $C_{\phi, \text{Lip}_0,\mathbb{K}(X, d)} : \text{Lip}_0,\mathbb{K}(X, d) \rightarrow \text{Lip}_0,\mathbb{K}(X, d)$ is compact if and only if $\phi$ supercontractive and $\phi(X)$ is totally bounded in $(X, d)$. 
In the following result, we characterize compact composition operators on real Lipschitz spaces \( \text{Lip}_0(X, d, \tau) \).

**Theorem 2.12** Let \((X, d)\) be a base pointed metric space, \(\tau\) be a base point preserving Lipschitz involution on \((X, d)\) and \(\phi : X \rightarrow X\) be a base point preserving Lipschitz mapping from \((X, d)\) into \((X, d)\) satisfying \(\phi \circ \tau = \tau \circ \phi\). Then the composition operator \(C_{\phi, \text{Lip}_0(X, d, \tau)} : \text{Lip}_0(X, d, \tau) \rightarrow \text{Lip}_0(X, d, \tau)\) is compact if and only if \(\phi\) is supercontractive and \(\phi(X)\) is totally bounded in \((X, d)\).

**Proof.** Since \(\tau\) is a Lipschitz involution on \((X, d)\), by Theorem 1.3, we deduce that \(\text{Lip}_0(X, d) = \text{Lip}_0(X, d, \tau) \oplus i \text{Lip}_0(X, d, \tau)\), there exists a constant \(C \geq 1\) such that

\[
\max\{L_{(X, d)}(f), L_{(X, d)}(g)\} \leq CL_{(X, d)}(f + ig) \leq 2C \max\{L_{(X, d)}(f), L_{(X, d)}(g)\}
\]

for all \(f, g \in \text{Lip}_0(X, d, \tau)\), and \((\text{Lip}_0(X, d, \tau), L_{(X, d)}(\cdot))\) is a real Banach space. Hence, by Theorem 2.1, the compactness of \(C_{\phi, \text{Lip}_0(X, d, \tau)} : \text{Lip}_0(X, d, \tau) \rightarrow \text{Lip}_0(X, d, \tau)\) is equivalent to the compactness of \((C_{\phi, \text{Lip}_0(X, d, \tau)}') : \text{Lip}_0(X, d) \rightarrow \text{Lip}_0(X, d)\) which is defined by

\[
(C_{\phi, \text{Lip}_0(X, d, \tau)})'(f + ig) = C_{\phi, \text{Lip}_0(X, d, \tau)}(f) + iC_{\phi, \text{Lip}_0(X, d, \tau)}(g)
\]

for all \(f, g \in \text{Lip}_0(X, d, \tau)\). It is easy to see that

\[
(C_{\phi, \text{Lip}_0(X, d, \tau)})' = C_{\phi, \text{Lip}_0(X, d)}.
\]

Therefore, the compactness of \(C_{\phi, \text{lip}(X, d, \tau)} : \text{Lip}(X, d, \tau) \rightarrow \text{Lip}(X, d, \tau)\) is equivalent to the compactness of \(C_{\phi, \text{Lip}_0(X, d)} : \text{Lip}_0(X, d) \rightarrow \text{Lip}_0(X, d)\) and this is equivalent to \(\phi\) is supercontractive and \(\phi(X)\) is totally bounded in \((X, d)\). Hence, the proof is complete.

\[\blacksquare\]

Note that Theorem 2.12 is a generalization of Theorem 2.11 whenever \(K = \mathbb{R}\).

### 3. Spectra of compact composition operators

We recall that if \(Y\) is a nonempty set, \(n \in \mathbb{N}\) and \(\psi : Y \rightarrow Y\) is a self-map of \(Y\), then a point \(y_0 \in Y\) is called a fixed point of \(\psi\) of order \(n\) if \(\psi^n(y_0) = y_0\) whenever \(n = 1\), and \(\psi^k(y_0) = y_0\) and \(\psi^k(y_0) \neq y_0\) for all \(k \in \{1, \ldots, n - 1\}\) whenever \(n \geq 2\).

Let \((X, d)\) be a metric space and the metric space \((\hat{X}, \hat{d})\) be the completion of \((X, d)\). It is known [10, Proposition 1.7.1] that if \((Y, \rho)\) is a complete metric space, then every Lipschitz mapping \(\phi : X \rightarrow Y\) from \((X, d)\) into \((Y, \rho)\) has a unique Lipschitz extension \(\tilde{\phi} : \hat{X} \rightarrow Y\) from \((\hat{X}, \hat{d})\) into \((Y, \rho)\), and

\[
\sup\{\rho(\tilde{\phi}(\tilde{x}), \tilde{\phi}(\tilde{y})) : \tilde{x}, \tilde{y} \in \hat{X}, \tilde{x} \neq \tilde{y}\} = \sup\{\rho(\phi(x), \phi(y)) : x, y \in X, x \neq y\}.
\]
Jiménez-Vargas and Villegas-Vallecillos [4] determined spectra of compact composition operators on Lipschitz spaces \( \text{Lip}_K(X,d), \| \cdot \|_{X,L} \) and little Lipschitz spaces \( \text{lip}_K(X,d), \| \cdot \|_{X,L} \) as the following.

**Theorem 3.1** (see [4, Theorem 1.4]). Let \((X,d)\) be a metric space, \(\phi : X \to X\) is a Lipschitz mapping from \((X,d)\) into \((X,d)\), \(\tilde{\phi} : \tilde{X} \to \tilde{X}\) its extension to the completion \((\tilde{X}, \tilde{d})\) of \((X,d)\) and \(A\) the set of all \(n \in \mathbb{N}\) such that \(\tilde{\phi}\) has a fixed point of order \(n\).

(i) If \(C_{\phi, \text{Lip}_K(X,d)} : \text{Lip}_K(X,d) \to \text{Lip}_K(X,d)\) is a compact operator, then \(A\) is finite and

\[
\sigma(C_{\phi, \text{Lip}_K(X,d)}) \setminus \{0\} = \bigcup_{n \in A} \{\lambda \in \mathbb{K} : \lambda^n = 1\}.
\]

Moreover, if \(X\) is infinite and connected in \((X,d)\), then

\[
\sigma(C_{\phi, \text{Lip}_K(X,d)}) = \{0, 1\}.
\]

(ii) Assume that \(\phi\) is bounded and \(\text{lip}_K(X,d)\) separates points uniformly on bounded subsets of \(X\). If \(C_{\phi, \text{lip}_K(X,d)} : \text{lip}_K(X,d) \to \text{lip}_K(X,d)\) is compact, then \(A\) is finite and

\[
\sigma(C_{\phi, \text{lip}_K(X,d)}) \setminus \{0\} = \bigcup_{n \in A} \{\lambda \in \mathbb{K} : \lambda^n = 1\}.
\]

Moreover, if \(X\) is infinite and connected in \((X,d)\), then

\[
\sigma(C_{\phi, \text{lip}_K(X,d)}) = \{0, 1\}.
\]

In the following theorem, we determine spectra of compact composition operators on \(\text{Lip}(X,d,\tau)\) and \(\text{lip}(X,d,\tau)\).

**Theorem 3.2** Let \((X,d)\) be a metric space, \(\tau\) a topological involution on \(X\), \(\phi : X \to X\) a Lipschitz mapping from \((X,d)\) into \((X,d)\) with \(\phi \circ \tau = \tau \circ \phi\), \(\tilde{\phi}\) the unique Lipschitz extension to completion \((\tilde{X}, \tilde{d})\) of \((X,d)\) and \(A\) the set of all \(n \in \mathbb{N}\) such that \(\tilde{\phi}\) has a fixed point of order \(n\).

(i) If \(C_{\phi, \text{Lip}(X,d,\tau)} : \text{Lip}(X,d,\tau) \to \text{Lip}(X,d,\tau)\) is a compact operator, then \(A\) is finite and

\[
\sigma(C_{\phi, \text{Lip}(X,d,\tau)}) \setminus \{0\} = \bigcup_{n \in A} \{\lambda \in \mathbb{R} : \lambda^n = 1\}.
\]

Moreover, if \(X\) is infinite and connected in \((X,d)\), then

\[
\sigma(C_{\phi, \text{Lip}(X,d,\tau)}) = \{0, 1\}.
\]
(ii) Assume that $\phi$ is bounded and $\text{lip}(X,d)$ separates points uniformly on bounded subsets of $X$. If $C_{\phi,\text{lip}(X,d,\tau)} : \text{lip}(X,d,\tau) \to \text{lip}(X,d,\tau)$ is compact, then $A$ is finite and

$$\sigma(C_{\phi,\text{lip}(X,d,\tau)}) \setminus \{0\} = \bigcup_{n \in A} \{ \lambda \in \mathbb{R} : \lambda^n = 1 \}.$$ 

Moreover, if $X$ is infinite and connected in $(X,d)$, then

$$\sigma(C_{\phi,\text{lip}(X,d,\tau)}) = \{0, 1\}.$$ 

**Proof.** Let $A = \text{Lip}(X,d,\tau)$ and $B = \text{Lip}(X,d)$ ($A = \text{lip}(X,d,\tau)$ and $B = \text{lip}(X,d)$, respectively). Suppose that $\phi$ is bounded and $B$ separates points uniformly on bounded subsets of $X$ whenever $A = \text{lip}(X,d,\tau)$.

Let $C_{\phi,A} : A \to A$ be a compact operator. Since $\tau$ is a topological involution on $(X,d)$, by Theorem 1.2, $B = A \oplus iA$ and there exists a constant $C \geq 1$ such that

$$\max\{\|f\|_{X,L}, \|g\|_{X,L}\} \leq C\|f + ig\|_{X,L} \leq 2C \max\{\|f\|_{X,L}, \|g\|_{X,L}\},$$

for all $f, g \in A$. By Theorem 2.1, $(C_{\phi,A})' : B \to B$ is a compact operator and

$$\sigma(C_{\phi,A}) = \mathbb{R} \cap \sigma((C_{\phi,A})').$$

By the argument given in the proofs of Theorem 2.3 for $A = \text{Lip}(X,d,\tau)$ and Theorem 2.8 for $A = \text{lip}(X,d,\tau)$, we have

$$(C_{\phi,A})' = C_{\phi,B}.$$ 

Therefore,

$$\sigma(C_{\phi,A}) = \mathbb{R} \cap \sigma(C_{\phi,B}). \quad (4)$$

On the other hand, by Theorem 3.1, we have

$$\sigma(C_{\phi,B}) \setminus \{0\} = \bigcup_{n \in A} \{ \lambda \in \mathbb{C} : \lambda^n = 1 \}. \quad (5)$$

From (4) and (5), we conclude that

$$\sigma(C_{\phi,A}) \setminus \{0\} = \bigcup_{n \in A} \{ \lambda \in \mathbb{R} : \lambda^n = 1 \}.$$ 

Moreover, if $X$ is infinite and connected in $(X,d)$, then

$$\sigma(C_{\phi,B}) = \{0, 1\}. $$
Spectrum of the compact composition operator $C$

Hence, the proof is complete.

In the following example which is a modification of [4, Example 1.1], we determine the spectrum of the compact composition operator $C_{\phi, Lip(X, d, \tau)}$ on $Lip(X, d, \tau)$, where $\tau$ is a suitable Lipschitz involution on $(X, d)$.

**Example 3.3** Take the sets $Z = [-1, -1/2] \cup [1/2, 1]$ and $Y = [-1/2, -1/4] \cup [1/4, 1/2]$ endowed, respectively, with the metrics endowed, respectively, with the metrics.

$$d_Z(x, y) = |x - y|, \quad (\forall x, y \in Z); \quad d_Y(x, y) = \sqrt{|x - y|}, \quad (\forall x, y \in Y).$$

Let $X = Y \cup Z$ and let $d : X \times X \to \mathbb{R}$ the distance on $X$ given by

$$d(x, y) = \begin{cases} d_Z(x, y) & \text{if } x, y \in Z; \\ d_Y(x, y) & \text{if } x, y \in Y; \\ d_Z(x, -1/2) + d_Y(-1/2, y) & \text{if } x \in [-1, -1/2], y \in Y; \\ d_Z(y, -1/2) + d_Y(-1/2, x) & \text{if } y \in [-1, -1/2], x \in Y; \\ d_Z(x, 1/2) + d_Y(1/2, y) & \text{if } x \in [1/2, 1], y \in Y; \\ d_Z(y, 1/2) + d_Y(1/2, x) & \text{if } y \in [1/2, 1], x \in Y. \end{cases}$$

Notice that $(X, d)$ is compact since the topology generated by $d$ is the usual topology of $X$. Define the map $\tau : X \to X$ by $\tau(x) = -x$. It is easy to see that

$$d(\tau(x), \tau(y)) = d(x, y),$$

for all $x, y \in X$, and so $\tau$ is a Lipschitz involution on $(X, d)$. Consider now $\phi : X \to X$ defined by

$$\phi(x) = \begin{cases} -2x & \text{if } x \in Y, \\ 1 & \text{if } x \in [-1, -1/2], \\ -1 & \text{if } x \in [1/2, 1]. \end{cases}$$

It is not hard to check that $\phi$ is Lipschitz mapping from $(X, d)$ into $(X, d)$ and $\phi \circ \tau = \tau \circ \phi$. Thus, $C_{\phi, Lip(X, d, \tau)} : Lip(X, d, \tau) \to Lip(X, d, \tau)$ is compact by Theorem 2.3. It is easy to see that $-1$ and $1$ are fixed point of $\phi$ of order 2 and if $x \in X \setminus \{-1, 1\}$, then $x$ is not fixed point of $\phi$ of order $n$ for all $n \in \mathbb{N}$. Since $(X, d)$ is a compact metric space, we deduce that $(\tilde{X}, \tilde{d}) = (X, d)$ and $\tilde{\phi} = \phi$. Thus, $A = \{2\}$ and so, by Theorem 3.2, we have

$$\sigma(C_{\phi, Lip(X, d, \tau)}) \setminus \{0\} = \bigcup_{n \in A} \{\lambda \in \mathbb{R} : \lambda^2 = 1\} = \{-1, 1\}. \quad (6)$$
On the other hand, $0 \in \sigma(C_{\phi,Lip}(X,d))$ since $X$ is infinite. Thus, $0 \in \mathbb{R} \cap \sigma(C_{\phi,Lip}(X,d))$.

By the argument given in the proof of Theorem 3.2, we conclude that $0 \in \sigma(C_{\phi,Lip}(X,d,\tau))$.

Now, from (6) we have

$$\sigma(C_{\phi,Lip}(X,d,\tau)) = \{-1, 0, 1\}.$$ 

References


