Commutativity degree of $\mathbb{Z}_p \wr \mathbb{Z}_p^n$

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Abstract. For a finite group $G$ the commutativity degree denote by $d(G)$ and defin:

$$d(G) = \frac{|\{(x,y) | x,y \in G, xy = yx\}|}{|G|^2}.$$ 

In [2] authors found commutativity degree for some groups, in this paper we find commutativity degree for a class of groups that have high nilpotencies.

Keywords: Presentation of groups, Finite groups, commutativity degree.

1. Introduction

For a finite group $G$ the commutativity degree

$$d(G) = \frac{|\{(x,y) | x,y \in G, xy = yx\}|}{|G|^2}.$$ 

is defined and studied by several authors (see for example [2, 3, 7]). When $d(G) \geq \frac{1}{2}$, it is proved by P. Lescot in 1995 that $G$ is abelian, or $\frac{G}{Z(G)}$ is elementary abelian with $|G| = 2$, or $G$ is isoclinic with $S_3$ and $d(G) = 1$.

Throughout this paper $n$ is positive integer and $p$ is odd prime number. We consider the wreath product $G_n = \mathbb{Z}_p \wr \mathbb{Z}_p^n$ where, the standard wreath product $G \wr H$ of the finite groups $G$ and $H$ is defined to be semidirect product of $G$ by direct product $B$ of $|G|$ copies of $H$.

In [1] it is proved that $G_n$ has efficient presentation as follows:

$$G_n = \langle x,y | y^p = x^{p^n} = 1, [x,x^y] = 1, 1 \leq i \leq \frac{p-1}{2} \rangle.$$ 

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Main theorems in this paper are:

**Theorem 1.1**

\[
d(G_n) = \frac{p^{(p-1)n} + (p^2 - 1)}{p^{(p-1)n+2}}.
\]

**Theorem 1.2**

\[
\lim_{n \to \infty} d(G_n) = \frac{1}{p^2}.
\]

**Theorem 1.3**

\[
\frac{1}{p^2} < d(G_n) < \frac{1}{p}.
\]

2. **Proofs**

We need some lemmas for proving Theorems 1.1, 1.2 and 1.3.

**Lemma 2.1** In group $G_n$ every element $z$ has an unique presentations as follows:

\[
z = y^{\alpha}(x)^{\beta_0}(x^y)^{\beta_1}(x^{y^2})^{\beta_2}...(x^{y^{p-1}})^{\beta_{p-1}}
\]

where $\alpha \in \{0, 1, 2, ..., p-1\}$ and $\beta_i \in \{0, 1, 2, ..., p^n - 1\}$ $(0 \leq i \leq p - 1)$.

**Proof** By presentation of $G_n$, it is clearly.

**Lemma 2.2** Let $z_1, z_2 \in G_n$ and $z_1 = y^{\alpha_1}(x)^{\beta_0}(x^y)^{\beta_1}(x^{y^2})^{\beta_2}...(x^{y^{p-1}})^{\beta_{p-1}}$ and $z_2 = y^{\alpha_2}(x)^{\gamma_0}(x^y)^{\gamma_1}(x^{y^2})^{\gamma_2}...(x^{y^{p-1}})^{\gamma_{p-1}}$. Then $z_1z_2 = z_2z_1$ if and only if:

\[
\beta_i + \gamma_{\alpha_2+i} = \beta_{\alpha_2+i} + \gamma_{\alpha_2-\alpha_1+i} \pmod{p^n}, (i = 0, 1, 2, ..., p - 1)
\]

where indices are reduced module of $p$.

**Proof** We have:

\[
z_2z_1 = y^{\alpha_1+\alpha_2}(x^y)^{\gamma_0}(x^{y^2})^{\gamma_1}...(x^{y^{p-1}})^{\gamma_{p-1}}(x)^{\beta_0}(x^y)^{\beta_1}(x^{y^2})^{\beta_2}...(x^{y^{p-1}})^{\beta_{p-1}}
\]

and

\[
z_1z_2 = y^{\alpha_1+\alpha_2}(x^y)^{\beta_0}(x^{y^2})^{\beta_1}...(x^{y^{p-1}})^{\beta_{p-1}}(x)^{\gamma_0}(x^y)^{\gamma_1}(x^{y^2})^{\gamma_2}...(x^{y^{p-1}})^{\gamma_{p-1}}.
\]

By lemma 2.1 every element in $G_n$ has unique presentation, so we have:
So we have:

\[
\begin{align*}
\beta_0 + \gamma_{\alpha_2} & \equiv \beta_{\alpha_2} + \gamma_{\alpha_2 - \alpha_1} \quad (\text{mod } p^n) \\
\beta_1 + \gamma_{\alpha_2+1} & \equiv \beta_{\alpha_2+1} + \gamma_{\alpha_2 - \alpha_1+1} \quad (\text{mod } p^n) \\
\vdots & \quad \vdots \\
\beta_{p-1} + \gamma_{\alpha_2+p-1} & \equiv \beta_{\alpha_2+p-1} + \gamma_{\alpha_2 - \alpha_1+p-1} \quad (\text{mod } p^n).
\end{align*}
\]

Then we have:

\[
\beta_i + \gamma_{\alpha_2+i} \equiv \beta_{\alpha_2+i} + \gamma_{\alpha_2 - \alpha_1+i} \quad (\text{mod } p^n), \quad (i = 0, 1, 2, \ldots, p - 1).
\]

\[\blacksquare\]

**Remark:** On set $G_n \times G_n$, we consider:

\[
\zeta(G_n) = \{(z_1, z_2) | z_1, z_2 \in G_n, z_1 z_2 = z_2 z_1\}.
\]

**Lemma 2.3**

\[
|\zeta(G_n)| = p^{(p+1)n}(p^{(p-1)n} + p^2 - 1).
\]

**Proof** Let $z \in G_n$ and $z = y^\alpha(x)^\beta_1(x^{y^\gamma_1})^\beta_2 \ldots (x^{y^{p-1}})^\beta_{p-1}$. We consider $\psi(z) = \alpha$. Now let

\[
\zeta_{\alpha_1, \alpha_2}(G_n) = \{(z_1, z_2) | z_1, z_2 \in G_n, z_1 z_2 = z_2 z_1, \psi(z_1) = \alpha_1, \psi(z_2) = \alpha_2\}.
\]

So we have:

\[
\bigcup_{\alpha_1=0}^{p-1} \bigcup_{\alpha_2=0}^{p-1} \zeta_{\alpha_1, \alpha_2}(G_n) = \zeta(G_n).
\]

More over:

\[
|\zeta(G_n)| = \sum_{\alpha_1=0}^{p-1} \sum_{\alpha_2=0}^{p-1} |\zeta_{\alpha_1, \alpha_2}(G_n)|.
\]

Now we have two cases.

**Case I:** $\alpha_1 = 0, \alpha_2 = 0$

Let $z_1 = x^{\beta_1}(x^{y^\gamma_1})^\beta_2 \ldots (x^{y^{p-1}})^\beta_{p-1}$ and $z_2 = x^{\gamma_0}(x^{y^\gamma_1})^\gamma_2 \ldots (x^{y^{p-1}})^\gamma_{p-1}$ where $\beta_i, \gamma_j \in \{0, 1, \ldots, p^n - 1\}$ and $0 \leq i, j \leq p - 1$.

Since $z_1 z_2 = z_2 z_1$ then:

\[
|\zeta_{0,0}(G_n)| = \frac{p^n \times p^n \times \cdots \times p^n}{2^p} = p^{(p+1)n}.
\]

**Case II:** $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$,

Let $z_1 = y^{\alpha_1}(x)^{\beta_1}(x^{y^{\gamma_1}})^{\beta_2} \ldots (x^{y^{p-1}})^{\beta_{p-1}}$ and $z_2 = y^{\alpha_2}(x)^{\gamma_0}(x^{y^{\gamma_1}})^{\gamma_2} \ldots (x^{y^{p-1}})^{\gamma_{p-1}}$. If $z_1 z_2 = z_2 z_1$ by lemma 2.2 we have:

\[
\beta_i + \gamma_{\alpha_2+i} \equiv \beta_{\alpha_2+i} + \gamma_{\alpha_2 - \alpha_1+i} \quad (\text{mod } p^n), \quad (i = 0, 1, 2, \ldots, p - 1) \quad (*)
\]
where indices are reduced module of \( p \).

Now we can choose \( \beta_0, \beta_1, \ldots, \beta_{p-1}, \gamma_0 \) and find \( \gamma_1, \gamma_2, \ldots, \gamma_{p-1} \) uniquely by \((*)\), then

\[
|\zeta_{\alpha_1,\alpha_2}(G_n)| = \frac{p^n \times p^n \times \ldots \times p^n}{p+1} = p^{n(p+1)}.
\]

Finally we have

\[
|\zeta(G_n)| = \sum_{\alpha_1=0}^{p-1} \sum_{\alpha_2=0}^{p-1} |\zeta_{\alpha_1,\alpha_2}(G_n)| = p^{2np} + (p^2 - 1)p^{n(p+1)} = p^{(p+1)n}(p^{(p-1)n} + p^2 - 1).
\]

**Proof theorems 1.1, 1.2 and 1.3:**

For 1.1 since \( d(G_n) = \frac{|\zeta(G_n)|}{|G_n|^2} \) so by lemma 2.3 we find \( d(G_n) = \frac{p^{(p-1)n} + (p^2 - 1)}{p^{(p-1)n+2}} \).

For 1.2 and 1.3 we have \( d(G_n) = \frac{1}{p^2} + \frac{p^2 - 1}{p^{p-1} + 2} \), so

\[
\lim_{n \to \infty} d(G_n) = \frac{1}{p^2}
\]

and \( d(G_n) > \frac{1}{p^2} \cdot \frac{1}{p^2} < \frac{1}{p} \) is simple . \( \square \)

**References**


[2] H. Doostie, M. Maghasedi, Certain classes of groups with commutativity degree \( d(G) < \frac{1}{2} \), Ars combinatorial, 89 (2008), 263–270.


